FORCE AND MOTION
PART - 2

10th Standard
Unit 1: Laws of Motion

INTRODUCTION

Human beings are so curious about things around them. Things around us are related to one another. Some bodies are at rest and some are in motion. Rest and motion are interrelated terms.

In the previous classes you have learnt about various types of motion such as linear motion, circular motion, oscillatory motion, and so on. So far, you have discussed the motion of bodies in terms of their displacement, velocity, and acceleration. In this unit, let us investigate the cause of motion.

When a body is at rest, starts moving, a question that arises in our mind is 'what causes the body to move?' Similarly, when a moving object comes to rest, you would like to know what brings it to rest? If a moving object speeds up or slows down or changes its direction. what speeds up or slows down the body? What changes the direction of motion?

One answer for all the above questions is 'Force'. In a common man's understanding of motion, a body needs a 'push' or 'pull' to move, or bring to rest or change its velocity. Hence, this 'push' or 'pull' is called as 'force'.

Let us define force in a more scientific manner using the three laws proposed by Sir Isaac Newton. These laws help you to understand the motion of a body and also to predict the future course of its motion, if you know the forces acting on it. Before Newton formulated his three laws of motion, a different perception about the force and motion of bodies prevailed. Let us first look at these ideas and then eventually learn about Newton’s laws in this unit.

Mechanics is the branch of physics that deals with the eff ect of force on bodies. It is divided into two branches, namely, statics and dynamics.

Statics: It deals with the bodies, which are at rest under the action of forces.

Dynamics: It is the study of moving bodies under the action of forces. Dynamics is further divided as follows.

Kinematics: It deals with the motion of bodies without considering the cause of motion.

Kinetics: It deals with the motion of bodies considering the cause of motion.

FORCE AND MOTION

According to Aristotle a Greek Philosopher and Scientist, the natural state of earthly bodies is ‘rest’. He stated that a moving body naturally comes to rest without any external infl uence of the force. Such motions are termed as ‘natural motion’ (Force independent). He also proposed that a force (a push or a pull) is needed to make the bodies to move from their natural state (rest) and behave contrary to their own natural state called as ‘violent motion’ (Force dependent). Further, he said, when two diff ernt mass bodies are dropped from a height, the heavier body falls faster than the lighter one.

Galileo proposed the following concepts about force, motion and inertia of bodies:

(i) Th e natural state of all earthly bodies is either the state of rest or the state of uniform motion.

(ii) A body in motion will continue to be in the same state of motion as long as no external force is applied.

(iii) When a force is applied on bodies, they resist any change in their state. Th is property of bodies is called 'inertia'.
(iv) When dropped from a height in vacuum, bodies of diff erent size, shape and mass fall at the same rate and reach the ground at the same time.

INERTIA

While you are travelling in a bus or in a car, when a sudden brake is applied, the upper part of your body leans in the forward direction. Similarly, when the vehicle suddenly is move forward from rest, you lean backward. Th is is due to, any body would like to continue to be in its state of rest or the state of motion. Th is is known as 'inertia'.

The inherent property of a body to resist any change in its state of rest or the state of uniform motion, unless it is infl uenced upon by an external unbalanced force, is known as 'inertia'.

In activity described above, the inertia of the coin keeps it in the state of rest when the cardboard moves. Th en, when the cardboard has moved, the coin falls into the tumbler due to gravity. Th is happen due to 'inertia of rest'.

Types of Inertia

a) Inertia of rest: The resistance of a body to change its state of rest is called inertia of rest.

b) Inertia of motion: The resistance of a body to change its state of motion is called inertia of motion.

c) Inertia of direction: The resistance of a body to change its direction of motion is called inertia of direction.

Examples of Inertia

An athlete runs some distance before jumping. Because, this will help him jump longer and higher. (Inertia of motion)

When you make a sharp turn while driving a car, you tend to lean sideways, (Inertia of direction).

When you vigorously shake the branches of a tree, some of the leaves and fruits are detached and they fall down, (Inertia of rest).

LINEAR MOMENTUM

The impact of a force is more if the velocity and the mass of the body is more. To quantify the impact of a force exactly, a new physical quantity known as linear momentum is defined. The linear momentum measures the impact of a force on a body.

The product of mass and velocity of a moving body gives the magnitude of linear momentum. It acts in the direction of the velocity of the object. Linear momentum is a vector quantity.

