ELECTRICITY

Part –2 (11^{th} to 12^{th})

11^{th} std (Term-I)

Unit - 4

WORK, ENERGY AND POWER

INTRODUCTION

The term work is used in diverse contexts in daily life. It refers to both physical as well as mental work. In fact, any activity can generally be called as work. But in Physics, the term work is treated as a physical quantity with a precise definition. Work is said to be done by the force when the force applied on a body displaces it. To do work, energy is required. In simple words, energy is defined as the ability to do work. Hence, work and energy are equivalents and have same dimension. Energy, in Physics exists in different forms such as mechanical, electrical, thermal, nuclear and so on. Many machines consume one form of energy and deliver energy in a different form. In this chapter we deal mainly with mechanical energy and its two types namely kinetic energy and potential energy. The next quantity in this sequence of discussion is the rate of work done or the rate of energy delivered. The rate of work done is called power. A powerful strike in cricket refers to a hit on the ball at a fast rate. This chapter aims at developing a good understanding of these three physical quantities namely work, energy and power and their physical significance.

WORK

Let us consider a force ($\overline{F}$), acting on a body which moves it by a displacement in some direction (dr)

The expression for work done (w) by the force on the body is mathematically written as,

$$W=\overline{F}.dr$$

Here, the product $\overline{F}.dr$ is a scalar product (or dot product). The scalar product of two vectors is a scalar. Thus, work done is a scalar quantity. It has only magnitude and no direction. In SI system, unit of work done is N m (or) joule (J). Its dimensional formula is $[ML^2T^{-2}]$.

The equation (4.1) is,

$$W= F\ dr\ \cos\theta$$

which can be realised using (as $a.\dot{b}$ ab$\cos\theta$) where, θ is the angle between applied force and the displacement of the body.

The work done by the force depends on the force (F), displacement (dr) and the angle (θ) between them Work done is zero in the following cases.

When the force is zero (F = 0). For example, a body moving on a horizontal smooth frictionless surface will continue to do so as no force (not even friction) is acting along the plane. (This is an ideal situation.)

When the displacement is zero (dr = 0). For example, when force is applied on a rigid wall it does not produce any displacement. Hence, the work done is zero

When the force and displacement are perpendicular (θ = 90o) to each other. when a body moves on a horizontal direction, the gravitational force (mg) does no work on the body, since it acts at right angles to the displacement as shown in Figure 4.3(b). In circular motion the centripetal force does not do work on the object moving on a circle as it is always perpendicular to the displacement.

For a given force (F) and displacement (dr), the angle (θ) between them decides the value of work done as consolidated.

There are many examples for the negative work done by a force. In a football game, the goalkeeper catches the ball coming towards him by applying a force such that the force is applied in a direction opposite to that of the motion of the ball till it comes to rest in his hands. During the time of applying the force, he does a negative work on the ball. We will discuss many more situations of negative work further in this unit.

A box is pulled with a force of 25 N to produce a displacement of 15 m. If the angle between the force and displacement is 30o, find the work done by the force.

- Force, F = 25 N
- Displacement, dr = 15 m
- Angle between F and dr, θ = 30°

Angle (θ)	cosθ	Work
θ =0 °	1	Positive, Maximum
0 θ<9 °(acti e)	0 c osθ<1	Positive
θ =9 °(rih a h e)	0	Zero
9 °<θ<8 °	-1 c osθ<0	Negative
θ =8 °	-1	Negative, Maximum

Work done, W= Fdr $\cos\theta$

$$W = 25 \times 15 \times \cos 30 = 25 \times 15 \times \frac{\sqrt{3}}{2}$$

W= 324.76 J

Work done by a constant force

When a constant force F acts on a body, the small work done (dW) by the force in producing a small displacement dr is given by the relation,

$$dW = (F\cos\theta)\, dr$$

The total work done in producing a displacement from initial position $\mathbf{r_i}$ to final position $\mathbf{r_f}$ is,

$$W = \int_{r_1}^{r_f} dW$$

$$W = \int_{r_f}^{r_f} (F\cos\theta) dr = (F\cos\theta)\int_{r_f}^{r_f} dr$$

$$= (F\cos\theta)(r_f - r_i)$$

The graphical representation of the work done by a constant force . The area under the graph shows the work done by the constant force.

An object of mass 2 kg falls from a height of 5 m to the ground. What is the work done by the gravitational force on the object? (Neglect air resistance; Take g = 10 m s^{-2})

In this case the force acting on the object is downward gravitational force $m\dot{g}$. This is a constant force. Work done by gravitational force is

$$W = \int_{r_1}^{r_f} \overline{F} d\overline{r}$$

$$W = (\cos\theta \int_{r_1}^{r_f} d\overline{r} = (mg.\cos\theta)(r_f - r_1)$$

The object also moves downward which is in the direction of gravitational force $\dot{F} = m\dot{g}$ as shown in figure. Hence, the angle between them is θ = 0°; cos θ =1 and the displacement, $(r_f - r_1)$ =5m.

$$W = mg(r_f - r_1)$$

$$W = 2 \times \phi \times 5 = \phi$$

The work done by the gravitational force on the object is positive.

An object of mass *m*=1 kg is sliding from top to bottom in the frictionless inclined plane of inclination angle θ =30° and the length of inclined plane is 10 m as shown in the figure. Calculate the work done by gravitational force and normal force on the object. Assume acceleration due to gravity, g = 10 m s-²

We calculated in the previous chapter that the acceleration experienced by the object in the inclined plane as g sinθ . According to Newton's second law, the force acting on the mass along the inclined plane F = mg sinθ. Note that this force is constant throughout the motion of the mass. The work done by the parallel component of gravitational force (mg sinθ) is given by

$$W = \vec{F} \cdot d\vec{r} = F dr \cos\phi$$

where ϕ is the angle between the force (mg sin θ) and the direction of motion (dr). In this case, force (mg sin θ) and the displacement (dr) are in the same direction. Hence $\phi = and \cos\phi = 1$

$$W = F\, dr = (mg \sin\theta)\,(dr)$$

(dr = length of the inclined place)

$$W = 1 \times 10 \times \sin(30°) \times 10 = 100 \times \frac{1}{2} = 50J$$

The component mg cosθ and the normal force *N* are perpendicular to the direction of motion of the object, so they do not perform any work.

If an object of mass 2 kg is thrown up from the ground reaches a height of 5 m and falls back to the Earth (neglect the air resistance). Calculate

1. The work done by gravity when the object reaches 5 m height
2. The work done by gravity when the object comes back to Earth
3. Total work done by gravity both in upward and downward motion and mention the physical significance of the result.

When the object goes up, the displacement points in the upward direction whereas the gravitational force acting on the object points in downward direction. Therefore, the angle between gravitational force and displacement of the object is 180°.

The work done by gravitational force in the upward motion.

Given that Δr =5m and F =mg

$$W_{up} = F\Delta r \cos\theta = mg\Delta r \cos 180^o$$

$$W_{up} = 2\times 10\times 5\times(-1) = -100\, joule.$$

$$[\cos 180^o = -1]$$

When the object falls back, both the gravitational force and displacement of the object are in the same direction. This implies that the angle between gravitational force and displacement of the object is 0°.

$$W_{down} = F\Delta r \cos 0^o$$

$$W_{down} = 2\times 10\times 5\times(1) = 100\, joule$$

$$[\cos 0^o = 1]$$

The total work done by gravity in the entire trip (upward and downward motion).

$$W_{total} = W_{up} + W_{down}$$

$$= -100\, joule + 100\, joule = 0$$

It implies that the gravity does not transfer any energy to the object. When the object is thrown upwards, the energy is transferred to the object by the external agency, which means that the object gains some energy. As soon as it comes back and hits the Earth, the energy gained by the object is transferred to the surface of the Earth (i.e., dissipated to the Earth).

A weight lifter lifts a mass of 250 kg with a force 5000 N to the height of 5 m.

1. What is the workdone by the weight lifter?
2. What is the workdone by the gravity?
3. What is the net workdone on the object?

When the weight lifter lifts the mass, force and displacement are in the same direction, which means that the angle between them $\theta = 0^o$. Therefore, the work done by the weight lifter,

$$W_{\text{weight lifter}} = F_w h\cos\theta = F_w h(\cos 0^0)$$
$$= 5000 \times 5 \times (1) = 25,000\, joule = 25kJ$$

When the weight lifter lifts the mass, the gravity acts downwards which means that the force and displacement are in opposite direction. Therefore, the angle between them $\theta = 180^{\circ}$

$$W_{\text{gravity}} = F_g h\cos\theta = mgh(\cos 180^{\circ})$$
$$= 250 \times 10 \times 5 \times (-1)$$
$$= -12,500\, joule = -12.5kJ$$

The net work done (or total work done) on the object

$$W_{\text{net}} = W_{\text{weight lifter}} + W_{\text{gravity}}$$
$$= 25kJ - 12.5kJ = +12.5kJ$$

Work done by a variable force

When the component of a variable force F acts on a body, the small work done (dW) by the force in producing a small displacement dr is given by the relation

$$dW = F\cos\theta\, dr$$

[F cos θ is the component of the variable force F] where, F and θ are variables. The total work done for a displacement from initial position r_i to final position r_f is given by the relation,

$$W = \int_{r_i}^{r_f} dW = \int_{r_i}^{r_f} F \cos\theta dr$$

A graphical representation of the work done by a variable force. The area under the graph is the work done by the variable force.

A variable force $F = k x^2$ acts on a particle which is initially at rest. Calculate the work done by the force during the displacement of the particle from $x = 0$ m to $x = 4$ m. (Assume the constant $k = 1$ N m^{-2})

Work done,

$$W = \int_{x_i}^{x_f} F(x)dx = k\int_{0}^{4} x^2 dx = \frac{64}{3} Nm$$

Energy is defined as the capacity to do work. In other words, work done is the manifestation of energy. That is why work and energy have the same dimension (ML^2T^{-2})

The important aspect of energy is that for an isolated system, the sum of all forms of energy i.e., the total energy remains the same in any process irrespective of whatever internal changes may take place. This means that the energy disappearing in one form reappears in another form. This is known as the law of conservation of energy. In this chapter we shall take up only the mechanical energy for discussion.

In a broader sense, mechanical energy is classified into two types

1. Kinetic energy
2. Potential energy

The energy possessed by a body due to its motion is called kinetic energy. The energy possessed by the body by virtue of its position is called potential energy.

The SI unit of energy is the same as that of work done i.e., N m (or) joule (J). The dimension of energy is also the same as that of work done. It is given by $[ML^2T^{-2}]$. The other units of energy and their SI equivalent values.

SI equivalent of other units of energy

Unit	Equivalent in joule
1 erg (CGS unit)	10^{-7} J
1 electron volt (eV)	$1.6x10^{-19}$ J
1 calorie (cal)	4.186 J
1 kilowatt hour (kWh)	$3.6x10^{6}$ J

Kinetic energy

Kinetic energy is the energy possessed by a body by virtue of its motion. All moving objects have kinetic energy. A body that is in motion has the ability to do work. For example a hammer kept at rest on a nail does not push the nail into the wood. Whereas the same hammer when it strikes the nail, draws the nail into the wood. Kinetic energy is measured by the amount of work that the body can perform before it comes to rest. The amount of work done by a moving body depends both on the mass of the body and the magnitude of its velocity. A body which is not in motion does not have kinetic energy.

Work–Kinetic Energy Theorem

Work and energy are equivalents. This is true in the case of kinetic energy also. To prove this, let us consider a body of mass m at rest on a frictionless horizontal surface.

The work (W) done by the constant force (F) for a displacement (s) in the same\ direction is,

$$W=Fs$$

The constant force is given by the equation,

$$F=ma$$

$$V^2 = u^2 + 2as$$

$$a = \frac{V^2 - u^2}{2s}$$

Substituting for a in equation

$$F = m\left(\frac{V^2 - u^2}{2s}\right)$$

$$W = m\left(\frac{V^2}{2s} S\right) - m\left(\frac{u^2}{2s} S\right)$$

$$W = \frac{1}{2} mv^2 - \frac{1}{2} mu^2$$

The term $\left(\frac{1}{2} mv^2\right)$ in the above equation is the kinetic energy of the body of mass (m) moving with velocity (v).

$$KE = \frac{1}{2} mv^2$$

Kinetic energy of the body is always positive. From equations

$$\Delta KE = \frac{1}{2} mv^2 - \frac{1}{2} mu^2$$

Thus, $W = \Delta KE$

The expression on the right hand side (RHS) of equation (4.12) is the change in kinetic energy (ΔKE) of the body.

This implies that the work done by the force on the body changes the kinetic energy of the body. This is called work-kinetic energy theorem.

The work-kinetic energy theorem implies the following.

1. If the work done by the force on the body is positive then its kinetic energy increases.
2. If the work done by the force on the body is negative then its kinetic energy decreases.
3. If there is no work done by the force on the body then there is no change in its kinetic energy, which means that the body has moved at constant speed provided its mass remains constant.

Relation between Momentum and Kinetic Energy

Consider an object of mass m moving with a velocity v. Then its linear momentum is $p = m\dot{v}$ and its kinetic energy, $KE = \frac{1}{2}mv^2$.

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\bar{v}.\bar{v})$$

Multiplying both the numerator and denominator of equation

$$KE = \frac{1}{2}\frac{m^2(\bar{v}.\bar{v})}{m}$$

$$= \frac{1}{2}\frac{(m\dot{v}).(m\dot{v})}{m}\left[\dot{p} = m\dot{v}\right]$$

$$= \frac{1}{2}\frac{\dot{p}.\dot{p}}{m}$$

$$= \frac{p^2}{2m}$$

$$KE = \frac{p^2}{2m}$$

where $|\dot{p}|$ is the magnitude of the momentum. The magnitude of the linear momentum can be obtained by

$$|\dot{p}| = p = \sqrt{2m(KE)}$$

Note that if kinetic energy and mass are given, only the magnitude of the momentum can be calculated but not the direction of momentum. It is because the kinetic energy and mass are scalars.

Two objects of masses 2 kg and 4 kg are moving with the same momentum of 20 kg m s^{-1}.

1. Will they have same kinetic energy?
2. Will they have same speed?

The kinetic energy of the mass is given by $KE = \frac{p^2}{2m}$

For the object of mass 2 kg, kinetic energy is

$$KE_1 = \frac{(20)^2}{2 \times 2} = \frac{400}{4} = 100J$$

For the object of mass 4 kg, kinetic energy is

$$KE_2 = \frac{(20)^2}{2 \times 4} = \frac{400}{8} = 50J$$

Note that $KE_1 \neq KE_2$ i.e., even though both are having the same momentum, the kinetic energy of both masses is not the same. The kinetic energy of the heavier object has lesser kinetic energy than smaller mass. It is because the kinetic energy is inversely proportional to the mass $\left(KE \alpha \frac{1}{m}\right)$ for a given momentum.

As the momentum, $p = mv$, the two objects will not have same speed.

Potential Energy

The potential energy of a body is associated with its position and configuration with respect to its surroundings. This is because the various forces acting on the body also depends on position and configuration.

"Potential energy of an object at a point P is defined as the amount of work done by an external force in moving the object at constant velocity from the point O (initial location) to the point P (final location). At initial point O potential energy can be taken as zero.

Mathematically, potential energy is defined as $U = \int \vec{F}_a . d\vec{r}$

where the limit of integration ranges from initial location point O to final location point P.

We have various types of potential energies. Each type is associated with a particular force.

1. The energy possessed by the body due to gravitational force gives rise to gravitational potential energy
2. The energy due to spring force and other similar forces give rise to elastic potential energy.
3. The energy due to electrostatic force on charges gives rise to electrostatic potential energy.

We will learn more about conservative forces in the section. Now, we continue to discuss more about gravitational potential energy and elastic potential energy.

Potential energy near the surface of the Earth

The gravitational potential energy (U) at some height *h* is equal to the amount of work required to take the object from ground to that height *h* with constant velocity.

Let us consider a body of mass *m* being moved from ground to the height h against the gravitational force.

The gravitational force F_g acting on the body is, $\vec{F}_g = -mg\hat{j}$ (as the force is in y direction, unit vector $\hat{j}$ is used). Here, negative sign implies that the force is acting vertically downwards. In order to move the body without acceleration (or with constant velocity), an external applied force $\dot{F_a}$ equal in magnitude but opposite to that of gravitational force F_g has to be applied on the body i.e., $\dot{F_a} = \dot{F_g}$. This implies that = $\dot{F_a} = +mg\hat{j}$. The positive sign implies that the applied force is in vertically upward direction. Hence, when the body is lifted up its velocity remains unchanged and thus its kinetic energy also remains constant.