Linear Momentum = mass × velocity
$p = m v$. . . . .

It helps to measure the magnitude of a force. Unit of momentum in SI system is kg m s–1 and in C.G.S system its unit is g cm s-1.

NEWTON'S LAWS OF MOTION

Newton's First Law

This law states that every body continues to be in its state of rest or the state of uniform motion along a straight line unless it is acted upon by some external force. It gives the definition of force as well as inertia.

Force

Force is an external effort in the form of push or pull, which:

produces or tries to produce the motion of a static body.

stops or tries to stop a moving body.

changes or tries to change the direction of motion of a moving body.

Force has both magnitude and direction. So, it is a vector quantity.

Types of forces

Based on the direction in which the forces act, they can be classified into two types as: (a) Like parallel forces and (b) Unlike parallel forces.

(a) Like parallel forces: Two or more forces of equal or unequal magnitude acting along the same direction, parallel to each other are called like parallel forces.

(b) Unlike parallel forces: If two or more equal forces or unequal forces act along opposite directions parallel to each other, then they are called unlike parallel forces. Action of forces are given in Table 1.1.

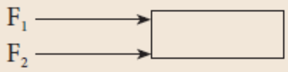
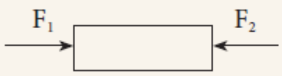
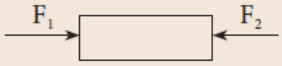
Resultant Force

When several forces act simultaneously on the same body, then the combined effect of the multiple forces can be represented by a single force, which is termed as '*resultant force*'. It is equal to the vector sum (adding the magnitude of the forces with their direction) of all the forces.

If the resultant force of all the forces acting on a body is equal to zero, then the body will be in equilibrium. Such forces are called balanced forces. If the resultant force is not equal to zero, then it causes the motion of the body due to unbalanced forces

Examples: Drawing water from a well, force applied with a crow bar, forces on a weight balance, etc.

A system can be brought to equilibrium by applying another force, which is equal to the resultant force in magnitude, but opposite in direction. Such force is called as 'Equilibrant'.

Action of forces	Diagram	Resultant force (F_{net})
Parallel forces are acting in the same direction		$F_{net} = F_1 + F_2$
Parallel unequal forces are acting in opposite directions		$F_{net} = F_1 - F_2$ (if $F_1 > F_2$) $F_{net} = F_2 - F_1$ (if $F_2 > F_1$) F_{net} is directed along the greater force.
Parallel equal forces are acting in opposite directions in the same line of action ($F_1 = F_2$)		$F_{net} = F_1 - F_2$ ($F_1 = F_2$) $F_{net} = 0$

Rotating Effect of Force

Have you observed the position of the handle in a door? It is always placed at the edge of door and not at some other place. Why? Have you tried to push a door by placing your hand closer to the hinges or the fixed edge? What do you observe?

The door can be easily opened or closed when you apply the force at a point far away from the fixed edge. In this case, the effect of the force you apply is to turn the door about the fixed edge. This turning effect of the applied force is more when the distance between the fixed edge and the point of application of force is more.

The axis of the fixed edge about which the door is rotated is called as the '*axis of rotation*'. Fix one end of a rod to the floor/wall, and apply a force at the other end tangentially.The rod will be turned about the fixed point is called as '*point of rotation*'.

Moment of the Force

The rotating or turning effect of a force about a fixed point or fixed axis is called moment of the force about that point or torque (τ). It is measured by the product of the force (F) and the perpendicular distance (d) between the fixed point or the fixed axis and the line of action of the force.

$\tau = F \times d$. . . . . (1.2)

Torque is a vector quantity. It is acting along the direction, perpendicular to the plane containing the line of action of force and the distance. Its SI unit is N m.

Couple: Two equal and unlike parallel forces applied simultaneously at two distinct points constitute a couple. The line of action of the two forces does not coincide. It does not produce any translatory motion since the resultant is zero. But, a couple results in causes the rotation of the body. Rotating effect of a couple is known as moment of a couple.

Examples: Turning a tap, winding or unwinding a screw, spinning of a top, etc.

Moment of a couple is measured by the product of any one of the forces and the perpendicular distance between the line of action of two forces. The turning effect of a couple is measured by the magnitude of its moment.

Moment of a couple = Force × perpendicular distance between the line of action of forces

$M = F \times S$. . . . .(1.3)

The unit of moment of a couple is newton metre (N m) in SI system and dyne cm in CGS system.

By convention, the direction of moment of a force or couple is taken as positive if the body is rotated in the anti-clockwise direction and negative if it is rotate in the clockwise direction. They are shown in Figures 1.4 (a and b)

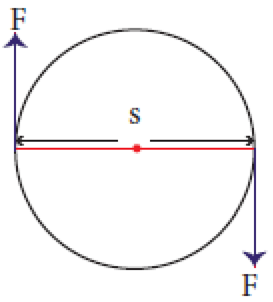


Figure 1.4 (a)
Clockwise moment

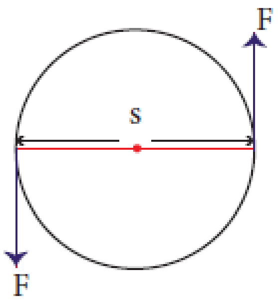


Figure 1.4 (b)
Anticlockwise moment

Application of Torque

1. Gears:

A gear is a circular wheel with teeth around its rim. It helps to change the speed of rotation of a wheel by changing the torque and helps to transmit power.

2. Seasaw

Most of you have played on the seasaw. Since there is a difference in the weight of the persons sitting on it, the heavier person lifts the lighter person. When the heavier person comes closer to the pivot point (fulcrum) the distance of the line of action of the force decreases. It causes less amount of torque to act on it. This enables the lighter person to lift the heavier person.

3. Steering Wheel

A small steering wheel enables you to manoeuore a car easily by transferring a torque to the wheels with less effort.

Principle of Moments

When a number of like or unlike parallel forces act on a rigid body and the body is in equilibrium, then the algebraic sum of the moments in the clockwise direction is equal to the algebraic sum of the moments in
the anticlockwise direction. In other words, at equilibrium, the algebraic sum of the moments of all the individual forces about any point is equal to zero.

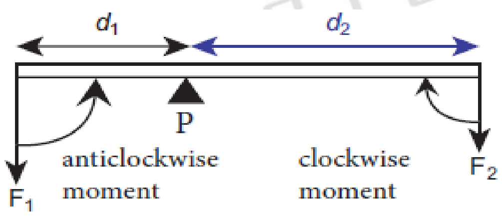


Figure 1.5 Principle of moments

In the illustration given in figures 1.5, the force F1 produces an anticlockwise rotation at a distance d1 from the point of pivot P (called fulcrum) and the force F2 produces a clockwise rotation at a distance d2 from the point of pivot P. The principle of moments can be written as follows:

Moment in clockwise direction	=	Moment in anticlockwise

direction

$F1 \times d1 = F2 \times d2$ . . . .

NEWTON'S SECOND LAW OF MOTION

According to this law, "the force acting on a body is directly proportional to the rate of change of linear momentum of the body and the change in momentum takes place in the direction of the force".

This law helps us to measure the amount of force. So, it is also called as '*law of force*'. Let, 'm' be the mass of a moving body, moving along a straight line with an initial speed 'u' After a time interval of 't', the velocity of the body changes to 'v' due to the impact of an unbalanced external force F.

Initial momentum of the body $P_i = mu$

Final momentum of the body $P_f = mv$

Change in momentum $\Delta p = P_f - P_i$
$= mv - mu$

By Newton's second law of motion,

Force, $F \propto$ rate of change of momentum

$F \propto$ change in momentum / time

$F \propto mv-mu/t$
$F=k\, m(v-u)/t$

Here, k is the proportionality constant. k = 1 in all systems of units. Hence,

$F = m(v-u)/t$

Since, acceleration = change in velocity/ time, a=(v-u)/t. Hence, we have

F = m × a

Force = mass × acceleration

No external force is required to maintain the motion of a body moving with uniform velocity. When the net force acting on a body is not equal to zero, then definitely the velocity of the body will change. Thus, change in momentum takes place in the direction of the force. The change may take place either in magnitude or in direction or in both.

Force is required to produce the acceleration of a body. In a uniform circular motion, even though the speed (magnitude of velocity) remains constant, the direction of the velocity changes at every point on the circular path. So, the acceleration is produced along the radius called as *centripetal acceleration*. The force, which produces this acceleration is called as centripetal force, about which you have learnt in class IX.