The gravitational potential energy (U) at some height *h* is equal to the amount of work required to take the object from the ground to that height *h*.

$$U = \int \overline{F_a} d\overline{r} = \int_0^h \left|\overline{F_a}\right| \left|d\overline{r}\right| \cos\theta$$

Since the displacement and the applied force are in the same upward direction, the angle between them, $\theta = 0^o$. Hence, cos0^o=1 and $\left|\dot{F_a}\right| = mg$ and $\left|d\dot{r}\right| = dr$.

$$U = mg \int_0^h dr$$

$$U = mg\left[r\right]_0^h = mgh$$

Note that the potential energy stored in the object is defined through work done by the external force which is positive. Physically this implies that the agency which is applying the external force is transferring the energy to the object which is then stored as potential energy. If the object is allowed to fall from a height *h* then the stored potential energy is converted into kinetic energy.

An object of mass 2 kg is taken to a height 5 m from the ground _*g* _10ms^{-2}.

1. Calculate the potential energy stored in the object.
2. Where does this potential energy come from?

3. What external force must act to bring the mass to that height?
4. What is the net force that acts on the object while the object is taken to the height 'h'?

The potential energy $U = mg\,h = 2\times10\times5 = 100$ J Here the positive sign implies that the energy is stored on the mass

This potential energy is transferred from external agency which applies the force on the mass.

The external applied force $\vec{F}_a$ which takes the object to the height 5 m is $\vec{F}_a = -\vec{F}_g$.

$$\vec{F}_a = -\left(-mg\hat{j}\right) = mg\hat{j}$$

where, $\hat{j}$ represents unit vector along vertical upward direction.

From the definition of potential energy, the object must be moved at constant velocity. So the net force acting on the object is zero.

$$\vec{F}_g + \vec{F}_a = 0$$

Elastic Potential Energy

When a spring is elongated, it develops a restoring force. The potential energy possessed by a spring due to a deforming force which stretches or compresses the spring is termed as elastic potential energy. The work done by the applied force against the restoring force of the spring is stored as the elastic potential energy in the spring.

Consider a spring-mass system. Let us assume a mass, m lying on a smooth horizontal. Here, $x = 0$ is the equilibrium position. One end of the spring is attached to a rigid wall and the other end to the mass.

As long as the spring remains in equilibrium position, its potential energy is zero. Now an external force $F_a^{'}$ is applied so that it is stretched by a distance (x) in the direction of the force.

There is a restoring force called spring force $F_s^{\cdot}$ developed in the spring which tries to bring the mass back to its original position. This applied force and the spring force are equal in magnitude but opposite in direction i.e., $F_a^{\cdot} = -F_s^{\cdot}$ According Hooke's law, the restoring force developed in the spring is

$$F_s^{'} = -k\dot{x}$$

The negative sign in the above expression implies that the spring force is always opposite to that of displacement x and k is the force constant. Therefore applied force is $F_a^{\cdot} = +k\dot{x}$. The positive sign implies that the applied force is in the direction of displacement x. The spring force is an example of variable force as it depends on the displacement x. Let the spring be stretched to a small distance

dx.The work done by the applied force on the spring to stretch it by a displacement x is stored as elastic potential energy.

$$U = \int \overline{F_a} d\overline{r} = \int_0^x \left|\overline{F_a}\right| \left|d\overline{r}\right| \cos\theta$$

$$= \int_0^x F_a dx \cos\theta$$

The applied force $F_a^{\cdot}$ and the displacement dr (i.e., here dx) are in the same direction. As, the initial position is taken as the equilibrium position or mean position, x=0 is the lower limit of integration.

$$U = \int_0^x kxdx$$

$$U = k\left[\frac{x^2}{2}\right]_0^x$$

$$U = \frac{1}{2}kx^2$$

If the initial position is not zero, and if the mass is changed from position x_i to x_f , then the elastic potential energy is

$$U = \frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

Force-displacement graph for a spring

Since the restoring spring force and displacement are linearly related as $F = -kx$, and are opposite in direction, the graph between F and x is a straight line with dwelling only in the second and fourth quadrant as shown in Figure 4.10. Th e elastic potential energy can be easily calculated by drawing a F - x graph. Th e shaded area (triangle) is the work done by the spring force.

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\times(x)\times(kx)$$
$$= \frac{1}{2}kx^2$$

Potential energy-displacement graph for a spring

A compressed or extended spring will transfer its stored potential energy into kinetic energy of the mass attached to the spring.

In a frictionless environment, the energy gets transferred from kinetic to potential and potential to kinetic repeatedly such that the total energy of the system remains constant. At the mean position,

$$\Delta KE = \Delta U$$

Let the two springs A and B be such that $k_A > k_B$. On which spring will more work has to be done if they are stretched by the same force?

$$F = K_A x_A = K_B x_B$$

$$x_A = \frac{F}{k_A}, x_B = \frac{F}{k_B}$$

The work done on the springs are stored as potential energy in the springs.

$$U_A = \frac{1}{2}k_A x_A^2 \; ; \quad U_B = \frac{1}{2}k_B x_B^2$$

$$\frac{U_A}{U_B} = \frac{k_A x_A^2}{k_B x_B^2} = \frac{k_A\left(\frac{F}{k_A}\right)^2}{k_B\left(\frac{F}{k_B}\right)^2} = \frac{\frac{1}{k_A}}{\frac{1}{k_B}}$$

$$\frac{U_A}{U_B} = \frac{k_B}{k_A}$$

$k_A > k_B$ implies that $U_B > U_A$. Thus, more work is done on B than A.

A body of mass m is attached to the spring which is elongated to 25 cm by an applied force from its equilibrium position.

1. Calculate the potential energy stored in the spring-mass system?
2. What is the work done by the spring force in this elongation?
3. Suppose the spring is compressed to the same 25 cm, calculate the potential energy stored and also the work done by the spring force during compression. (The spring constant, k = 0.1 N m^{-1}).

The spring constant, k = 0.1 N m^{-1}
The displacement, x = 25 cm = 0.25 m

The potential energy stored in the spring is given by

$$U = \frac{1}{2}kx^2 = \frac{1}{2} \times 0.1 \times (0.25)^2 = 0.0031J$$

The work done W_s by the spring force F_s is given by,

$$W_s = \int_0^x \vec{F}_s \cdot d\vec{r} = \int_0^x \left(-kx\hat{i}\right) \cdot (dx\,\hat{i})$$

The spring force F_s acts in the negative x direction while elongation acts in th positive x direction.

$$W_s = \int_0^x (-kx)dx = -\frac{1}{2}kx^2$$

$$W_s = -\frac{1}{2} \times 0.1 \times (0.25)^2 = -0.0031J$$

Note that the potential energy is defined through the work done by the external agency. The positive sign in the potential energy implies that the energy is transferred from the agency to the object. But the work done by the restoring force in this case is negative since restoring force is in the opposite direction to the displacement direction.

During compression also the potential energy stored in the object is the same.

$$U = \frac{1}{2}kx^2 = 0.0031J$$

Work done by the restoring spring force during compression is given by

$$W_s = \int_0^x \vec{F}_s . d\vec{r} = \int_0^x \left(kx\hat{i}\right).\left(-dx\,\hat{i}\right)$$

In the case of compression, the restoring spring force acts towards positive x-axis and displacement is along negative x direction.

$$W_s = \int_0^x (-kx)\,dx = -\frac{1}{2}kx^2 = -0.0031\,J$$

Conservative and nonconservative forces

Conservative force

A force is said to be a conservative force if the work done by or against the force in moving the body depends only on the initial and final positions of the body and not on the nature of the path followed between the initial and final positions. Let us consider an object at point A on the Earth. It can be taken to another point B at a height *h* above the surface of the Earth by three paths.

Whatever may be the path, the work done against the gravitational force is the same as long as the initial and final positions are the same. This is the reason why gravitational force is a conservative force. Conservative force is equal to the negative gradient of the potential energy. In one dimensional case, Examples for conservative forces are elastic spring force, electrostatic force, magnetic force, gravitational force, etc.

S.No	Conservative forces	Non-conservative forces
1.	Work done is independent of the path	Work done depends upon the path
2.	Work done in a round trip is zero	Work done in a round trip is not zero
3	Total energy remains constant	Energy is dissipated as heat energy
4	Work done is completely recoverable	Work done is not completely recoverable.
5	Force is the negative gradient of potential energy	No such relation exists.

Non-conservative force

A force is said to be non-conservative if the work done by or against the force in moving a body depends upon the path between the initial and final positions. This means that the value of work done is different in different paths.

1. Frictional forces are non-conservative forces as the work done against friction depends on the length of the path moved by the body.
2. The force due to air resistance, viscous force are also non-conservative forces as the work done by or against these forces depends upon the velocity of motion.

Compute the work done by the gravitational force for the following cases

$$\text{Force } \vec{F} = mg(-\hat{j}) = -mg\hat{j}$$

Displacement vector $d\dot{r} = dx\hat{i} + dy\hat{j}$

(As the displacement is in two dimension; unit vectors $\hat{i}$ and $\hat{j}$ are used)

Since the motion is only vertical, horizontal displacement component dx is zero. Hence, work done by the force along path 1 (of distance h).

$$W_{path\,1} = \int_A^B \vec{F}.d\vec{r} = \int_A^B (-mg\hat{j}).(dy\,\hat{j})$$

$$= -mg\int_0^h dy = -mgh$$

Total work done for path 2 is

$$W_{path\,2} = \int_A^B \vec{F}.d\vec{r} = \int_A^C \vec{F}.d\vec{r} + \int_C^D \vec{F}.d\vec{r} + \int_D^B \vec{F}.d\vec{r}$$

But

$$\int_A^C \vec{F}.d\vec{r} = \int_A^C (-mg\hat{j}).(dx\hat{i}) = 0$$

$$\int_C^D \vec{F}.d\vec{r} = \int_C^D (-mg\hat{j}).(dy\hat{j})$$

$$= -mg\int_0^h dy = -mgh$$

$$\int_D^B \vec{F}.d\vec{r} = \int_D^B (-mg\hat{j}).(-dx\hat{i}) = 0$$

Therefore, the total work done by the force along the path 2 is

$$W_{\text{path 2}} = \int_A^B \vec{F}.d\vec{r} = -mgh$$

Note that the work done by the conservative force is independent of the path.

Consider an object of mass 2 kg moved by an external force 20 N in a surface having coefficient of kinetic friction 0.9 to a distance 10 m. What is the work done by the external force and kinetic friction ? Comment on the result. (Assume g = 10 ms^{-2})

m = 2 kg, d = 10 m, F_{ext} = 20 N, μ_k = 0.9. When an object is in motion on the horizontal surface, it experiences two forces.

1. External force, F_{ext} =20 N
2. Kinetic friction

$$f_k = \mu_k mg = 0.9 \times (2) \times 10 = 18\,\text{N}.$$

The work done by the external force $W_{ext} = F\ d \times 20 \times 10 \times 200$J

The work done by the force of kinetic friction $W_k = f_k = (-18) \times 10 = -180$ J. Here the negative sign implies that the force of kinetic friction is opposite to the direction of displacement.

The total work done on the object $W_{total} = W_{ext} + W_k = 200$ J -180 J = 20 J.

Since the friction is a non-conservative force, out of 200 J given by the external force, the 180 J is lost and it can not be recovered.

Law of conservation of energy

When an object is thrown upwards its kinetic energy goes on decreasing and consequently its potential energy keeps increasing (neglecting air resistance). When it reaches the highest point its energy is completely potential. Similarly, when the object falls back from a height its kinetic energy increases whereas its potential energy decreases. When it touches the ground its energy is completely kinetic. At the intermediate points the energy is both kinetic and potential When the body reaches the ground the kinetic energy is completely dissipated into some other form of energy like sound, heat, light and deformation of the body etc.

In this example the energy transformation takes place at every point. The sum of kinetic energy and potential energy i.e., the total mechanical energy always remains constant, implying that the total energy is conserved. This is stated as the law of conservation of energy.

The law of conservation of energy states that energy can neither be created nor destroyed. It may be transformed from one form to another but the total energy of an isolated system remains constant.

illustrates that, if an object starts from rest at height h, the total energy is purely potential energy (U=mgh) and the kinetic energy (KE) is zero at h. When the object falls at some distance y, the potential energy and the kinetic energy are not zero whereas, the total energy remains same as measured at height h. When the object is about to touch the ground, the potential energy is zero and total energy is purely kinetic.

An object of mass 1 kg is falling from the height h = 10 m. Calculate

1. The total energy of an object at h =10 m
2. Potential energy of the object when it is at h = 4 m
3. Kinetic energy of the object when it is at h = 4 m
4. What will be the speed of the object when it hits the ground? (Assume g =10 ms^{-2})

The gravitational force is a conservative force. So the total energy remains constant throughout the motion. At h =10 m, the total energy E is entirely potential energy.

$$E = U = mgh = 1 \times 10 \times 10 = 100 \text{ J}$$

The potential energy of the object at h =4 m is

$$U = mgh = 1 \times 10 \times 4 = 40 \text{ J}$$

Since the total energy is constant throughout the motion, the kinetic energy at h=4m must be $KE = E - U = 100 - 40 = 60$J

Alternatively, the kinetic energy could also be found from velocity of the object at 4 m. At the height 4 m, the object has fallen through a height of 6 m.

The velocity after falling 6 m is calculated from the equation of motion,

$$v = \sqrt{2gh} = \sqrt{2 \times 10 \times 6} = \sqrt{120} \text{ m s}^{-1};$$

$$v^2 = 120$$

The kinetic energy is $KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 1 \times 120 = 60 \text{ J}$

When the object is just about to hit the ground, the total energy is completely kinetic and the potential energy, U=0.

$$E = KE = \frac{1}{2}mv^2 = 100\ J$$

$$v = \sqrt{\frac{2}{m}KE} = \sqrt{\frac{2}{1} \times 100} = \sqrt{200} = 14.12\ m\ s^{-1}$$

A body of mass 100 kg is lifted to a height 10 m from the ground in two different ways as shown in the figure. What is the work done by the gravity in both the cases? Why is it easier to take the object through a ramp?

m = 100 kg, h = 10 m
Along path (1):

The minimum force $F1$ required to move the object to the height of 10 m should be equal to the gravitational force, $F_1 = mg = 100 \times 10 = 1000$ N

The distance moved along path (1) is, h=10 m

$$W = F\,h = 1000 \times 10 = 10{,}000\ J$$

Along path (2):

In the case of the ramp, the minimum force F_2 that we apply on the object to take it up is not equal to mg, it is rather equal to $mg \sin\theta$. (mg $\sin\theta < mg$) .

Here, angle $\theta = 30^o$

Therefore, $F_2 = mg \sin\theta = 100 \times 10 \times \sin 30o = 100 \times 10 \times 0.5 = 500N$

Hence, ($mg \sin\theta < mg$).

$$l = \frac{h}{\sin 30} = \frac{10}{0.5} = 20\,m$$

The work done on the object along path (2) is, $W = F_2\,l = 500 \times 20 = 10{,}000$ J

Since the gravitational force is a conservative force, the work done by gravity on the object is independent of the path taken.

In both the paths the work done by the gravitational force is 10,000 J

Along path (1): more force needs to be applied against gravity to cover lesser distance .

Along path (2): lesser force needs to be applied against the gravity to cover more distance.

As the force needs to be applied along the ramp is less, it is easier to move the object along the ramp.

An object of mass m is projected from the ground with initial speed v_0. Find the speed at height h.

Since the gravitational force is conservative; the total energy is conserved throughout the motion.

	Initial	Final
Kinetic energy	$\frac{1}{2}mv_0^2$	$\frac{1}{2}mv^2$
Potential energy	0	mgh
Total energy	$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_0^2$	$\frac{1}{2}mv^2 + mgh$

Final values of potential energy, kinetic energy and total energy are measured a the height h.

By law of conservation of energy, the initial and final total energies are the same.

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgh$$

$$v_0^2 = v^2 + 2gh$$

$$v = \sqrt{v_0^2 - 2gh}$$

Note that in section similar result is obtained using kinematic equation based on calculus method. However, calculation through energy conservation method is much easier than calculus method.

An object of mass 2 kg attached to a spring is moved to a distance x=10 m from its equilibrium position. The spring constant k=1 N m^{-1} and assume that the surface is frictionless.

1. When the mass crosses the equilibrium position, what is the speed of the mass?
2. What is the force that acts on the object when the mass crosses the equilibrium position and extremum position x = ±10 m.