Units of force:

SI unit of force is newton (N) and in C.G.S system its unit is dyne.

Definition of 1 newton (N):

The amount of force required for a body of mass 1 kg produces an acceleration of 1 m s^{-2}, 1 N = 1 kg m s^{-2}

Definition of 1 dyne:

The amount of force required for a body of mass 1 gram produces an acceleration of 1 cm s^{-2}, 1 dyne = 1 g cm s^{-2}; also 1 N = 10^5 dyne.

Unit force:

The amount of force required to produce an acceleration of 1 m s^{-2} in a body of mass 1 kg is called '*unit force*'.

Gravitational unit of force:

In the SI system of units, gravitational unit of force is kilogram force, represented by kg f. In the CGS system its unit is gram force, represented by g f.

1 kg f = 1 kg × 9.8 m s^{-2} = 9.8 N;

1 g f = 1 g × 980 cm s^{-2} = 980 dyne

Impulse

A large force acting for a very short interval of time is called as 'Impulsive force'. When a force F acts on a body for a period of time t, then the product of force and time is known as 'impulse' represented by 'J'

Impulse, $J = F \times t$ (1.7)

By Newton's second law

$F = \Delta p \,/\, t$ (Δ refers to change)

$\Delta p = F \times t$ (1.8)

From 1.7 and 1.8

$J = \Delta p$

Impulse is also equal to the magnitude of change in momentum. Its unit is kg m s–1 or N s.

Change in momentum can be achieved in two ways. They are:

i. a large force acting for a short period of time and

ii. a smaller force acting for a longer period of time.

Examples:

Automobiles are fitted with springs and shock absorbers to reduce jerks while moving on uneven roads.

In cricket, a fielder pulls back his hands while catching the ball. He experiences a smaller force for a longer interval of time to catch the ball, resulting in a lesser impulse on his hands.

NEWTON'S THIRD LAW OF MOTION

Newton's third law states that 'for every action, there is an equal and opposite reaction. They always act on two different bodies'.

If a body A applies a force F_A on a body B, then the body B reacts with force F_B on the body A, which is equal to F_A in magnitude, but opposite in direction. $F_B = -F_A$

Examples:

When birds fly they push the air downwards with their wings (Action) and the air pushes the bird upwards (Reaction).
When a person swims he pushes the water using the hands backwards (Action), and the water pushes the swimmer in the forward direction (Reaction).

When you fire a bullet, the gun recoils backward and the bullet is moving forward (Action) and the gun equalises this forward action by moving backward (Reaction).

PRINCIPLE OF CONSERVATION OF LINEAR MOMENTUM

There is no change in the linear momentum of a system of bodies as long as no net external force acts on them.

Let us prove the law of conservation of linear momentum with the following illustration:

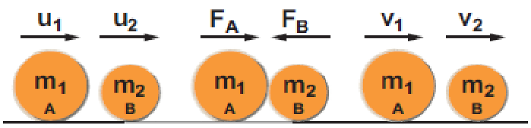


Figure 1.7 Conservation of linear momentum

Proof:

Let two bodies A and B having masses m1 and m2 move with initial velocity u1 and u2 in a straight line. Let the velocity of the first body be higher than that of the second body. i.e., u1>u2. During an interval of time t second, they tend to have a collision. After the impact, both of them move along the same straight line with a velocity v1 and v2 respectively.

Force on body B due to A,

$F_B = m_2 (v_2 – u_2)/t$

Force on body A due to B,

$F_A = m_1 (v_1 – u_1)/t$

By Newton's III law of motion,

Action force = Reaction force

$F_A \quad = –F_B$

$m_1 (v_1 - u_1)/t = –m_2 (v_2 - u_2)/t$

$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$ (1.9)

The above equation confirms in the absence of an external force, the algebraic sum of the momentum after collision is numerically equal to the algebraic sum of the momentum before collision.

Hence the law of conservation linear momentum is proved.

ROCKET PROPULSION

Propulsion of rockets is based on the law of conservation of linear momentum as well as Newton's III law of motion. Rockets are filled with a fuel (either liquid or solid) in the propellant tank. When the rocket is fired, this fuel is burnt and a hot gas is ejected with a high speed from the nozzle of the rocket, producing a huge momentum. To balance this momentum, an equal and opposite reaction force is produced in the combustion chamber, which makes the rocket project forward.