Since the spring force is a conservative force, the total energy is constant. At x=10 m, the total energy is purely potential.

$$E = U = \frac{1}{2}kx^2 = \frac{1}{2}\times(1)\times(10)^2 = 50\ J$$

When the mass crosses the equilibrium position (x=0) , the potential energy

$$U = \frac{1}{2}\times 1\times(0) = 0\ J$$

The entire energy is purely kinetic energy at this position.

$$E = KE = \frac{1}{2}mv^2 = 50\,J$$

The speed

$$v = \sqrt{\frac{2KE}{m}} = \sqrt{\frac{2 \times 50}{2}} = \sqrt{50}\,ms^{-1} = 7.07\,ms^{-1}$$

Since the restoring spring force is F = - kx, when the object crosses the equilibrium position, it experiences no force. Note that at equilibrium position, the object moves very fast. When the object is at x = +10 m (elongation), the force F = − k x

F = − (1) (10) = − 10 N. Here the negative sign implies that the force is towards equilibrium i.e., towards negative x-axis and when the object is at x = − 10m (compression), it experiences a forces F = − (1) (− 10) = +10 N. Here the positive sign implies that the force points towards positive x-axis.

The object comes to momentary rest at x = ±10m even though it experiences a maximum force at both these points.

Motion in a vertical circle

Imagine that a body of mass (m) attached to one end of a massless and inextensible string executes circular motion in a vertical plane with the other end of the string fixed. The length of the string becomes the radius (r) of the circular path

Let us discuss the motion of the body by taking the free body diagram (FBD) at a position where the position vector (r) makes an angle θ with the vertically downward direction and the instantaneous velocity.

There are two forces acting on the mass.

1. Gravitational force which acts downward

2. Tension along the string.

Applying Newton's second law on the mass, In the tangential direction,

$$mg\sin\theta = m a_t$$

$$mg\sin\theta = -m\left(\frac{dv}{dt}\right)$$

where, $a_t = -\frac{dv}{dt}$ is tangential retardation

In the radial direction,

$$T - mg\cos\theta = m a_r$$

$$T - mg\cos\theta = \frac{mv^2}{r}$$

where, $a_r = \frac{v^2}{r}$ is the centripetal acceleration.

The circle can be divided into four sections A, B, C, D for better understanding of the motion. The four important facts to be understood from the two equations are as follows:

1. The mass is having tangential acceleration (g sin θ) for all values of θ (except θ = 0o), it is clear that this vertical cirular motion is not a uniform circular motion.
2. From the equations (4.28) and (4.29) it is understood that as the magnitude of velocity is not a constant in the course of motion, the tension in the string is also not constant

The equation (4.29), $T = mg\cos\theta + \frac{mv^2}{r}$ highlights that in sections A and D of the circle, $\left(\text{for} -\frac{\pi}{2} < \theta < \frac{\pi}{2};\ \cos\theta \text{ is positive}\right)$, the term $mg\cos\theta$ is always

greater than zero. Hence the tension cannot vanish even when the velocity vanishes.

The equation (4.29), $\frac{mv^2}{r} = T - mg\cos\theta$; further highlights that in sections B and C of the circle, $\left(\text{for } \frac{\pi}{2} < \theta < \frac{3\pi}{2};\ \cos\theta \text{ is negative}\right)$, the second term

is always greater than zero. Hence velocity cannot vanish, even when the tension vanishes.

These points are to be kept in mind while solving problems related to motion in vertical circle.

To start with let us consider only two positions, say the lowest point 1 and the highest point 2 as shown in Figure 4.15 for further analysis. Let the velocity of the body at the lowest point 1 be v_1, at the highest point 2 be v_2 and v at any other point. The direction of velocity is tangential to the circular path at all points. Let T_1 be the tension in the string at the lowest point and T_2 be the tension at the highest point and $\dot{T}$ be the tension at any other point. Tension at each point acts towards the center. The tensions and velocities at these two points can be found by applying the law of conservation of energy.

$$T_1 - mg = \frac{mv_1^2}{r}$$

$$T_1 = \frac{mv_1^2}{r} + mg$$

At the highest point 2, both the gravitational force mg on the body and the tension T_2 act downwards, i.e. towards the center again.

$$T_2 + mg = \frac{mv_2^2}{r}$$

$$T_2 = \frac{mv_2^2}{r} - mg$$

$$T_1 - T_2 = \frac{mv_1^2}{r} + mg - \left(\frac{mv_2^2}{r} - mg\right)$$

$$= \frac{mv_1^2}{r} + mg - \frac{mv_2^2}{r} + mg$$

$$T_1 - T_2 = \frac{m}{r}\left[v_1^2 - v_2^2\right] + 2mg \qquad (3)$$

The term $[v_1^2 - v_2^2]$ can be found easily by applying law of conservation of energy.

Total Energy at point 1(E_1) is same as the total energy at a point 2 (E_2)

$$E_1 = E_2$$

Potential Energy at point 1, $U_1=0$ (by taking reference as point 1)

Kinetic Energy at point 1,

$$KE_1 = \frac{1}{2}mv_1^2$$

Total Energy at point 1,

$$E_1 = U_1 + KE_1 = 0 + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_1^2$$

Similarly, Potential Energy at point 2, U_2 =mg (2r)

Kinetic Energy at point 2,

$$KE_2 = \frac{1}{2}mv_2^2$$

Total Energy at point 2,

$$E_2 = U_2 + KE_2 = 2\text{ mg r} + \frac{1}{2}mv_2^2$$

From the law of conservation of energy given in equation

$$\frac{1}{2}mv_1^2 = 2mgr + \frac{1}{2}mv_2^2$$

After rearranging,

$$\frac{1}{2}m\left(v_1^2 - v_2^2\right) = 2mgr$$

$$v_1^2 - v_2^2 = 4gr$$

Substituting equation

$$T_1 - T_2 = \frac{m}{r}[4gr] + 2mg$$

Therefore, the difference in tension is

$$T_1 - T_2 = 6\ mg$$

The body must have a minimum speed at point 2 otherwise, the string will slack before reaching point 2 and the body will not loop the circle. To find this minimum speed let us take the tension T_2 = 0 in equation

$$0 = \frac{mv_2^2}{r} - mg$$

$$\frac{mv_2^2}{r} = mg$$

$$v_2^2 = rg$$

$$v_2 = \sqrt{gr}$$

The body must have a speed at point 2, $v_2 \geq \sqrt{gr}$ to stay in the circular path.

To have this minimum speed $\left(v_2 = \sqrt{gr}\right)$ at point 2, the body must have minimum speed also at point 1.

By making use of equation (4.36) we can find the minimum speed at point 1.

$$v_1^2 - v_2^2 = 4gr$$

Substituting equation

$$v_1^2 - gr = 4gr$$

$$v_1^2 = 5gr$$

$$v_1 = \sqrt{5gr}$$

The body must have a speed at point 1, $v_1 \geq \sqrt{5gr}$ to stay in the circular path.

It is clear that the minimum speed at the lowest point 1 should be $\sqrt{5}$ times more than the minimum speed at the highest point 2, so that the body loops without leaving the circle.

Water in a bucket tied with rope is whirled around in a vertical circle of radius 0.5 m. Calculate the minimum velocity at the lowest point so that the water does not spill from it in the course of motion. (g = 10 ms^{-2})

Radius of circle r = 0.5 m

The required speed at the highest point

$v_2 = \sqrt{gr} = \sqrt{10 \times 0.5} = \sqrt{5}$ ms^{-1}. The speed at lowest point $v_1 = \sqrt{5gr} = \sqrt{5} \times \sqrt{gr} = \sqrt{5} \times \sqrt{5} = 5$ ms^{-1}

POWER

Definition of Power

Power is a measure of how fast or slow a work is done. Power is defined as the rate of work done or energy delivered.

$$\text{Power } (P) = \frac{\text{work done}(W)}{\text{time taken}(t)}$$

$$P = \frac{W}{t}$$

Average power

The average power (P_{av}) is defined as the ratio of the total work done to the total time taken.

$$P_{av} = \frac{\text{total work done}}{\text{total time taken}}$$

Instantaneous power

The instantaneous power (P_{inst}) is defined as the power delivered at an instant (as time interval approaches zero),

$$P_{inst} = \frac{dW}{dt}$$

Unit of Power

Power is a scalar quantity. Its dimension is $[ML^2T^{-3}]$. The SI unit of power is watt (W), named after the inventor of the steam engine James Watt. One watt is defined as the power when one joule of work is done in one second,(1 W = 1 J s^{-1}).

The higher units are kilowatt(kW), megawatt(MW), and Gigawatt(GW).

$$1kW = 1000\ W = 10^3\ watt$$

$$1MW = 10^6\ watt$$

$$1GW = 10^9\ watt$$

For motors, engines and some automobiles an old unit of power still commercially in use which is called as the horse-power (hp). We have a conversion for horse-power (hp) into watt (W) which is,

$$1\ hp = 746\ W$$

All electrical goods come with a definite power rating in watt printed on them. A 100 watt bulb consumes 100 joule of electrical energy in one second. The energy measured in joule in terms of power in watt and time in second is written as, 1 J =1 W s. When electrical appliances are put in use for long hours, they consume a large amount of energy. Measuring the electrical energy in a small unit watt. second (W s) leads to handling large numerical values. Hence, electrical energy is measured in the unit called kilowatt hour (kWh).

$$1\ \text{electrical unit} = 1\ kWh = 1 \times (10^3\ W) \times (3600\ s)$$

$$1\ \text{electrical unit} = 3600 \times 10^3\ W\ s$$

$$1\ \text{electrical unit} = 3.6 \times 10^6\ J$$

$$1\ kWh = 3.6 \times 10^6\ J$$

Electricity bills are generated in units of kWh for electrical energy consumption. 1 unit of electrical energy is 1 kWh. (Note: kWh is unit of energy and not of power.)

Calculate the energy consumed in electrical units when a 75 W fan is used for 8 hours daily for one month (30 days).

Power, P = 75 W

- ❖ Time of usage, t = 8 hour × 30 days = 240 hours
- ❖ Electrical energy consumed is the product of power and time of usage.

Electrical energy = power × time of usage = P × t

$$= 75 \text{ watt} \times 240 \text{ hour}$$
$$= 18000 \text{ watt hour}$$
$$= 18 \text{ kilowatt hour} = 18\,\text{kWh}$$
$$1 \text{ electrical unit} = 1\,\text{kWh}$$
$$\text{Electrical energy} = 18 \text{ unit}$$

Incandescent lamps glow for 1000 hours. CFL lamps glow for 6000 hours. But LED lamps glow for 50000 hrs (almost 25 years at 5.5 hour per day).

Relation between power and velocity

The work done by a force $\vec{F}$ for a displacement $d\vec{r}$ is

$$W = \int \vec{F} \cdot d\vec{r}$$

Left hand side of the equation (4.40) can be written as

$$W = \int dW = \int \frac{dW}{dt} dt$$

Since, velocity is $\bar{v} = \frac{d\bar{r}}{dt}; d\bar{r} = \bar{v}dt.$. Right hand side of the equation (4.40) can be written as

$$\int \vec{F} \cdot d\vec{r} = \int \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt = \int \left(\vec{F} \cdot \vec{v} \right) dt \quad \left[\vec{v} = \frac{d\vec{r}}{dt} \right]$$

$$\int \frac{dW}{dt} dt = \int \left(\vec{F} \cdot \vec{v} \right) dt$$

$$\int \left(\frac{dW}{dt} - \vec{F} \cdot \vec{v} \right) dt = 0$$

This relation is true for any arbitrary value of dt. This implies that the term within the bracket must be equal to zero, i.e.,

$$\frac{dW}{dt} - \vec{F} \cdot \vec{v} = 0$$

Or

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

A vehicle of mass 1250 kg is driven with an acceleration 0.2 ms^{-2} along a straight level road against an external resistive force 500 N. Calculate the power delivered by the vehicle's engine if the velocity of the vehicle is 30 ms^{-1}.

The vehicle's engine has to do work against resistive force and make vechile to move with an acceleration. Therefore, power delivered by the vehicle engine is

$$P = (\text{resistive force} + \text{mass} \times \text{acceleration})\,(\text{velocity})$$

$$P = \vec{F}_{tot}\vec{v} = (F_{resistive} + F)\vec{v}$$

$$P = \vec{F}_{tot} \cdot \vec{v} = (F_{resistive} + ma)\vec{v}$$

$$= (500\ N + (1250\ kg) \times (0.2\ ms^{-2}))\,(30\ ms^{-1}) = 22.5\ kW$$

COLLISIONS

Collision is a common phenomenon that happens around us every now and then. For example, carom, billiards, marbles, etc.,. Collisions can happen between two bodies with or without physical contacts.

Linear momentum is conserved in all collision processes. When two bodies collide, the mutual impulsive forces acting between them during the collision time (Δt) produces a change in their respective momenta. That is, the first body exerts a force F_{12} on the second body. From Newton's third law, the second body exerts a force F_{21} on the first body. This causes a change in momentum $\Delta \dot{p}_1$ and Δp_2 of the first body and second body respectively. Now, the relations could be written as,

$$\Delta\vec{p}_1 = \vec{F}_{12}\Delta t$$
$$\Delta\vec{p}_2 = \vec{F}_{21}\Delta t$$

Adding equation

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = \vec{F}_{12}\Delta t + \vec{F}_{21}\Delta t = \left(\vec{F}_{12} + \vec{F}_{21}\right)\Delta t$$

According to Newton's third law, $\dot{F}_{12} = -\dot{F}_{21}$

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = 0$$
$$\Delta\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

Dividing both sides by Δt and taking limit Δ t→0 , we get

$$\lim_{\Delta t \to 0} \frac{\Delta(\vec{p}_1 + \vec{p}_2)}{\Delta t} = \frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = 0$$

The above expression implies that the total linear momentum is a conserved quantity. Note: The momentum is a vector quantity. Hence, vector addition has to be followed to find the total momentum of the individual bodies in collision.

Types of collections

In any collision process, the total linear momentum and total energy are always conserved whereas the total kinetic energy need not be conserved always. Some part of the initial kinetic energy is transformed to other forms of energy. This is because, the impact of collisions and deformation occurring due to collisions may in general, produce heat, sound, light etc. By taking these effects into account, we classify the types of collisions as follows:

1. Elastic collision
2. Inelastic collision

Elastic collision

In a collision, the total initial kinetic energy of the bodies (before collision) is equal to the total final kinetic energy of the bodies (after collision) then, it is called as elastic collision. i.e.,

Total kinetic energy before collision = Total kinetic energy after collision

Inelastic collision

In a collision, the total initial kinetic energy of the bodies (before collision) is not equal to the total final kinetic energy of the bodies (after collision) then, it is called as inelastic collision. i.e., Total kinetic energy before collision ≠ Total kinetic energy after collision

$$\begin{pmatrix}\text{Total kinetic energy}\\ \text{after collision}\end{pmatrix} - \begin{pmatrix}\text{Total kinetic energy}\\ \text{before collision}\end{pmatrix} = \begin{pmatrix}\text{loss in energy}\\ \text{during collision}\end{pmatrix} = \Delta Q$$

Even though kinetic energy is not conserved but the total energy is conserved. This is because the total energy contains the kinetic energy term and also a term ΔQ, which includes all the losses that take place during collision. Note that loss in kinetic energy during collision is transformed to another form of energy like sound, thermal, etc. Further, if the two colliding bodies stick together after collision such collisions are known as completely inelastic collision or perfectly inelastic collision. Such a collision is found very often. For example when a clay putty is thrown on a moving vehicle, the clay putty (or Bubblegum) sticks to the moving vehicle and they move together with the same velocity.

Elastic collisions in one dimension

Consider two elastic bodies of masses m_1 and m_2 moving in a straight line (along positive x direction) on a frictionless horizontal surface.

Mass	Initial velocity	Final velocity
Mass m_1	u_1	v_1
Mass m_2	u_2	v_2

In order to have collision, we assume that the mass m_1 moves faster than mass m_2 i.e., $u_1 > u_2$. For elastic collision, the total linear momentum and kinetic energies of the two bodies before and after collision must remain the same.

S.No.	Elastic Collision	Inelastic Collision
1.	Total momentum is conserved	Total momentum is conserved
2.	Total kinetic energy is conserved	Total kinetic energy is not conserved
3.	Forces involved are conservative forces	Forces involved are non-conservative forces
4.	Mechanical energy is not dissipated.	Mechanical energy is dissipated into heat, light, sound etc.