While in motion, the mass of the rocket gradually decreases, until the fuel is completely burnt out. Since, there is no net external force acting on it, the linear momentum of the system is conserved. The mass of the rocket decreases with altitude, which results in the gradual increase in velocity of the rocket. At one stage, it reaches a velocity, which is sufficient to just escape from the gravitational pull of the Earth. This velocity is called *escape velocity*. (This topic will be discussed in detail in higher classes).

GRAVITATION

Newton's universal law of gravitation

This law states that every particle of matter in this universe attracts every other particle with a force. This force is directly proportional to the product of their masses and inversely proportional to the square of the distance between the centers of these masses. The direction of the force acts along the line joining the masses.

Force between the masses is always attractive and it does not depend on the medium where they are placed.

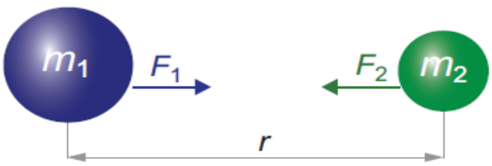


Figure 1.8 Gravitational force between two masses

Let, m_1 and m_2 be the masses of two bodies A and B placed *r* metre apart in space

Force $F \propto m_1 \times m_2$

$F \propto 1/ r_2$

On combining the above two expressions

$F \propto m_1 \times m_2/r^2$

$F= G\, m_1\, m_2/r^2$…………(1.10)

Where G is the universal gravitational constant. Its value in SI unit is 6.674×10^{-11} N $m^2\, kg^{-2}$.

Acceleration due to gravity (g)

When you throw any object upwards, its velocity ceases at a particular height and then it falls down due to the gravitational force of the Earth.

The velocity of the object keeps changing as it falls down. This change in velocity must be due to the force acting on the object. The acceleration of the body is due to the Earth's gravitational force. So, it is called as 'acceleration due to the gravitational force of the Earth' or 'acceleration due to gravity of the Earth'. It is represented as 'g'. Its unit is m s^{-2}

Mean value of the acceleration due to gravity is taken as 9.8 m s^{-2} on the surface of the Earth. This means that the velocity of a body during the downward free fall motion varies by 9.8 m s^{-1} for every 1 second. However, the value of 'g' is not the same at all points on the surface of the earth.

Relation between g and G

When a body is at rests on the surface of the Earth, it is acted upon by the gravitational force of the Earth. Let us compute the magnitude of this force in two ways. Let, M be the mass of the Earth and m be the mass of the body. The entire mass of the Earth is
assumed to be concentrated at its centre. The radius of the Earth is R = 6378 km (= 6400 km approximately). By Newton's law of gravitation, the force acting on the body is given by

$F = G M m/R^2$ ……….. (1.11)

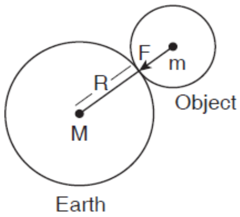


Figure 1.9 Relation between g and G

Here, the radius of the body considered is negligible when compared with the Earth's radius. Now, the same force can be obtained from Newton's second law of motion. According to this law, the force acting on the body is given by the product of its mass and acceleration (called as weight). Here, acceleration of the body is under the action of gravity hence a = g

$F = m a = m g$
$F = \text{weight} = mg$ --------------- (1.12)

Comparing equations (1.7) and (1.8), we get

$mg = GMm/R^2$ ------------(1.13)

Acceleration due to gravity

$g = GM/R^2$ --------------- (1.14)

Mass of the Earth (M)

Rearranging the equation (1.14), the mass of the Earth is obtained as follows:

Mass of the Earth $M = g R^2/G$

Substituting the known values of g, R and G,
you can calculate the mass of the Earth as

M = 5.972 × 1024 kg

Variation of acceleration due to gravity (g):

Since, g depends on the geometric radius of the Earth, ($g \propto 1/R2$), its value changes from one place to another on the surface of the Earth. Since, the geometric radius of the Earth is maximum in the equatorial region and minimum in the polar region, the value of g is maximum in the polar region and minimum at the equatorial region.

When you move to a higher altitude from the surface of the Earth, the value of g reduces. In the same way, when you move deep below the surface of the Earth, the value of g reduces. (This topic will be discussed in detail in the higher classes). Value of g is zero at the centre of the Earth.

MASS AND WEIGHT

Mass: Mass is the basic property of a body. Mass of a body is defined as the quantity of matter contained in the body. Its SI unit is kilogram (kg).

Weight: Weight of a body is defined as the gravitational force exerted on it due to the Earth's gravity alone.

Weight = Gravitational Force

= mass (m)× acceleration due to gravity(g).

g = acceleration due to gravity for Earth (at sea level) = 9.8 m s^{-2}.

Weight is a vector quantity. Direction of weight is always towards the centre of the Earth. SI unit of weight is newton (N). Weight of a body varies from one place to another place on the Earth since it depends on the acceleration due to gravity of the Earth (g) weight of a body is more at the poles than at the equatorial region.

The value of acceleration due to gravity on the surface of the moon is 1.625 ms^{-2}. This is about 0.1654 times the acceleration due to gravity of the Earth. If a person whose mass is 60 kg stands on the surface of Earth, his weight would be 588 N ($W = mg = 60 \times 9.8$). If the same person goes to the surface of the Moon, he would weigh only 97.5 N ($W = 60 \times 1.625$). But, his mass remains the same (60 kg) on both the Earth and the Moon.

APPARENT WEIGHT

The weight that you feel to possess during up and down motion, is not same as your actual weight. Apparent weight is the weight of the body acquired due to the action of gravity and other external forces acting on the body.

Let us see this from the following illustration:

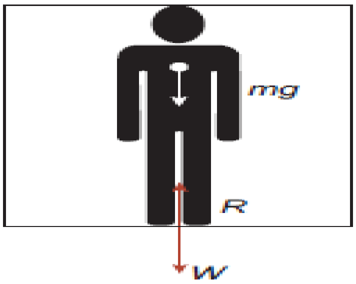


Figure 1.10

Let us consider a person of mass m, who is travelling in lift. The actual weight of the person is W = mg, which is acting vertically downwards. The reaction force exerted by the lift's surface 'R', taken as apparent weight is acting vertically upwards.

Let us see different possibilities of the apparent weight 'R' of the person that arise, depending on the motion of the lift; upwards or downwards which are given in Table 1.2

Weightlessness

Have you gone to an amusement park and taken a ride in a roller coaster? or in a giant wheel? During the fast downward and upward movement, how did you feel?

Table 1.2 Apparent weight of a person in a moving lift

Case 1: Lift is moving upward with an	Case 2: Lift is moving downward with an acceleration	Case 3: Lift is at rest.	Case 4: Lift is falling down freely

acceleration 'a'	'a'		
R – W = Fnet = ma R = W + ma R = mg + ma R = m(g+a)	W – R = Fnet = ma R = W - ma R = mg - ma R = m(g-a)	Here, the acceleration is zero a = 0 R = W R= mg	Here, the acceleration is equal to g a = g R = m(g – g)
R > W	R < W	R = W	R = 0
Apparent weight is greater than the actual weight.	Apparent weight is lesser than the actual weight.	Apparent weight is equal to the actual weight.	Apparent weight is equal to zero.

Its amazing!!. You actually feel as if you are falling freely without having any weight. This is due to the phenomenon of 'weightlessness'. You seem to have lost your weight when you move down with a certain acceleration. Sometimes, you experience the same feeling while travelling in a lift.

When the person in a lift moves down with an acceleration (a) equal to the acceleration due to gravity (g), i.e., when a = g, this motion is called as '*free fall*'. Here, the apparent weight (R = m (g – g) = 0) of the person is zero. This condition or state refers to the state of weightlessness. (Refer case 4 from Table 1.2).

The same effect takes place while falling freely in a roller coaster or on a swing or in a vertical giant wheel. You feel an apparent weight loss and weight gain when you are moving up and down in such rides.

Weightlessness of the astronauts

Some of us believe that the astronauts in the orbiting spacestation do not experience any gravitational force of the Earth. So they float. But this is absolutely wrong.

Astronauts are not floating but falling freely around the earth due to their huge oribital velocity. Since spacestation and astronauts have equal acceleration, they are under free fall condition. (R = 0 refer case 4 in Table 1.2). Hence, both the astronauts and the spacestation are in the state of weightlessness.

Application of Newton's law of gravitation

1) Dimensions of the heavenly bodies can be measured using the gravitation law. Mass of the Earth, radius of the Earth, acceleration due to gravity, etc. can be calculated with a higher accuracy.