	Momentum of mass m_1	Momentum of mass m_2	Total linear momentum
Before collision	$p_{i1} = m_1u_1$	$p_{i2} = m_2u_2$	$p_i = p_{i1} + p_{i2}$ $p_i = m_1u_1 + m_2u_2$
After collision	$p_{f1} = m_1v_1$	$p_{f2} = m_2v_2$	$p_f = p_{f1} + p_{f2}$ $p_f = m_1v_1 + m_2v_2$

From the law of conservation of linear momentum,

Total momentum before collision (p_i) = Total momentum after collision (p_f)

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$\text{Or } m_1(u_1 - v_1) = m_2(v_2 - u_2)$$

	Kinetic energy of mass m_1	Kinetic energy of mass m_2	Total kinetic energy
Before collision	$KE_{i1} = \frac{1}{2}m_1u_1^2$	$KE_{i2} = \frac{1}{2}m_2u_2^2$	$KE_i = KE_{i1} + KE_{i2}$ $KE_i = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$
After collision	$KE_{f1} = \frac{1}{2}m_1v_1^2$	$KE_{f2} = \frac{1}{2}m_2v_2^2$	$KE_f = KE_{f1} + KE_{f2}$ $KE_f = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

Total kinetic energy before collision KE_i = Total kinetic energy after collision KE_f

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

After simplifying and rearranging the terms,

$$m_1\left(u_1^2 - v_1^2\right) = m_2\left(v_2^2 - u_2^2\right)$$

Using the formula $a^2 - b^2 = (a+b)(a-b)$ we can rewrite the above equation as

$$m_1(u_1 + v_1)(u_1 - v_1) = m_2(v_2 + u_2)(v_2 - u_2)$$

$$\frac{m_1(u_1+v_1)(u_1-v_1)}{m_1(u_1-v_1)} = \frac{m_2(v_2+u_2)(v_2-u_2)}{m_2(v_2-u_2)}$$

$$u_1 + v_1 = v_2 + u_2$$

$$u_1 - u_2 = v_2 - v_1 \quad \text{Rearranging,} \quad (4.50)$$

Equation (4.50) can be rewritten as

$$u_1 - u_2 = -(v_1 - v_2)$$

This means that for any elastic head on collision, the relative speed of the two elastic bodies after the collision has the same magnitude as before collision but in opposite direction. Further note that this result is independent of mass.

Rewriting the above equation for v_1 and v_2,

$$v_1 = v_2 + u_2 - u_1$$

Or

$$v_2 = u_1 + v_1 - u_2$$

To find the final velocities v_1 and v_2:

Substituting equation (4.52) in equation (4.47) gives the velocity of m_1 as

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - u_2 - u_2)$$

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - 2u_2)$$

$$m_1u_1 - m_1v_1 = m_2u_1 + m_2v_1 - 2m_2u_2$$

$$m_1u_1 - m_2u_1 + 2m_2u_2 = m_1v_1 + m_2v_1$$

$$(m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$$

$$\text{or } v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

Similarly, by substituting (4.51) in equation (4.47) or substituting equation (4.53) in equation (4.52), we get the final velocity of m_2 as

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

When bodies has the same mass i.e., $m_1 = m_2$,

$$\Rightarrow v_1 = (0)\ u_1 + \left(\frac{2m_2}{2m_2}\right)u_2$$

$$v_1 = u_2 \qquad (5$$

$$\Rightarrow v_2 = \left(\frac{2m_1}{2m_1}\right)u_1 + (0)u_2$$

$$v_2 = u_1$$

The equations (4.55) and (4.56) show that in one dimensional elastic collision, when two bodies of equal mass collide after the collision their velocities are exchanged.

When bodies have the same mass i.e., $m_1 = m_2$ and second body (usually called target) is at rest ($u_2 = 0$),

By substituting $m_1 = m_2$ and $u_2 = 0$ in equations (4.53) and equations (4.54) we get, body moves with the initial velocity of the first body.

$$\Rightarrow v_1 = 0$$

$$\Rightarrow v_2 = u_1$$

The first body is very much lighter than the second body

$$\left(m_1 \ll m_2, \frac{m_1}{m_2} \ll 1\right) \text{ then the ratio } \frac{m_1}{m_2} \approx 0$$

Dividing numerator and denominator of equation (4.53) by m2, we get

$$v_1 = \left(\frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1}\right) u_1 + \left(\frac{2}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_1 = \left(\frac{0 - 1}{0 + 1}\right) u_1$$

$$v_1 = -u_1$$

Dividing numerator and denominator of equation (4.54) by m2, we get

$$v_2 = \left(\frac{2\frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right) u_1 + \left(\frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_2 = (0) u_1 + \left(\frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_2 = 0$$

The equation (4.59) implies that the first body which is lighter returns back (rebounds) in the opposite direction with the same initial velocity as it has a negative sign. The equation (4.60) implies that the second body which is heavier in mass continues to remain at rest even after collision. For example, if a ball is thrown at a fixed wall, the ball will bounce back from the wall with the same velocity with which it was thrown but in opposite direction.

The second body is very much lighter than the first body

$$\left(m_2 \ll m_1, \frac{m_2}{m_1} \ll 1\right) \text{ then the ratio } \frac{m_2}{m_1} \approx 0$$

$$v_1 = \left(\frac{1-\frac{m_2}{m_1}}{1+\frac{m_2}{m_1}}\right)u_1 + \left(\frac{2\frac{m_2}{m_1}}{1+\frac{m_2}{m_1}}\right)(0)$$

$$v_1 = \left(\frac{1-0}{1+0}\right)u_1 + \left(\frac{0}{1+0}\right)(0)$$

$$v_1 = u_1$$

Dividing numerator and denominator of equation (4.58) by m1, we get

$$v_2 = \left(\frac{2}{1+\frac{m_2}{m_1}}\right)u_1 + \left(\frac{\frac{m_2}{m_1}-1}{1+\frac{m_2}{m_1}}\right)(0)$$

$$v_2 = \left(\frac{2}{1+0}\right)u_1$$

$$v_2 = 2u_1$$

The equation (4.61) implies that the first body which is heavier continues to move with the same initial velocity. The equation (4.62) suggests that the second body which is lighter will move with twice the initial velocity of the first body. It means that the lighter body is thrown away from the point of collision.

A lighter particle moving with a speed of 10 m s^{-1} collides with an object of double its mass moving in the same direction with half its speed. Assume that the collision is a one dimensional elastic collision. What will be the speed of both particles after the collision?

Let the mass of the first body be m which moves with an initial velocity, u1 = 10 m s-1. Therefore, the mass of second body is 2m and its initial velocity is

$$u_2 = \frac{1}{2} u_1 = \frac{1}{2}(10 \text{ m s}^{-1}),$$

Then, the final velocities of the bodies can be calculated from the equation (4.53) and equation (4.54)

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = \left(\frac{m-2m}{m+2m}\right)10 + \left(\frac{2\times 2m}{m+2m}\right)5$$

$$v_1 = -\left(\frac{1}{3}\right)10 + \left(\frac{4}{3}\right)5 = \frac{-10+20}{3} = \frac{10}{3}$$

$$v_1 = 3.33 \text{ ms}^{-1}$$

$$v_2 = \left(\frac{2m_1}{m_1+m_2}\right)u_1 + \left(\frac{m_2-m_1}{m_1+m_2}\right)u_2$$

$$v_2 = \left(\frac{2m}{m+2m}\right)10 + \left(\frac{2m-m}{m+2m}\right)5$$

$$v_2 = \left(\frac{2}{3}\right)10 + \left(\frac{1}{3}\right)5 = \frac{20+5}{3} = \frac{25}{3}$$

$$v_2 = 8.33 \text{ ms}^{-1}$$

As the two speeds v_1 and v_2 are positive, they move in the same direction with the velocities, 3.33 m s^{-1} and 8.33 m s^{-1} respectively.

Perfect inelastic collision

In a perfectly inelastic or completely inelastic collision, the objects stick together permanently after collision such that they move with common velocity. Let the two bodies with masses m_1 and m_2 move with initial velocities u_1 and u_2 respectively before collision. After perfect inelastic collision both the objects move together with a common velocity v

Since, the linear momentum is conserved during collisions,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$$

	Velocity		Linear momentum	
	Initial	Final	Initial	Final
Mass m_1	u_1	v	$m_1 u_1$	$m_1 v$
Mass m_2	u_2	v	$m_2 u_2$	$m_2 v$
	Total		$m_1 u_1 + m_2 u_2$	$(m_1 + m_2) v$

The common velocity can be computed by

$$v = \frac{m_1 u_1 + m_2 u_2}{(m_1 + m_2)}$$

A bullet of mass 50 g is fired from below into a suspended object of mass 450 g. The object rises through a height of 1.8 m with bullet remaining inside the object. Find the speed of the bullet. Take g = 10 ms^{-2}.

$$m_1 = 50\ g = 0.05\ kg; \quad m_2 = 450\ g = 0.45\ kg$$

The speed of the bullet is u1. The second body is at rest $u_2 = 0$. Let the common velocity of the bullet and the object after the bullet is embedded into the object is v.

$$v = \frac{m_1 u_1 + m_2 u_2}{(m_1 + m_2)}$$

$$v = \frac{0.05 u_1 + (0.45 \times 0)}{(0.05 + 0.45)} = \frac{0.05}{0.50} u_1$$

The combined velocity is the initial velocity for the vertical upward motion of the combined bullet and the object. From second equation of motion,

$$v = \sqrt{2gh}$$

$$v = \sqrt{2 \times 10 \times 1.8} = \sqrt{36}$$

$$v = 6\,ms^{-1}$$

Substituting this in the above equation, the value of u_1 is

$$6 = \frac{0.05}{0.50} u_1 \quad \text{or} \quad u_1 = \frac{0.50}{0.05} \times 6 = 10 \times 6$$

$$u_1 = 60\,ms^{-1}$$

Loss of kinetic energy in perfect inelastic collision

In perfectly inelastic collision, the loss in kinetic energy during collision is transformed to another form of energy like sound, thermal, heat, light etc. Let KE_i be the total kinetic energy before collision and KE_f be the total kinetic energy after collision.

Total kinetic energy before collision,

$$KE_i = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$KE_f = \frac{1}{2}(m_1 + m_2) v^2$$

Then the loss of kinetic energy is Loss of KE, $\Delta Q = KE_i - KE_f$

$$= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2)v^2$$

Substituting equation (4.63) in equation (4.66), and on simplifying (expand v by using the algebra

$$(a+b)^2 = a^2 + b^2 + 2ab,$$

$$\text{Loss of KE, } \Delta Q = \frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)(u_1 - u_2)^2$$

Coefficient of restitution (e)

Suppose we drop a rubber ball and a plastic ball on the same floor. The rubber ball will bounce back higher than the plastic ball. This is because the loss of kinetic energy for an elastic ball is much lesser than the loss of kinetic energy for a plastic ball. The amount of kinetic energy after the collision of two bodies, in general, can be measured through a dimensionless number called the coefficient of restitution (COR).

It is defined as the ratio of velocity of separation (relative velocity) after collision to the velocity of approach (relative velocity) before collision, i.e.,

$$e = \frac{\text{velocity of separation(after collision)}}{\text{velocity of approach(before collision)}}$$

$$= \frac{(v_2 - v_1)}{(u_1 - u_2)}$$

In an elastic collision, we have obtained the velocity of separation is equal to the velocity of approach i.e.,

$$(u_1 - u_2) = (v_2 - v_1) \rightarrow \frac{(v_2 - v_1)}{(u_1 - u_2)} = 1 = e$$

This implies that, coefficient of restitution for an elastic collision, e=1. Physically, it means that there is no loss of kinetic energy after the collision. So, the body bounces back with the same kinetic energy which is usually called as perfect elastic.

In any real collision problems, there will be some losses in kinetic energy due to collision, which means e is not always equal to unity. If the ball is perfectly plastic, it will never bounce back and therefore their separation of velocity is zero after the collision. Hence, the value of coefficient of restitution, e=0.

In general, the coefficient of restitution for a material lies between

$$0 < e < 1.$$

Show that the ratio of velocities of equal masses in an inelastic collision when one of the masses is stationary is

$$\frac{v_1}{v_2} = \frac{1-e}{1+e}.$$

$$e = \frac{\text{velocity of separation (after collision)}}{\text{velocity of approach (before collision)}}$$

$$= \frac{(v_2 - v_1)}{(u_1 - u_2)} = \frac{(v_2 - v_1)}{(u_1 - 0)} = \frac{(v_2 - v_1)}{u_1}$$

$$\Rightarrow v_2 - v_1 = e\,u_1$$

From the law of conservation of linear momentum,

$$m u_1 = m v_1 + m v_2 \Rightarrow u_1 = v_1 + v_2 \ (2)$$

Using the equation (2) for u1 in (1), we get

$$v_2 - v_1 = e(v_1 + v_2)$$

On simplification, we get

$$\frac{v_1}{v_2} = \frac{1-e}{1+e}$$

12th-std
Unit 1- Electrostatics

INTRODUCTION

- Electromagnetism is one of the most important branches of physics. The technological developments of the modern 21st century are primarily due to our understanding of electromagnetism. The forces we experience in everyday life are electromagnetic in nature except gravity.

- In standard XI, we studied about the gravitational force, tension, friction, normal force etc. Newton treated them to be independent of each other with each force being a separate natural force. But what is the origin of all these forces? It is now understood that except gravity, all forces which we experience in every day life (tension in the string, normal force from the surface, friction etc.) arise from electromagnetic forces within the atoms. Some examples are

- When an object is pushed, the atoms in our hand interact with the atoms in the object and this interaction is basically electromagnetic in nature.

(ii) When we stand on Earth's surface, the gravitational force on us acts downwards and the normal force acts upward to counter balance the gravitational force. What is the origin of this normal force?
It arises due to the electromagnetic interaction of atoms on the surface of the Earth with the atoms present in the feet of the person. Though, we are attracted by the gravitational force of the Earth, we stand on Earth only because of electromagnetic force of atoms.

(iii) When an object is moved on a surface, static friction resists the motion of the object. This static friction arises due to electromagnetic interaction between the atoms present in the object and atoms on the surface. Kinetic friction also has similar origin.

From these examples, it is clear that understanding electromagnetism is very essential to understand the universe in a holistic manner. The basic principles of electromagnetism are dealt in XII physics volume 1. This unit deals with the behaviour and other related phenomena of charges at rest. This **branch of electricity which deals with stationary charges is called Electrostatics.**

Historical background of electric charges

- Two millenniums ago, Greeks noticed that amber (a solid, translucent material formed from the resin of a fossilized tree) after rubbing with animal fur attracted small pieces of leaves and dust. The amber possessing this property is said to be 'charged'. It was initially thought that amber has this special property. Later people found that not only amber but even a glass rod rubbed with silk cloth, attracts pieces of papers. So glass rod also becomes 'charged' when rubbed with a suitable material.

- Consider a charged rubber rod hanging from a thread as shown in Figure 1.1. Suppose another charged rubber rod is brought near the first rubber rod; the rods repel each other. Now if we bring a charged glass rod close to the charged rubber rod, they attract each other. At the same time, if a charged glass rod is brought near another charged glass rod, both the rods repel each other.

From these observations, the following inferences are made

(i) The charging of rubber rod and that of glass rod are different from one another.

(ii) The charged rubber rod repels another charged rubber rod, which implies that 'like charges repel each other'. We can also arrive at the same inference by observing that a charged glass rod repels another charged glass rod.

(iii) The charged amber rod attracts the charged glass rod, implying that the charge in the glass rod is not the same kind of charge present in the rubber. Thus unlike charges attract each other.
Therefore, two kinds of charges exist in the universe. In the 18th century,

Benjamin Franklin called one type of charge as positive (+) and another type of charge as negative (-). Based on Franklin's convention, rubber and amber rods are negatively charged while the glass rod is positively charged. **If the net charge is zero in the object, it is said to be electrically neutral.**

- Following the pioneering work of J. J. Thomson and E. Rutherford, in the late 19th century and in the beginning of 20th century, we now understand that the atom is electrically neutral and is made up of the negatively charged electrons, positively charged protons, and neutrons which have zero charge. The material objects made up of atoms are neutral in general. When an object is rubbed with another object (for example rubber with silk cloth), some amount of charge is transferred from one object to another due to the friction between them and the object is then said to be electrically charged. **Charging the objects through rubbing is called triboelectric charging.**

Basic properties of charges

(i) Electric charge

- Most objects in the universe are made up of atoms, which in turn are made up of protons, neutrons and electrons. These particles have mass, an inherent property of particles. Similarly, the electric charge

is another intrinsic and fundamental property of particles. The nature of charges is understood through various experiments performed in the 19th and 20th century. The SI unit of charge is coulomb.

(ii) Conservation of charges

- Benjamin Franklin argued that when one object is rubbed with another object, charges get transferred from one to the other. Before rubbing, both objects are electrically neutral and rubbing simply transfers the charges from one object to the other. (For example, when a glass rod is rubbed against silk cloth, some negative charge are transferred from glass to silk. As a result, the glass rod is positively charged and silk cloth becomes negatively charged). From these observations, he concluded that charges are neither created or nor destroyed but can only be transferred from one object to other. This is called conservation of total charges and is one of the fundamental conservation laws in physics. It is stated more generally in the following way.

'**The total electric charge in the universe is constant and charge can neither be created nor be destroyed. In any physical process, the net change in charge will always be zero**.