2) Helps in discovering new stars and planets.

3) One of the irregularities in the motion of stars is called 'Wobble' lead to the disturbance in the motion of a planet nearby. In this condition the mass of the star can be calculated using the law of gravitation.

4) Helps to explain germination of roots is due to the property of geotropism which is the property of a root responding to the gravity.

5) Helps to predict the path of the astronomical bodies.

Points to Remember

Mechanics is divided into statics and dynamics.

Ability of a body to maintain its state of rest or motion is called Inertia.

Moment of the couple is measured by the product of any one of the forces and the perpendicular distance between two forces.

SI unit of force is newton (N). C.G.S unit is dyne.

When a force F acts on a body for a period of time t, then the product of force and time is known as 'impulse'.

The unit of weight is newton or kg f

The weight of a body is more at the poles than at the equatorial region.

Mass of a body is defined as the quantity of matter contained in the object. Its SI unit is kilogram (kg).

Apparent weight is the weight of the body acquired due to the action of gravity and other external forces on the body.

Whenever a body or a person falls freely under the action of Earth's gravitational force alone, it appears to have zero weight. This state is referred to as 'weightlessness'.

SOLVED PROBLEMS

Problem-1: Calculate the velocity of a moving body of mass 5 kg whose linear momentum is 2.5 $kg\ m\ s^{-1}$.

Solution: Linear momentum = mass × velocity
Velocity = linear momentum / mass. V = 2.5 / 5 = 0.5 $m\ s^{-1}$

Problem 2: A door is pushed, at a point whose distance from the hinges is 90 cm, with a force of 40 N. Calculate the moment of the force about the hinges.

Solution:

Formula: The moment of a force M = F × d

Given: F = 40 N and d = 90 cm = 0.9 m.

Hence, moment of the force = 40 × 0.9 = 36 N m.

Problem 3 : At what height from the centre of the Earth the acceleration due to gravity will be 1/4th of its value as at the Earth.

Solution:

Data: Height from the centre of the Earth, $R' = R + h$

The acceleration due to gravity at that height, $g' = g/4$

Formula: $g = GM / R^2$

$g/g' = (R'/R)^2 = (R+h/R)^2 = (1+h/R)^2$

$4 = (1+h/R)^2$,

$2 = 1+h/R$ or $h=R$. $R' = 2R$

From the centre of the Earth, the object is placed at twice the radius of the earth.

UNIT - 2 KINEMATICS

INTRODUCTION

Physics is basically an experimental science and rests on two pillars– Experiments and mathematics. Two thousand three hundred years ago the Greek librarian Eratosthenes measured the radius of the Earth. The size of the atom was measured much later, only in the beginning of the 20th century. The central aspect in physics is motion. Motion is found at all levels–from microscopic level (within the atom) to macroscopic and galactic level (planetary system and beyond). In short the entire Universe is governed by various types of motion. The study of various types of motion is expressed using the language of mathematics.

How do objects move? How fast or slow do they move? For example, when ten athletes run in a race, all of them do not run in the same manner. Their performance cannot be qualitatively recorded by usage of words like 'fastest', 'faster', 'average', 'slower' or 'slowest'. It has to be quantified. Quantifying means assigning numbers to each athlete's motion. Comparing these numbers one can analyse how fast or slow each athlete runs when compared to others. In this unit, the basic mathematics needed for analyzing motion in terms of its direction and magnitude is covered.

Kinematics is the branch of mechanics which deals with the motion of objects without taking force into account. The Greek word "kinema" means "motion".

CONCEPT OF REST AND MOTION

The concept of rest and motion can be well understood by the following elucidation. A person sitting in a moving bus is at rest with respect to a fellow passenger but is in motion with respect to a person outside the bus. The concepts of rest and motion have meaning only with respect to some reference frame. To understand rest or motion we need a convenient fixed reference frame.

Frame of Reference:

If we imagine a coordinate system and the position of an object is described relative to it, then such a coordinate system is called frame of reference. At any given instant of time, the frame of reference with respect to which the position of the object is described in terms of position coordinates (x, y, z) (i.e., distances of the given position of an object along the x, y, and z–axes.) is called "Cartesian coordinate system".