(iii) Quantisation of charges

- What is the smallest amount of charge that can be found in nature? Experiments show that the charge on an electron is $-e$ and the charge on the proton is $+e$. Here, e denotes the fundamental unit of charge. The charge q on any object is equal to an integral multiple of this fundamental unit of charge e.

$q = ne$ (1.1)

Here n is any integer (0, ±1, ±2, ±3, ±4………..). This is called quantisation of electric charge.

- Robert Millikan in his famous experiment found that the value of e = 1.6×10^{-19}C. The charge of an electron is −1.6 × 10-19 C and the charge of the proton is +1.6 × 10-19C.

When a glass rod is rubbed with silk cloth, the number of charges transferred is usually very large, typically of the order of 1010. So the charge quantisation is not appreciable at the macroscopic level. Hence the charges are treated to be continuous (not discrete). But at the microscopic level, quantisation of charge plays a vital role.

E X A M P L E

Calculate the number of electrons in one coulomb of negative charge.

Solution

According to the quantisation of charge

$$q = ne$$

Here q = 1C. So the number of electrons in 1 coulomb of charge is

n= $\frac{q}{e} = \frac{1c}{1.6\times10-19}\times$ **=6. 25× 10** ^{18}. . electrons

COULOMB'S LAW

- In the year 1786, Coulomb deduced the expression for the force between two stationary point charges in vacuum or free space. Consider two point charges q1 and q2 at rest in vacuum, and separated by a distance of r, as shown in Figure 1.2. According to Coulomb, the force on the point charge q2 exerted by another point charge q1

$$\vec{F}_{21} = k\frac{q_1 q_2}{r^2}\hat{r}_{12}$$

Where r_{12} is the unit vector directed from charge q1 to charge q2 and k is the proportionality constant.

Important aspects of Coulomb's law

(i) Coulomb's law states that the electrostatic force is directly proportional to the product of the magnitude of the two point charges and is inversely proportional to the square of the distance between the two point charges.

(ii) The force on the charge q2 exerted by the charge q1 always lies along the line joining the two charges. is the unit
vector pointing from charge q1 to q2 .It is shown in the Figure 1.2. Likewise, the force on the charge q1 exerted by q2 is along (i.e., in the direction opposite to).

(iii) In SI units, $K = \frac{1}{4\pi\varepsilon 0}$ and its value is 9×10^9 N m^2 C^{-2}. Here ε_o is the permittivity of free space or vacuum and the value of $\varepsilon_o = \frac{1}{4\pi k} =$ $8.85\times10^{12}C^2N^1m^2$.

(iv) The magnitude of the electrostatic force between two charges each of one coulomb and separated by a distance of 1 m is calculated as follows:

$$\frac{9\times10\times1\times1}{1} = 9\times 10^9 \text{ N}$$

This is a huge quantity, almost equivalent to the weight of one million ton. We never come across 1 coulomb of charge in practice. Most of the electrical phenomena in day-to-day life involve electrical charges of the order of μC (micro coulomb) or nC (nano coulomb).

(v) In SI units, Coulomb's law in vacuum takes the form

$$\vec{F}_{21} = k\frac{q_1 q_2}{r^2}\hat{r}_{12}$$

In a medium of permittivity ε, the force between two point charges is given,

$$\vec{F}_{21} = k\frac{q_1 q_2}{r^2}\hat{r}_{12}$$

the force between two point charges in a medium other than vacuum is always less than that in vacuum. We define the relative permittivity for a given medium as $\varepsilon_r = \frac{\varepsilon}{\varepsilon_o}$

For vacuum or air, $\varepsilon_r = 1$ and for all other media $\varepsilon_r > 1$.

(vi) Coulomb's law has same structure as Newton's law of gravitation. Both are inversely proportional to the square of the distance between the particles. The electrostatic force is directly proportional to the product of the magnitude of two point charges and gravitational force is directly proportional to the product of two masses. But there are some important differences between these two laws.

• The gravitational force between two masses is always attractive but Coulomb force between two charges can be attractive or repulsive, depending on the nature of charges.

• The value of the gravitational constant G = 6.626 × 10-11 N m2 kg-2. The value of the constant k in Coulomb law is k = 9 × 109 N m2 C-2. Since k is much more greater than G, the electrostatic force is always greater in magnitude than gravitational force for smaller size objects.

• The gravitational force between two masses is independent of the medium. For example, if 1 kg of two masses are kept in air or inside water, the gravitational force between two masses remains the same. But the electrostatic force between the two charges depends on nature of the medium in which the two charges are kept at rest.

• The gravitational force between two point masses is the same whether two masses are at rest or in motion. If the charges are in motion, yet another force (Lorentz force) comes into play in addition to coulomb force.

(vii) The force on a charge q1 exerted by a point charge q2 is given by

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r^2}\hat{r}_{21}$$

Here is the unit vector from charge q2 to q1. But $\hat{r}_{21} = -\hat{r}_{12}$,

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r^2}(-\hat{r}_{12}) = -\frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r^2}(\hat{r}_{12})$$

$$\text{(or)} \quad \vec{F}_{12} = -\vec{F}_{21}$$

Therefore, the electrostatic force obeys Newton's third law.

(viii) The expression for Coulomb force is true only for point charges. But the point charge is an ideal concept. However we can apply Coulomb's law for two charged objects whose sizes are very much smaller than the distance between them. In fact, Coulomb discovered his law by considering the charged spheres in the torsion balance as point charges. The distance between the two charged spheres is much greater than the radii of the spheres.

They are separated by a distance of 1m. Calculate the force experienced by the two charges for the following cases:

(a) q1 = +2μC and q2 = +3μC
(b) q1 = +2μC and q2 = -3μC
(c) q1= +2μC and q2 = -3μC kept in water (εr = 80)

(a) q1 = +2 μC, q2 = +3 μC, and r = 1m. Both are positive charges. so the force will be repulsive

Force experienced by the charge q2 due to q1 is given by

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_o} \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

Here $\hat{r}_{12}$ is the unit vector from q1 to q2. Since q2 is located on the right of q1, we have

$\hat{r}_{12} = \hat{i}$, so that

$$\vec{F}_{21} = \frac{9\times10^9 \times 2\times10^{-6} \times 3\times10^{-6}}{1^2}\hat{i} \quad \left|\frac{1}{4\pi\varepsilon_o} = 9\times10^9\right|$$

$$= 54\times10^{-3} N\hat{i}$$

rding to ew on's thi d law the for e exp rienced by he cha ge q^1 ue to q^2 is

$$\vec{F}_{12} = -\vec{F}_{21}$$

So that $\vec{F}_{12} = -54\times10^{-3} N\hat{i}$.

The dirctions of $F_{21}$and F_{12} are sho n in he abo e fig re in as (a)

q1 + C, q2 - C, and r = 1 . The are unl ke cha ges. So he for e wil be tt active. Force exprienced by he chage q2 due to q_1 is given by

$$\vec{F}_{21} = \frac{9\times10^9 \times (2\times10^{-6}) \times (-3\times10^{-6})}{1^2}\hat{r}_{12}$$

$$= -54\times10^{-3} N\hat{i} \quad (\text{Using } \hat{r}_{12} = \hat{i})$$

Therefore,

$$\vec{F}_{21}^{W} = -\frac{54\times10^{-3} N}{80}\hat{i} = -0.675\times10^{-3} N\hat{i}$$

The charge q_2 will experience an attractive force towards q_1 which is in the negative x direction.

According to Newton's third law, the force experienced by the charge q_1 due to q_2 is

$$\vec{F}_{12} = -\vec{F}_{21}$$

so that $\vec{F}_{12} = 54 \times 10^{-3} N \hat{i}$

The directions of $\vec{F}_{21}$ and $\vec{F}_{12}$ are shown in the figure (case (b)).

(c) If these two charges are kept inside the water, then the force experienced by q_2 due to q_1

$$\vec{F}_{21}^{w} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}\hat{r}_{12}$$

since $\varepsilon = \varepsilon_r \varepsilon_o$,

we have $\vec{F}_{21}^{w} = \frac{1}{4\pi\varepsilon_r\varepsilon_o}\frac{q_1 q_2}{r^2}\hat{r}_{12} = \frac{\vec{F}_{21}}{\varepsilon_r}$

EXA M P L E

Two sma l-sized ide tical equ lly changed sph res, eac hav ng mas 1 m re han ing in qu librium as ho n in he fig re. The lenth of ac str ng is 0 m nd the ang e θ i ° it the ver ical. Calculate the mag itude of he chacnge in ac sphere

Solution

If the two spheres are neutral, the angle between them will be 0o when hanged vertically. Since they are positively charged spheres, there will be a repulsive force between them and they will be at equilibrium with each other at an angle of 7° with the vertical. At equilibrium, each charge experiences zero net force in each direction. We can draw a free body diagram for one of the charged spheres and apply Newton's second law for both vertical and horizontal directions.

The free body diagram is shown below. In the x-direction, the acceleration of the charged sphere is zero.

Using Newton's second law $\left(\vec{F}_{tot} = m\vec{a}\right)$.

$$T\sin\theta\hat{i} - F_e\hat{i} = 0$$

$$T\sin\theta = F_e$$

Here T is the tension acting on the charge due to the string and Fe is the electrostatic force between the two charges.
In the y-direction also, the net acceleration experienced by the charge is zero.

$$T\cos\theta\hat{j} - mg\hat{j} = 0$$

Therefore, $T\cos\theta = mg$.

By dividing equation (1) by equation (2),

$$\tan\theta = \frac{F_e}{mg} \qquad (3)$$

Since they are equally charged, the magnitude of the electrostatic force is

$$F_e = k\frac{q^2}{r^2} \text{ where } k = \frac{1}{4\pi\varepsilon_0}$$

Here r = 2a = 2Lsinθ. By substituting these values in equation (3),

$$\tan\theta = k\frac{q^2}{mg\left(2L\sin\theta\right)^2} \qquad (4)$$

Rearranging the equation (4) to get q

$$q = 2L\sin\theta\sqrt{\frac{mg\tan\theta}{k}}$$

$$= 2\times0.1\times\sin7^\circ\times\sqrt{\frac{10^{-3}\times10\times\tan7^\circ}{9\times10^{9}}}$$

$q = 8.9 \times 10^{-9}$ C = 8.9 nC

Calculate the electrostatic force and gravitational force between the proton and the electron in a hydrogen atom. They are separated by a distance of 5.3×10^{-11} m. The magnitude of charges on the electron and proton are 1.6×10^{-19} C. Mass of the electron is $m_e = 9.1 \times 10^{-31}$ kg and mass of proton is $m_p = 1.6 \times 10^{-27}$ kg.

Solution

The proton and the electron attract each other. The magnitude of the electrostatic force between these two particles is given by

$$F_e = \frac{ke^2}{r^2} = \frac{9\times10^9 \times \left(1.6\times10^{-19}\right)^2}{\left(5.3\times10^{-11}\right)^2}$$

$$= \frac{9\times2.56}{28.09}\times10^{-7} = 8.2 \times 10^{-8}\ \text{N}$$

The gravitational force between the proton and the electron is attractive. The magnitude of the gravitational force between these particles is

$$F_G = \frac{Gm_e m_p}{r^2}$$

$$= \frac{6.67\times10^{-11}\times9.1\times10^{-31}\times1.6\times10^{-27}}{\left(5.3\times10^{-11}\right)^2}$$

$$= \frac{97.11}{28.09}\times10^{-47} = 3.4 \times 10^{-47}\ \text{N}$$

The ratio of the two forces $\frac{F_e}{F_G} = \frac{8.2\times10^{-8}}{3.4\times10^{-47}}$

$= 2.41 \times 10^{39}$

Note that $F_e \approx 10^{39} F_G$

- The electrostatic force between a proton and an electron is enormously greater than the gravitational force between them. Thus the gravitational force is negligible when compared with the electrostatic force in many situations such as for small size objects and in the atomic domain. This is the reason why a charged comb attracts an uncharged piece of paper with greater force even though the piece of paper is attracted downward by the Earth. This is shown

Superposition principle

- Coulomb's law explains the interaction between two point charges. If there are more than two charges, the force on one charge due to all the other charges needs to be calculated. Coulomb's law alone does not give the answer. The superposition principle explains the interaction between multiple charges.

- According to this superposition principle, the total force acting on a given charge is equal to the vector sum of forces exerted on it by all the other charges.

- Consider a system of n charges, namely q1, q2, q3 ….qn. The force on q1 exerted by the charge q2

$$\vec{F}_{12} = k\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}$$

- Here is the unit vector from q2 to q1 along the line joining the two charges and

- 21 is the distance between the charges q1 and q2. The electrostatic force between two charges is not affected by the presence of other charges in the neighbourhood.

The force on q1 exerted by the charge q3 is

$$\vec{F}_{13} = k\frac{q_1 q_3}{r_{31}^2}\hat{r}_{31}$$

- By continuing this, the total force acting on the charge q1 due to all other charges is given by

$$\vec{F}_1^{tot} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \dots\dots\dots \vec{F}_{1n}$$

$$\vec{F}_1^{tot} = k\left\{\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21} + \frac{q_1 q_3}{r_{31}^2}\hat{r}_{31} + \frac{q_1 q_4}{r_{41}^2}\hat{r}_{41} + \dots \right.$$
$$\left. \dots + \frac{q_1 q_n}{r_{n1}^2}\hat{r}_{n1}\right\} \qquad (1.3)$$

E X A M P L E 1.5

- Consider four equal charges q1,q2, q3 and q4 = q = +1μC located at four different points on a circle of radius 1m, as shown in the figure. Calculate the total force acting on the charge q1 due to all the other charges.

Solution

According to the superposition principle, the total electrostatic force on charge q1 is the vector sum of the forces due to the other charges,

$$\vec{F}_1^{tot} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}$$

following dia ram sho s the dir ction of ac for e on he cha ge q1.

The charges q2 and q4 are equi-distant from q1. As a result the strengths (magnitude) of the forces F^{12} and F^{14} are the same even though their directions are different. Therefore the vectors representing these two forces are drawn with equal lengths. But the charge q3 is located farther compared to q2 and q4. Since the strength of the electrostatic force decreases as distance increases, the strength of the force F^{13} is lesser than that of forces F^{12} and F^{14}. Hence the vector representing the force F^{13} is drawn with smaller length compared to that for forces F^{12} and F^{14}

From the figure, $r_{21} = \sqrt{2}\,m = r_{41}$ and $r_{31} = 2m$

The magnitudes of the forces are given by

$$F_{13} = \frac{kq^2}{r_{31}^2} = \frac{9\times10^9\times10^{-12}}{4}$$

$F_{13} = 2.25 \times 10^{-3}$ N

$$F_{12} = \frac{kq^2}{r_{12}^2} = F_{14} = \frac{9\times10^9\times10^{-12}}{2}$$

$= 4.5 \times 10^{-3}$ N

From the figure, the angle θ = 45°. In terms of the components, we have

$$\vec{F}_{12} = F_{12}\cos\theta\hat{i} - F_{12}\sin\theta\hat{j}$$

$$= 4.5\times10^{-3}\times\frac{1}{\sqrt{2}}\hat{i} - 4.5\times10^{-3}\times\frac{1}{\sqrt{2}}\hat{j}$$

$$\vec{F}_{13} = F_{13}\hat{i} = 2.25\times10^{-3}N\hat{i}$$

$$\vec{F}_{14} = F_{14}\cos\theta\hat{i} + F_{14}\sin\theta\hat{j}$$

$$= 4.5\times10^{-3}\times\frac{1}{\sqrt{2}}\hat{i} + 4.5\times10^{-3}\times\frac{1}{\sqrt{2}}\hat{j}$$

Then the total force on q_1 is,

$$\vec{F}_1^{tot} = \left(F_{12}\cos\theta\hat{i} - F_{12}\sin\theta\hat{j}\right) + F_{13}\hat{i} + \left(F_{14}\cos\theta\hat{i} + F_{14}\sin\theta\hat{j}\right)$$

$$\vec{F}_1^{tot} = \left(F_{12}\cos\theta + F_{13} + F_{14}\cos\theta\right)\hat{i} + \left(-F_{12}\sin\theta + F_{14}\sin\theta\right)\hat{j}$$

Since $F_{12} = F_{14}$, the j^{th} component is zero. Hence we have

$$\vec{F}_1^{tot} = \left(F_{12}\cos\theta + F_{13} + F_{14}\cos\theta\right)\hat{i}$$

substituting the values in the above equation,

$$= \left(\frac{4.5}{\sqrt{2}} + 2.25 + \frac{4.5}{\sqrt{2}}\right)\hat{i} = \left(4.5\sqrt{2} + 2.25\right)\hat{i}$$

$$\vec{F}_1^{tot} = 8.61\times10^{-3}N\hat{i}$$

The resultant force is along the positive x axis.

ELECTRIC FIELD AND ELECTRIC FIELD LINES

Electric Field

- The interaction between two charges is determined by Coulomb's law. How does the interaction itself occur? Consider a point charge kept at a point in space. If another point charge is placed at some distance from the first point charge, it experiences either an attractive force or repulsive force. This is called 'action at a distance'. But how does the second charge know about existence of the first charge which is located at some distance away from it? To answer this question, Michael Faraday introduced the concept of field.

- According to Faraday, every charge in the universe creates an electric field in the surrounding space, and if another charge is brought into its field, it will interact with the electric field at that point and will experience a force. It may be recalled that the interaction of two masses is similarly explained using the concept of gravitational field (Refer unit 6, volume 2, XI physics). Both the electric and gravitational forces are non-contact forces, hence the field concept is required to explain action at a distance.

- Consider a source point charge q located at a point in space. Another point charge qo (test charge) is placed at some point P which is at a distance r from the charge q. The electrostatic force experienced by the charge qo due to q is given by Coulomb's law.

$$\vec{F} = \frac{kqq_0}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r} \quad \text{where} \quad k = \frac{1}{4\pi\varepsilon_0}$$

- The charge q creates an electric field in the surrounding space. The electric field at the point P at a distance r from the point charge q is the force experienced by a unit charge and is given by

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{kq}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \quad (1.4)$$

Here is the unit vector pointing from q to the point of interest P. The electric field is a vector quantity and its SI unit is Newton per Coulomb (NC-1).

Important aspects of Electric field

(i) If the charge q is positive then the electric field points away from the source charge and if q is negative, the electric field points towards the source charge q. This is shown in the Figure 1.4.

(ii) If the electric field at a point P is E then the force experienced by the test charge qo placed at the point P is

(iii) The equation (1.4) implies that the electric field is independent of the test charge qo and it depends only on the source charge q.

(iv) Since the electric field is a vector quantity, at every point in space, this field has unique direction and magnitude as shown in Figures 1.6(a) and (b). From equation (1.4), we can infer that as distance increases, the electric field decreases in magnitude.

Note that in Figures 1.6 (a) and (b) the length of the electric field vector is shown for three different points. The strength or magnitude of the electric field at point P is stronger than at the points Q and R because the point P is closer to the source charge.

(v) In the definition of electric field, it is assumed that the test charge $q0$ is taken sufficiently small, so that bringing this test charge will not move the source charge. In other words, the test charge is made sufficiently small such that it will not modify the electric field of the source charge.

(vi) The expression (1.4) is valid only for point charges. For continuous and finite size charge distributions, integration techniques must be used These will be explained later in the same section. However, this expression can be used as an approximation for a finite-sized charge if the test point is very far away from the finite sized source charge. Note that we similarly treat the Earth as a point mass when we calculate the gravitational field of the Sun on the Earth (refer unit 6, volume 2, XI physics).

(vii) There are two kinds of the electric field: uniform (constant) electric field and non-uniform electric field. Uniform electric field will have the same direction and constant magnitude at all points in space. Non-uniform electric field will have different directions or different magnitudes or both at different points in space. The electric field created by a point charge is basically a non uniform electric field. This non-uniformity arises, both in direction and magnitude, with the direction being radially outward (or inward) and the magnitude changes as distance increases. These are shown in Figure 1.7.

EXAMPLE 1.6

Calculate the electric field at points P, Q for the following two cases, as shown in the figure.

(a) A positive point charge +1 μC is placed at the origin
(b) A negative point charge -2 μC is placed at the origin

Solution

Case (a)

The magnitude of the electric field at point P is

$$E_P = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{9\times10^9\times1\times10^{-6}}{4}$$
$$= 2.25\times10^3\, NC^{-1}$$

Since the source charge is positive, the electric field points away from the charge. So the electric field at the point P is given by

$$\vec{E}_P = 2.25\times10^3 NC^{-1}\,\hat{i}$$

For the point Q

$$\left|\vec{E}_Q\right| = \frac{9\times10^9\times1\times10^{-6}}{16} = 0.56\times10^3\, NC^{-1}$$

Hence $\vec{E}_Q = 0.56\times10^3\,\hat{j}$

Case (b)

The magnitude of the electric field at point P

$$\left|\vec{E}_P\right| = \frac{kq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{9\times10^9\times2\times10^{-6}}{4}$$

$= 4.5 \times 10^3$ N C^{-1}

Since the source charge is negative, the electric field points towards the charge. So the electric field at the point P is given by

$$\vec{E}_P = -4.5\times10^3\,\hat{i}\ NC^{-1}$$

For the point Q, $\left|\vec{E}_Q\right| = \frac{9\times10^9\times2\times10^{-6}}{36}$

$= 0.5 \times 10^3$ N C^{-1}

$$\vec{E}_R = 0.56\times10^3\,\hat{i}\ NC^{-1}$$

At the point Q the electric field is directed along the positive x-axis.

Electric field due to the system of point charges

- Suppose a number of point charges are distributed in space. To find the electric field at some point P due to this collection of point charges, superposition principle is used. The electric field at an arbitrary point due to a collection of point charges is simply equal to the vector sum of the electric fields created by the individual point charges. This is called superposition of electric fields. Consider a collection of point charges located at various points in space. The total electric field at some point P due to all these n chargesisgivenby $q_1, q_2, q_3 \cdots\cdots\cdots q_n$

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots\ldots + \vec{E}_n \quad (1.6)$$

$$\vec{E}_{tot} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_1}{r_{1P}^2}\hat{r}_{1P} + \frac{q_2}{r_{2P}^2}\hat{r}_{2P} + \frac{q_3}{r_{3P}^2}\hat{r}_{3P} + \ldots \ldots + \frac{q_n}{r_{nP}^2}\hat{r}_{nP}\right\} \quad (1.7)$$

Here $r_{1P}, r_{2P}, r_{3P} \ldots\ldots\ldots r_{nP}$ are the distance of the the charges $q_1, q_2, q_3 \ldots\ldots\ldots q_n$ from the point P respectively. Also $\hat{r}_{1P}, \hat{r}_{2P}, \hat{r}_{3P} \ldots\ldots\ldots \hat{r}_{nP}$ are the corresponding unit vectors directed from $q_1, q_2, q_3 \ldots\ldots\ldots q_n$ to P.

Equation (1.7) can be re-written as,

$$\vec{E}_{tot} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\left(\frac{q_i}{r_{iP}^2}\hat{r}_{iP}\right) \quad (1.8)$$

For example in Figure 1.8, the resultant electric field due to three point charges at point P is shown. $q_1, q_2, q_3 \ldots\ldots\ldots q_n$

Note that the relative lengths of the electric field vectors for the charges depend on relative distances of the charges to the point P.

EXAMPLE 1.7

- Consider the charge configuration as shown in the figure. Calculate the electric field at point A. If an electron is placed at points A, what is the acceleration experienced by this electron? (mass of the electron = 9.1×10^{-31} kg and charge of electron = −1.6 × 10-19 C)

Solution

By using superposition principle, the net electric field at point A is

$$\vec{E}_A = \frac{1}{4\pi\varepsilon_o}\frac{q_1}{r_{1A}^2}\hat{r}_{1A} + \frac{1}{4\pi\varepsilon_o}\frac{q_2}{r_{2A}^2}\hat{r}_{2A},$$

where r_{1A} and r_{2A} are the distances of point A from the two charges respectively.

$$\vec{E}_A = \frac{9\times10^9\times1\times10^{-9}}{(2\times10^{-3})^2}(\hat{j}) + \frac{9\times10^9\times1\times10^{-9}}{(2\times10^{-3})^2}(\hat{i})$$

$$= 2.25\times10^9\ \hat{j} + 2.25\times10^9\hat{i} = 2.25\times10^9(\hat{i}+\hat{j})$$

The magnitude of electric field

$$|\vec{E}_A| = \sqrt{(2.25\times10^9)^2 + (2.25\times10^9)^2}$$
$$= 2.25\times\sqrt{2}\times10^9\,NC^{-1}$$

The direction of $\vec{E}_A$ is given by $\frac{\vec{E}_A}{|\vec{E}_A|} = \frac{2.25\times10^9(\hat{i}+\hat{j})}{2.25\times\sqrt{2}\times10^9} = \frac{(\hat{i}+\hat{j})}{\sqrt{2}}$, which is the unit vector along OA as shown in the figure.

The acceleration experienced by an electron placed at point A is

$$\vec{a}_A = \frac{\vec{F}}{m} = \frac{q\vec{E}_A}{m}$$

$$= \frac{(-1.6\times10^{-19})\times(2.25\times10^{9})(\hat{i}+\hat{j})}{9.1\times10^{-31}}$$

$$= -3.95\times10^{20}(\hat{i}+\hat{j})N$$

The electron is accelerated in a direction exactly opposite to $\vec{E}_A$.

Electric field due to continuous charge distribution

- The electric charge is quantized microscopically. The expressions (1.2), (1.3), (1.4) are applicable to only point charges. While dealing with the electric field due to a charged sphere or a charged wire etc., it is very difficult to look at individual charges in these charged bodies. Therefore, it is assumed that charge is distributed continuously on the charged bodies and the discrete nature of charges is not considered here. The electric field due to such continuous charge distributions is found by invoking the method of calculus.

- Consider the following charged object of irregular shape as shown in Figure 1.9. The entire charged object is divided into a large number of charge elementsand each charge element $\Delta q_1, \Delta q_2, \Delta q_3, \ldots\ldots \Delta q_n$ is taken as a point charge.

- The electric field at a point P due to a charged object is approximately given by the sum of the fields at P due to all such charge elements.

$$\vec{E} \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{\Delta q_1}{r_{1P}^2}\hat{r}_{1P} + \frac{\Delta q_2}{r_{2P}^2}\hat{r}_{2P} + \dots\dots + \frac{\Delta q_n}{r_{nP}^2}\hat{r}_{nP}\right)$$

$$\approx \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{\Delta q_i}{r_{iP}^2}\hat{r}_{iP} \qquad (1.9)$$

- Here is the ith charge element, riP is the distance of the point P from the ith charge element and is the unit vector from ith charge element to the point P.
- However the equation (1.9) is only an approximation. To incorporate the continuous distribution of charge, we take the limit In this limit, the summation in the equation (1.9) becomes an integration and takes the following form

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2}\hat{r} \qquad (1.10)$$

.

,

Here r is the distance of the point P from the infinitesimal charge *dq* and $\hat{r}$ is the unit vector from *dq* to point P. Even though the electric field for a continuous charge distribution is difficult to evaluate, the force experienced by some test charge q in this electric field is still given by $\vec{F} = q\vec{E}$.

If the charge Q is uniformly distributed along the wire of length L, then linear charge density (charge per unit length) is $\lambda = \frac{Q}{L}$. Its unit is coulomb per meter

The charge present in the infinitesimal length *dl* is *dq* = λd*l*. This is shown in Figure 1.10 (a).

The electric field due to the line of total charge Q is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{r^2}\hat{r} = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{dl}{r^2}\hat{r}$$

(b) If the charge Q is uniformly distributed on a surface of area A, then surface charge density (charge per unit area) is $\sigma = \frac{Q}{A}$. Its unit is coulomb per square meter ($C\ m^{-2}$).

The charge present in the infinitesimal area dA is $dq = \sigma dA$. This is shown in the figure 1.10 (b).

The electric field due to a of total charge Q is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_\circ}\int \frac{\sigma da}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_\circ}\sigma\int \frac{da}{r^2}\hat{r}$$

This is shown in Figure 1.10(b).

(c) If the charge Q is uniformly distributed in a volume V, then volume charge density (charge per unit volume) is given by $\rho = \frac{Q}{V}$. Its unit is coulomb per cubic meter ($C\ m^{-3}$).

The charge present in the infinitesimal volume element dV is $dq = \rho dV$. This is shown in Figure 1.10(c).

The electric field due to a volume of total charge Q is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_\circ}\int \frac{\rho dV}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_\circ}\rho\int \frac{dV}{r^2}\hat{r}.$$

EXAMPLE 1.8

A block of mass m and positive charge q is placed on an insulated frictionless inclined plane as shown in the figure. A uniform electric field E is applied parallel to the inclined surface such that the block is at rest. Calculate the magnitude of the electric field E.

Solution

Note: A similar problem is solved in XIth Physics volume I, unit 3 section 3.3.2.
There are three forces that acts on the mass m:

(i) The downward gravitational force exerted by the Earth (mg)
(ii) The normal force exerted by the inclined surface (N)
(iii) The Coulomb force given by uniform electric field (qE)
The free body diagram for the mass m is drawn below.

A convenient inertial coordinate system is located in the inclined surface as shown in the figure. The mass m has zero net acceleration both in x and y-direction.

Along x-direction, applying Newton's second law, we have

$$mg\sin\theta\hat{i} - qE\hat{i} = 0$$

$$mg\sin\theta - qE = 0$$

$$\text{or,} \quad E = \frac{mg\sin\theta}{q}$$

Note that the magnitude of the electric field is directly proportional to the mass m and inversely proportional to the charge q. It implies that, if the mass is increased by keeping the charge constant, then a strong electric field is required to stop the object from sliding. If the charge is increased by keeping the mass constant, then a weak electric field is sufficient to stop the mass from sliding down the plane.

The electric field also can be expressed in terms of height and the length of the inclined surface of the plane.

$$E = \frac{mg\,h}{qL}$$

Electric field lines

Electric field vectors are visualized by the concept of electric field lines. They form a set of continuous lines which are the visual representation of the electric field in some region of space. The following rules are followed while drawing electric field lines for charges.

• The electric field lines start from a positive charge and end at negative charges or at infinity. For a positive point charge the electric field lines point radially outward and for a negative point charge, the electric field lines point radially inward.

Note that for an isolated positive point charge the electric field line starts from the charge and ends only at infinity. For an isolated negative point charge the electric field lines start at infinity and end at the negative charge.

• The electric field vector at a point in space is tangential to the electric field line at that point.

• The electric field lines are denser (more closer) in a region where the electric field has larger magnitude and less dense in a region where the electric field is of smaller magnitude. In other words, the number of lines passing through a given surface area perpendicular to the lines is proportional to the magnitude of the electric field in that region. electric field lines from a positive point charge. The magnitude of the electric field for a point charge decreases as the distance increases $\left(|\vec{E}| \propto \frac{1}{r^2}\right)$. So the

electric field has greater magnitude at the surface A than at B. Therefore, the number of lines crossing the surface A is greater than the number of lines crossing the surface B. Note that at surface B the electric field lines are farther apart compared to the electric field lines at the surface A.

As a consequence, if some charge is placed in the intersection point, then it has to move in two different directions at the same time, which is physically impossible. Hence, electric field lines do not intersect.

• The number of electric field lines that emanate from the positive charge or end at a negative charge is directly proportional to the magnitude of the charges.

For example in the Figure 1.15, the electric field lines are drawn for charges +q and -2q. Note that the number of field lines emanating from +q is 8 and the number of field lines ending at -2q is 16. Since the magnitude of the second charge is twice that

• No two electric field lines intersect each other. If two lines cross at a point, then there will be two different electric field vectors at the same point,

of the first charge, the number of field lines drawn for -2q is twice in number than that for charge +q.

EXAMPLE 1.9

The following pictures depict electric field lines for various charge configurations.

(i) In figure (a) identify the signs of two charges and find the ratio
(ii) In figure (b), calculate the ratio of two positive charges and identify the strength of the electric field at three points A, B, and C
(iii) Figure (c) represents the electric field lines for three charges. If q2 = -20 nC, then calculate the values of q1 and q3

Solution

(i) The electric field lines start at q2 and end at q1. In figure (a), q2 is positive and q1 is negative. The number of lines starting from q2 is 18 and number of the lines ending at q1 is 6. So q2 has greater magnitude.

ratio of $\left|\frac{q_1}{q_2}\right| = \frac{N_1}{N_2} = \frac{6}{18} = \frac{1}{3}$. It implies

that $|q_2| = 3|q_1|$

(ii) In figure (b), the number of field lines emanating from both positive charges are equal (N=18). So the charges are equal. At point A, the electric field lines are denser compared to the lines at point B. So the electric field at point A is greater in magnitude compared to the field at point B. Further, no electric field line passes through C, which implies that the resultant electric field at C due to these two charges is zero.

(iii) In the figure (c), the electric field lines start at q1 and q3 and end at q2. This implies that q1 and q3 are positive charges. The ratio of the number

of field lines is $\left|\frac{q_1}{q_2}\right| = \frac{8}{16} = \left|\frac{q_3}{q_2}\right| = \frac{1}{2}$, implying that q_1 and q_3 are half of the magnitude of q_2. So $q_1 = q_3 = +10$ nC.

ELECTRIC DIPOLE AND ITS PROPERTIES

Electric dipole

Two equal and opposite charges separated by a small distance constitute an electric dipole. In many molecules, the centers of positive and negative charge do not coincide. Such molecules behave as permanent dipoles. Examples: CO, water, ammonia, HCl etc.
Consider two equal and opposite point charges (+q, -q) that are separated by a distance 2a as shown in Figure 1.16(a).
The electric dipole moment is defined as

$p = qr_+ - qr_-$. Here $\vec{r}_+$ is the position vector of +q from the origin and $\vec{r}_-$ is the position vector of -q

from the origin. Then, from Figure 1.16 (a),

$$\vec{p} = qa\hat{i} - qa(-\hat{i}) = 2qa\hat{i} \qquad (1.11)$$

The electric dipole moment vector lies along the line joining two charges and is directed from –q to +q. The SI unit of dipole moment is coulomb meter (Cm). The electric field lines for an electric dipole are shown in Figure 1.16 (b).

• For simplicity, the two charges are placed on the x-axis. Even if the two charges are placed on y or z-axies, dipole moment will point from –q to +q. The magnitude of the electric dipole moment is equal to the product of the magnitude of one of the charges and the distance between them,

• Though the electric dipole moment for two equal and opposite charges is defined, it is very general. It is possible to define and calculate the electric dipole moment for a single charge, two positive charges, two negative charges and also for more than two charges.

For a collection of n point charges, the electric dipole moment is defined as follows:

pqa=2

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i \qquad (1.12)$$

where $\vec{r}_i$ is the position vector of charge q_i from the origin.

Solution

Case (a) The position vector for the +q on the positive x-axis is $a\hat{i}$ and position vector for the +q charge the negative x axis is $-a\hat{i}$. So the dipole moment is,

$$\vec{p} = (+q)(a\hat{i}) + (+q)(-a\hat{i}) = 0$$

Case (b) In this case one charge is placed at the origin, so its position vector is zero. Hence only the second charge +q with position vector $a\hat{i}$ contributes to the dipole moment, which is $\vec{p} = qa\hat{i}$.

From both cases (a) and (b), we can infer that in general the electric dipole moment depends on the choice of the origin and charge configuration. But for one special case, the electric dipole moment is independent of the origin. If the total charge is zero, then the electric dipole moment will be the same irrespective of the choice of the origin. It is because of this reason that the electric dipole moment of an electric dipole (total charge is zero) is always directed from –q to +q, independent of the choice of the origin

Case (c) $\vec{p} = (-2q)a\hat{j} + q(2a)(-\hat{j}) = -4qa\hat{j}$.

Note that in this case $\vec{p}$ is directed from -2q to +q.

Case (d) $\vec{p} = -2qa(-\hat{i}) + qa\hat{j} + qa(-\hat{j})$

$$= 2qa\hat{i}$$

- The water molecule (H2O) has this charge configuration. The water molecule has three atoms (two H atom and one O atom). The centers of positive (H) and negative (O) charges of a water molecule lie at different points, hence it possess permanent dipole moment. The O-H bond length is 0.958 × 10-10 m due to which the electric dipole moment of water molecule has the magnitude p = 6.1 x 10-30 Cm. The electric dipole moment is directed from center of negative charge to the center of positive charge, as shown in the figure.

Electric field due to a dipole

- Case (i) Electric field due to an electric dipole at points on the axial line Consider an electric dipole placed on the x-axis as shown in Figure 1.17. A point C is located at a distance of r from the midpoint O of the dipole along the axial line.

The electric field at a point C due to +q is

$$\vec{E}_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-a)^2} \text{ along BC}$$

Since the electric dipole moment vector $\vec{p}$ is from –q to +q and is directed along BC, the above equation is rewritten as

$$\vec{E}_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-a)^2}\hat{p} \qquad (1.13)$$

where $\hat{p}$ is the electric dipole moment unit vector from –q to +q.

The electric field at a point C due to –q is

$$\vec{E}_- = -\frac{1}{4\pi\varepsilon_0}\frac{q}{(r+a)^2}\hat{p} \qquad (1.14)$$

Since +q is located closer to the point C than –q, $\vec{E}_+$ is stronger than $\vec{E}_-$. Therefore, the length of the $\vec{E}_+$ vector is drawn larger than that of $\vec{E}_-$ vector.

The total electric field at point C is calculated using the superposition principle of the electric field.

$$\vec{E}_{tot} = \vec{E}_+ + \vec{E}_-$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-a)^2}\hat{p} - \frac{1}{4\pi\varepsilon_0}\frac{q}{(r+a)^2}\hat{p}$$

$$\vec{E}_{tot} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right)\hat{p} \quad (1.15)$$

$$\vec{E}_{tot} = \frac{1}{4\pi\varepsilon_0} q\left(\frac{4ra}{(r^2-a^2)^2}\right)\hat{p} \quad (1.16)$$

Note that the total electric field is along $\vec{E}_+$, since $+q$ is closer to C than $-q$.

The direction of $\vec{E}_{tot}$ is shown in Figure 1.18.

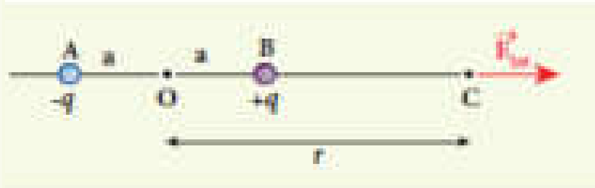


Figure 1.18 Total electric field of the dipole on the axial line

If the point C is very far away from the dipole then (r >> a). Under this limit the term $(r^2 - a^2)^2 \approx r^4$. Substituting this into equation (1.16), we get

$$\vec{E}_{tot} = \frac{1}{4\pi\varepsilon_0}\left(\frac{4aq}{r^3}\right)\hat{p} \quad (r >> a)$$

since $2aq\hat{p} = \vec{p}$

$$\vec{E}_{tot} = \frac{1}{4\pi\varepsilon_0}\frac{2\vec{p}}{r^3} \quad (r >> a) \quad (1.17)$$

If the point C is chosen on the left side of the dipole, the total electric field is still in the direction of $\vec{p}$. We infer this result by examining the electric field lines of the dipole shown in Figure 1.16(b).

Case (ii) Electric field due to an electric dipole at a point on the equatorial plane

Consider a point C at a distance r from the midpoint O of the dipole on the equatorial plane as shown in Figure 1.19.

Since the point C is equi-distant from +q and −q, the magnitude of the electric fields of +q and −q are the same. The direction of $\vec{E}_+$ is along BC and the direction of $\vec{E}_-$ is along CA. $\vec{E}_+$ and $\vec{E}_-$ are resolved into two components; one component parallel to the dipole axis and the other perpendicular to it. The perpendicular components $|\vec{E}_+|\sin\theta$ and $|\vec{E}_-|\sin\theta$ are oppositely directed and cancel each other. The magnitude of the total electric field at point C is the sum of the parallel components of $\vec{E}_+$ and $\vec{E}_-$ and its direction is along $-\hat{p}$ as shown in the Figure 1.19.

$$\vec{E}_{tot} = -|\vec{E}_+|\cos\theta\,\hat{p} - |\vec{E}_-|\cos\theta\,\hat{p} \quad (1.18)$$

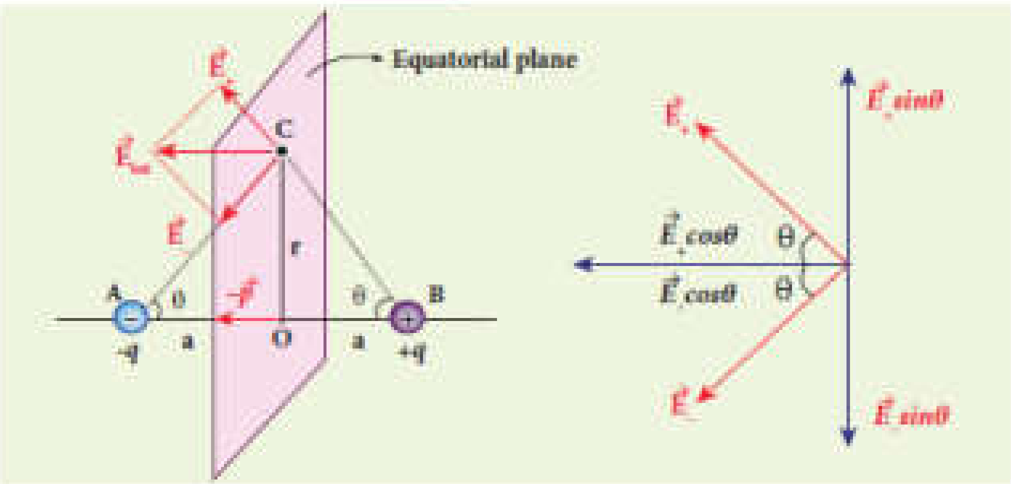


Figure 1.19 Electric field due to a dipole at a point on the equatorial plane

The magnitudes $\vec{E}_+$ and $\vec{E}_-$ are the same and are given by

$$|\vec{E}_+| = |\vec{E}_-| = \frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+a^2)} \qquad (1.19)$$

By substituting equation (1.19) into equation (1.18), we get

$$\vec{E}_{tot} = -\frac{1}{4\pi\epsilon_0}\frac{2q\cos\theta}{(r^2+a^2)}\hat{p}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{2qa}{(r^2+a^2)^{\frac{3}{2}}}\hat{p}$$

$$\text{since } \cos\theta = \frac{a}{\sqrt{r^2+a^2}}$$

$$\vec{E}_{tot} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{(r^2+a^2)^{\frac{3}{2}}}$$

$$\text{since } \vec{p} = 2qa\hat{p} \qquad (1.20)$$

At very large distances (r>>a), the equation (1.20) becomes

$$\vec{E}_{tot} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{r^3} \qquad (r >> a) \qquad (1.21)$$

Important inferences

(i) From equations (1.17) and (1.21), it is inferred that for very large distances, the magnitude of the electric field at points on the dipole axis is twice the magnitude of the electric field at points on the equatorial plane. The direction of the electric field at points on the dipole axis is directed along the direction of dipole moment vector $\vec{p}$ but at points on the equatorial plane it is directed opposite to the dipole moment vector, that is along $-\vec{p}$.

(ii) At very large distances, the electric field due to a dipole varies as $\frac{1}{r^3}$. Note that for a point charge, the electric field varies as $\frac{1}{r^2}$. This implies that the electric field due to a dipole at very large distances goes to zero faster than the electric field due to a point charge. The reason for this behavior is that at very large distance, the two charges appear to be close to each other and neutralize each other.

(iii) The equations (1.17) and (1.21) are valid only at very large distances (r>>a). Suppose the distance 2a approaches zero and q approaches infinity such that the product of $2aq = p$ is finite, then the dipole is called a point dipole. For such point dipoles, equations (1.17) and (1.21) are exact and hold true for any r.

1.4.3 Torque experienced by an electric dipole in the uniform electric field

Consider an electric dipole of dipole moment $\vec{p}$ placed in a uniform electric field $\vec{E}$ whose field lines are equally spaced and point in the same direction. The charge +q will experience a force $q\vec{E}$ in the direction of the field and charge –q will experience a force $-q\vec{E}$ in a direction opposite to the field. Since the external field $\vec{E}$ is uniform, the total force acting on the dipole is zero. These two forces acting at different points will constitute a couple and the dipole experience a torque as shown in Figure 1.20. This torque tends to rotate the dipole. (Note that electric field lines of a uniform field are equally spaced and point in the same direction).

The total torque on the dipole about the point O

$$\vec{\tau} = \vec{OA}\times(-q\vec{E}) + \vec{OB}\times q\vec{E} \qquad (1.22)$$

Using right-hand corkscrew rule (Refer XI, volume 1, unit 2), it is found that total

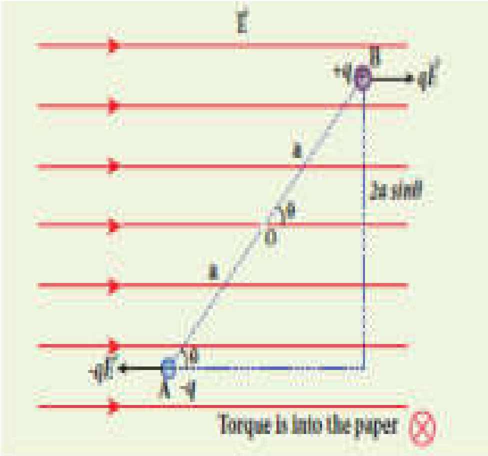


Figure 1.20 Torque on dipole

torque is perpendicular to the plane of the paper and is directed into it.

The magnitude of the total torque $\tau = |\overrightarrow{OA}||(-q\vec{E})|\sin\theta + |\overrightarrow{OB}||q\vec{E}|\sin\theta$

$$\tau = qE \cdot 2a\sin\theta \qquad (1.23)$$

where θ is the angle made by $\vec{p}$ with $\vec{E}$. Since p = 2aq, the torque is written in terms of the vector product as

$$\vec{\tau} = \vec{p} \times \vec{E} \qquad (1.24)$$

The magnitude of this torque is $\tau = pE\sin\theta$ and is maximum when $\theta = 90^\circ$.

This torque tends to rotate the dipole and align it with the electric field $\vec{E}$. Once $\vec{p}$ is aligned with $\vec{E}$, the total torque on the dipole becomes zero.

If the electric field is not uniform, then the force experienced by +q is different from that experienced by -q. In addition to the torque, there will be net force acting on the dipole. This is shown in Figure 1.21.

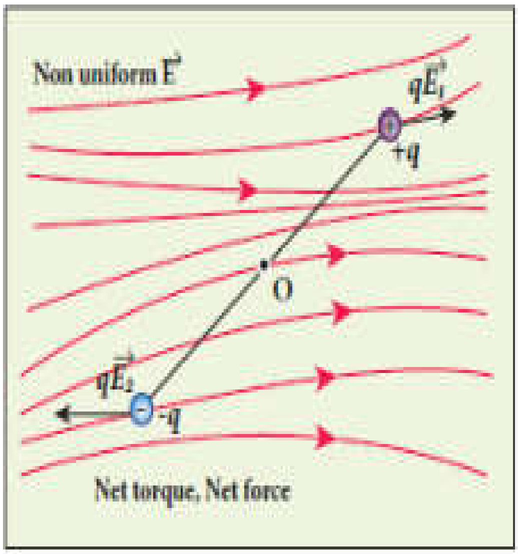


Figure 1.21 The dipole in a non-uniform electric field

EXAMPLE 1.11

A sample of HCl gas is placed in a uniform electric field of magnitude 3 × 10⁴ N C⁻¹. The dipole moment of each HCl molecule is 3.4 × 10⁻³⁰ Cm. Calculate the maximum torque experienced by each HCl molecule.

Solution

The maximum torque experienced by the dipole is when it is aligned perpendicular to the applied field.

$$\tau_{max} = pE\sin 90^\circ = 3.4\times10^{-30}\times3\times10^{4}\,Nm$$

$$\tau_{max} = 10.2\times10^{-26}\,Nm$$

ELECTROSTATIC POTENTIAL AND POTENTIAL ENERGY

Introduction

- In mechanics, potential energy is defined for conservative forces. Since gravitational force is a conservative force, its gravitational potential energy is defined in XI standard physics (Unit 6). Since Coulomb force is an inverse-square-law force, its also a conservative force like gravitational force. Therefore, we can define potential energy for charge configurations

Electrostatic Potential energy and Electrostatic potential

- Consider a positive charge q kept fixed at the origin which produces an electric field around it. A positive test charge q′ is brought from point R to point P against the repulsive force between q and q′ as shown in Figure 1.22. Work must be done to overcome this repulsion. This work done is stored as potential energy.

- The test charge q′ is brought from R to P with constant velocity which means that external force used to bring the test charge q′ from R to P must be equal and opposite

to the coulomb force $(\vec{F}_{ext} = -\vec{F}_{coulomb})$. The work done is

$$W = \int_R^P \vec{F}_{ext} \cdot d\vec{r} \qquad (1.25)$$

Since coulomb force is conservative, work done is independent of the path and it depends only on the initial and final positions of the test charge. If potential energy associated with q′ at P is U_P and that at R is U_R, then difference in potential energy is defined as the work done to bring a test charge q′ from point R to P and is given as $U_P - U_R = W = \Delta U$

$$\Delta U = \int_R^P \vec{F}_{ext} \cdot d\vec{r} \qquad (1.26)$$

$$\text{Since } \vec{F}_{ext} = -\vec{F}_{coulomb} = -q'\vec{E} \qquad (1.27)$$

$$\Delta U = \int_R^P -(q'\vec{E}) \cdot d\vec{r} = q' \int_R^P (-\vec{E}) \cdot d\vec{r} \qquad (1.28)$$

The potential energy difference per unit charge is given by

$$\frac{\Delta U}{q'} = \frac{q'\int_{R}^{P}\left(-\vec{E}\right)\cdot d\vec{r}}{q'} = -\int_{R}^{P}\vec{E}\cdot d\vec{r} \qquad (1.29)$$

The above equation (1.29) is independent of q′. The quantity $\frac{\Delta U}{q'} = -\int_{R}^{P}\vec{E}\cdot d\vec{r}$ is called electric potential difference between P and R and is denoted as $V_P - V_R = \Delta V$.

In otherwords, the electric potential difference is defined as the work done by an external force to bring unit positive charge from point R to point P.

$$V_P - V_R = \Delta V = \int_{R}^{P} -\vec{E}\cdot d\vec{r} \qquad (1.30)$$

The electric potential energy difference can be written as ΔU = q' ΔV. Physically potential difference between two points is a meaningful quantity. The value of the potential itself at one point is not meaningful. Therefore the point R is taken at infinity and its potential is considered as zero ($V_\infty = 0$).

Then the electric potential at a point P is equal to the work done by an external force to bring a unit positive charge with constant velocity from infinity to the point P in the region of the external electric field $\vec{E}$. Mathematically this is written as

$$V_P = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r} \qquad (1.31)$$

Important points

1. Electric potential at point P depends only on the electric field which is due to the source charge q and not on the test charge q'. Unit positive charge is brought from infinity to the point P with constant velocity because external agency should not impart any kinetic energy to the test charge.
2. From equation (1.29), the unit of electric potential is Joule per coulomb. The practical unit is volt (V) named after Alessandro Volta (1745-1827) who invented the electrical battery. The potential difference between two points is expressed in terms of voltage.

1.5.2 Electric potential due to a point charge

Consider a positive charge q kept fixed at the origin. Let P be a point at distance r from the charge q. This is shown in Figure 1.23.

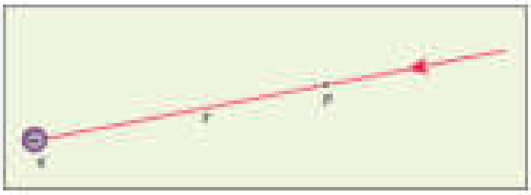


Figure 1.23 Electrostatic potential at a point P

The electric potential at the point P is

$$V = \int_{\infty}^{r}(-\vec{E})\cdot d\vec{r} = -\int_{\infty}^{r}\vec{E}\cdot d\vec{r} \qquad (1.32)$$

Electric field due to positive point charge q is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$V = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r^2}\hat{r}.d\vec{r}$$

The infinitesimal displacement vector, $d\vec{r} = dr\hat{r}$ and using $\hat{r}.\hat{r} = 1$, we have

$$V = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r^2}\hat{r}\cdot dr\hat{r} = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r^2}dr$$

After the integration,

$$V = -\frac{1}{4\pi\varepsilon_0}q\left\{-\frac{1}{r}\right\}_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Hence the electric potential due to a point charge q at a distance r is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad (1.33)$$

Important points

(i) If the source charge q is positive, V > 0. If q is negative, then V is negative and equal to $V = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

(ii) The description of motion of objects using the concept of potential or potential energy is simpler than that using the concept of field.

(iii) From expression (1.33), it is clear that the potential due to positive charge decreases as the distance increases, but for a negative charge the potential increases as the distance is increased. At infinity $(r = \infty)$ electrostatic potential is zero (V = 0).

In the case of gravitational force, mass moves from a point of higher gravitational potential to a point of lower gravitational potential. Similarly a positive charge moves from a point of higher electrostatic potential to lower electrostatic potential. However a negative charge moves from lower electrostatic potential to higher electrostatic potential. This comparison is shown in Figure 1.24.

(iv) The electric potential at a point P due to a collection of charges $q_1, q_2, q_3, \ldots q_n$ is equal to sum of the electric potentials due to individual charges.

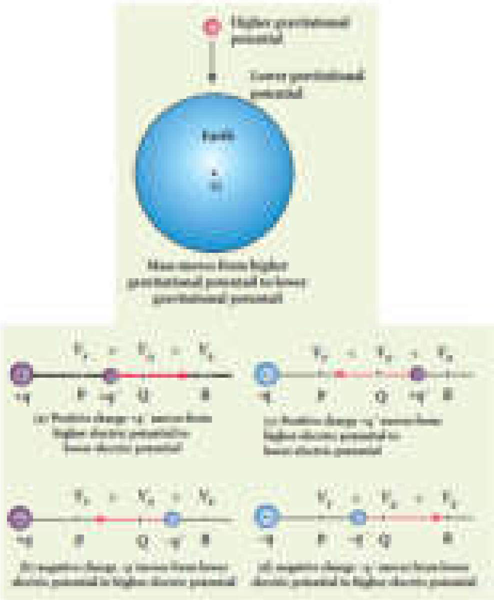


Figure 1.24 Motion of charges in terms of electric potential

$$V_{tot} = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3} + \ldots + \frac{kq_n}{r_n} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} \quad (1.34)$$

where $r_1, r_2, r_3 \ldots r_n$ are the distances of q_1, q_2, q_3 ….. q_n respectively from P (Figure 1.25).

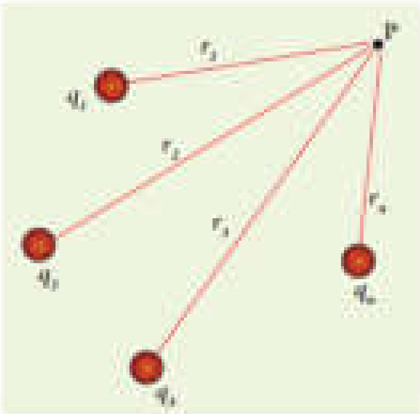
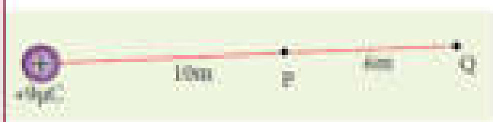


Figure 1.25 Electrostatic potential due to collection of charges

EXAMPLE 1.12

(a) Calculate the electric potential at points P and Q as shown in the figure below.

(b) Suppose the charge +9μC is replaced by -9μC find the electrostatic potentials at points P and Q



(c) Calculate the work done to bring a test charge +2μC from infinity to the point P. Assume the charge +9μC is held fixed at origin and +2μC is brought from infinity to P.

Solution

(a) Electric potential at point P is given by

$$V_P = \frac{1}{4\pi\varepsilon_o}\frac{q}{r_P} = \frac{9\times10^9\times9\times10^{-6}}{10} = 8.1\times10^3 V$$

Electric potential at point Q is given by

$$V_Q = \frac{1}{4\pi\varepsilon_o}\frac{q}{r_Q} = \frac{9\times10^9\times9\times10^{-6}}{16} = 5.06\times10^3 V$$

Note that the electric potential at point Q is less than the electric potential at point P. If we put a positive charge at P, it moves from P to Q. However if we place a negative charge at P it will move towards the charge +9μC.

The potential difference between the points P and Q is given by

$$\Delta V = V_P - V_Q = +3.04\times10^3 V$$

Suppose we replace the charge +9 μC by -9 μC, then the corresponding potentials at the points P and Q are,

$$V_P = -8.1\times10^3 V,\ V_Q = -5.06\times10^3 V$$

Note that in this case electric potential at the point Q is higher than at point P.

The potential difference or voltage between the points P and Q is given by

$$\Delta V = V_P - V_Q = -3.04\times10^3 V$$

(c) The electric potential V at a point P due to some charge is defined as the work done by an external force to bring a unit positive charge from infinity to P. So to bring the q amount of charge from infinity to the point P, work done is given as follows.

$$W = qV$$

$$W_Q = 2\times10^{-8}\times5.06\times10^{3}\,J = 10.12\times10^{-5}\,J.$$

EXAMPLE 1.13

Consider a point charge +q placed at the origin and another point charge -2q placed at a distance of 9 m from the charge +q. Determine the point between the two charges at which electric potential is zero.

Solution

According to the superposition principle, the total electric potential at a point is equal to the sum of the potentials due to each charge at that point.

Consider the point at which the total potential zero is located at a distance x from the charge +q as shown in the figure.

The total electric potential at P is zero.

$$V_{tot} = \frac{1}{4\pi\epsilon_o}\left(\frac{q}{x} - \frac{2q}{(9-x)}\right) = 0$$

which gives $\frac{q}{x} = \frac{2q}{(9-x)}$

or $\frac{1}{x} = \frac{2}{(9-x)}$

Hence, $x = 3$ m

Electrostatic potential at a point due to an electric dipole

Consider two equal and opposite charges separated by a small distance 2a as shown in Figure 1.26. The point P is located at a distance r from the midpoint of the dipole. Let θ be the angle between the line OP and dipole axis AB.

Let r_1 be the distance of point P from +q and r_2 be the distance of point P from –q.

Potential at P due to charge $+q = \frac{1}{4\pi\varepsilon_o}\frac{q}{r_1}$

Potential at P due to charge $-q = -\frac{1}{4\pi\varepsilon_o}\frac{q}{r_2}$

Total potential at the point P,

$$V = \frac{1}{4\pi\varepsilon_o}q\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad (1.35)$$

Suppose if the point P is far away from the dipole, such that r>>a, then equation (1.35) can be expressed in terms of r. By the cosine law for triangle BOP

$$r_1^2 = r^2 + a^2 - 2ra\cos\theta$$

$$r_1^2 = r^2\left(1 + \frac{a^2}{r^2} - \frac{2a}{r}\cos\theta\right)$$

Since the point P is very far from dipole, then r>>a. As a result the term $\frac{a^2}{r^2}$ is very small and can be neglected. Therefore

$$r_1^2 = r^2\left(1 - 2a\frac{\cos\theta}{r}\right)$$

$$\text{(or)} \quad r_1 = r\left(1 - \frac{2a}{r}\cos\theta\right)^{\frac{1}{2}}$$

Similarly applying the cosine law for triangle AOP,

$$r_2^2 = r^2 + a^2 - 2ra\cos(180-\theta)$$

since $\cos(180-\theta) = -\cos\theta$ we get

$$r_2^2 = r^2 + a^2 + 2ra\cos\theta$$

Neglecting the term $\frac{a^2}{r^2}$ (because r>>a)

$$r_2^2 = r^2\left(1+\frac{2a\cos\theta}{r}\right)$$

$$r_2 = r\left(1+\frac{2a\cos\theta}{r}\right)^{\frac{1}{2}}$$

Using Binomial theorem, we get

$$\frac{1}{r_2} = \frac{1}{r}\left(1-a\frac{\cos\theta}{r}\right) \qquad (1.37)$$

Substituting equation (1.37) and (1.36) in equation (1.35),

$$V = \frac{1}{4\pi\varepsilon_o}q\left(\frac{1}{r}\left(1+a\frac{\cos\theta}{r}\right)-\frac{1}{r}\left(1-a\frac{\cos\theta}{r}\right)\right)$$

$$V = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\left(1+a\frac{\cos\theta}{r}-1+a\frac{\cos\theta}{r}\right)\right)$$

$$V = \frac{1}{4\pi\varepsilon_o}\frac{2aq}{r^2}\cos\theta$$

But the electric dipole moment p = 2qa and we get,

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\cos\theta}{r^2}\right)$$

Now we can write p cosθ = $\vec{p}\cdot\hat{r}$, where $\hat{r}$ is the unit vector from the point O to point P. Hence the electric potential at a point P due to an electric dipole is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad (r>>a) \quad (1.38)$$

Equation (1.38) is valid for distances very large compared to the size of the dipole. But for a point dipole, the equation (1.38) is valid for any distance.

Special cases

Case (i) If the point P lies on the axial line of the dipole on the side of +q, then θ = 0. Then the electric potential becomes

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} \qquad (1.39)$$

Case (ii) If the point P lies on the axial line of the dipole on the side of –q, then θ = 180o, then

$$V = -\frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} \qquad (1.40)$$

Case (iii) If the point P lies on the equatorial line of the dipole, then θ = 90o. Hence

$$V = 0 \qquad (1.41)$$

Important points

I. The potential due to an electric dipole falls as $\frac{1}{r^2}$ and the potential due to a single point charge falls as $\frac{1}{r}$. Thus the potential due to the dipole falls faster than that due to a monopole (point charge). As the distance increases from electric dipole, the effects of positive and negative charges nullify each other.

II. The potential due to a point charge is spherically symmetric since it depends only on the distance r. But the potential due to a dipole is not spherically symmetric because the potential depends on the angle between $\vec{p}$ and position vector $\vec{r}$ of the point.

However the dipole potential is axially symmetric. If the position vector $\vec{r}$ is rotated about $\vec{p}$ by keeping θ fixed, then all points on the cone at the same distance r will have the same potential as shown in Figure 1.27. In this figure, all the points located on the blue curve will have the same potential.

Equi-potential Surface

- Consider a point charge q located at some point in space and an imaginary sphere of radius r is chosen by keeping the charge q at its center (Figure 1.28(a)). The electric potential at all points on the surface of the given sphere is the same. Such a surface is called an equipotential surface.

- An equipotential surface is a surface on which all the points are at the same potential. For a point charge the equipotential surfaces are concentric spherical surfaces as shown in Figure 1.28(b). Each spherical surface is an equipotential surface but the value of the potential is different for different spherical surfaces.

- For a uniform electric field, the equipotential surfaces form a set of planes normal to the electric field $\vec{E}$. This is shown in the Figure 1.29.

Properties of equipotential surfaces

(i) The work done to move a charge q between any two points A and B, W = q (VB – VA). If the points A and B lie on the same equipotential surface, work done is zero because VA = VB.

(ii) The electric field is normal to an equipotential surface. If it is not normal, then there is a component of the field parallel to the surface. Then work must be done to move a charge between two points on the same surface. This is a contradiction. Therefore the electric field must always be normal to equipotential surface.

Relation between electric field and potential

- Consider a positive charge q kept fixed at the origin. To move a unit positive charge by a small distance dx in the electric field E, the work done is given by dW = −E dx. The minus sign implies that work is done against the electric field. This work done is equal to electric potential difference. Therefore,

$$dW = dV.$$

$$\text{(or)} \quad dV = -E\,dx \qquad (1.42)$$

$$\text{Hence } E = -\frac{dV}{dx} \qquad (1.43)$$

The electric field is the negative gradient of the electric potential. In general,

$$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial k}\hat{k}\right] \qquad (1.44)$$

Electrostatic potential energy for collection of point charges

- The electric potential at a point at a distance r from point charge q1 is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r}$$

This potential V is the work done to bring a unit positive charge from infinity to the point. Now if the charge q2 is brought from infinity to that point at a distance r from q1, the work done is the product of q2 and the electric potential at that point. Thus we have

$$W = q_2 V$$

This work done is stored as the electrostatic potential energy U of a system of charges q1 and q2 separated by a distance r. Thus we have

$$U = q_2 V = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \qquad (1.45)$$

- The electrostatic potential energy depends only on the distance between the two point charges. In fact, the expression (1.45) is derived by assuming that q1 is fixed and q2 is brought from infinity. The equation (1.45) holds true when q2 is fixed and q1 is brought from infinity or both q1 and q2 are simultaneously brought from infinity to a distance r between them.

- Three charges are arranged in the following configuration as shown in Figure 1.30.

- To calculate the total electrostatic potential energy, we use the following procedure. We bring all the charges one by one and arrange them according to the configuration as shown in Figure 1.30.

- Bringing a charge q1 from infinity to the point A requires no work, because there are no other charges already present in the vicinity of charge q1.

- To bring the second charge q2 to the point B, work must be done against the electric field created by the charge q1. So the work done on the charge q2 is W = q2 V1B. Here V1B is the electrostatic potential due to the charge q1 at point B.

$$U = \frac{1}{4\pi\varepsilon_o}\frac{q_1 q_2}{r_{12}} \qquad (1.46)$$

Note that the expression is same when q_2 is brought first and then q_1 later.

- Similarly to bring the charge q_3 to the point C, work has to be done against the total electric field due to both charges q_1 and q_2. So the work done to bring the charge q_3 is = q_3 (V_{1C} + V_{2C}). Here V_{1C} is the electrostatic potential due to charge q_1 at point C and V_{2C} is the electrostatic potential due to charge q_2 at point C.

The electrostatic potential is

$$U = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) \qquad (1.47)$$

- Adding equations (1.46) and (1.47), the total electrostatic potential energy for the system of three charges q1, q2 and q3 is

$$U = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) \qquad (1.48)$$

- Note that this stored potential energy U is equal to the total external work done to assemble the three charges at the given