It is to be noted that if the x, y and z axes are drawn in anticlockwise direction then the coordinate system is called as "right– handed Cartesian coordinate system". Though other coordinate systems do exist, in physics we conventionally follow the right–handed coordinate system.

Illustrates the difference between left and right handed coordinate systems.

Point mass

To explain the motion of an object which has finite mass, the concept of "point mass" is required and is very useful. Let the mass of any object be assumed to be concentrated at a point. Then this idealized mass is called "point mass". It has no internal structure like shape and size. Mathematically a point mass has finite mass with zero dimension. Even though in reality a point mass does not exist, it often simplifies our calculations. It is to be noted that the term "point mass" is a relative term. It has meaning only with respect to a reference frame and with respect to the kind of motion that we analyse.

Examples

- To analyse the motion of Earth with respect to Sun, Earth can be treated as a point mass. This is because the distance between the Sun and Earth is very large compared to the size of the Earth.

- If we throw an irregular object like a small stone in the air, to analyse its motion it is simpler to consider the stone as a point mass as it moves in space. The size of the stone is very much smaller than the distance through which it travels.

Types of motion

In our day–to–day life the following kinds of motion are observed:

Linear motion

An object is said to be in linear motion if it moves in a straight line.

Examples

1. An athlete running on a straight track
2. A particle falling vertically downwards to the Earth.

Circular motion

Circular motion is defined as a motion described by an object traversing a circular path.

Examples

1. The whirling motion of a stone attached to a string
2. The motion of a satellite around the Earth

Rotational motion

If any object moves in a rotational motion about an axis, the motion is called 'rotation'. During rotation every point in the object transverses a circular path about an axis, (except the points located on the axis).

Examples

1. Rotation of a disc about an axis through its center
2. Spinning of the Earth about its own axis.

Vibratory motion

If an object or particle executes a to–and– fro motion about a fixed point, it is said to be in vibratory motion. This is sometimes also called oscillatory motion.

Examples

1. Vibration of a string on a guitar
2. Movement of a swing

Other types of motion like elliptical motion and helical motion are also possible.

Motion in One, Two and Three Dimensions

Let the position of a particle in space be expressed in terms of rectangular coordinates x, y and z. When these coordinates change with time, then the particle is said to be in motion. However, it is not necessary that all the three coordinates should together change with time. Even if one or two coordinates changes with time, the particle is said to be in motion. Then we have the following classification.

Motion in one dimension

One dimensional motion is the motion of a particle moving along a straight line.

This motion is sometimes known as rectilinear or linear motion.

In this motion, only one of the three rectangular coordinates specifying the position of the object changes with time.

For example, if a car moves from position A to position B along x–direction, as shown in Figure 2.8, then the variation in x–coordinate alone is noticed.

Examples

1. Motion of a train along a straight railway track.
2. An object falling freely under gravity close to Earth

Motion in two dimensions

If a particle is moving along a curved path in a plane, then it is said to be in two dimensional motion.

In this motion, two of the three rectangular coordinates specifying the position of object change with time.

Examples

1. Motion of a coin on a carrom board.
2. An insect crawling over the floor of a room.

Motion in three dimensions

A particle moving in usual three dimensional space has three dimensional motion.

In this motion, all the three coordinates specifying the position of an object change with respect to time. When a particle moves in three dimensions, all the three coordinates x, y and z will vary.

Examples

1. A bird flying in the sky.
2. Random motion of a gas molecule.
3. Flying of a kite on a windy day.

ELEMENTARY CONCEPTS OF VECTOR ALGEBRA

In physics, some quantities possess only magnitude and some quantities possess both magnitude and direction. To understand these physical quantities, it is very important to know the properties of vectors and scalars.

Scalar

It is a property which can be described only by magnitude. In physics a number of quantities can be described by scalars.

Examples

Distance, mass, temperature, speed and energy

Vector

It is a quantity which is described by both magnitude and direction. Geometrically a vector is a directed line segment which is shown in Figure 2.10. In physics certain quantities can be described only by vectors.

Examples

Force, velocity, displacement, position vector, acceleration, linear momentum and angular momentum.

Magnitude of a Vector

The length of a vector is called magnitude of the vector. It is always a positive quantity. Sometimes the magnitude of a vector is also called 'norm' of the vector. For a vector $\vec{A}$ the magnitude or norm is denoted by $|\vec{A}|$ or simply 'A'.

Different types of Vectors
Equal vectors:

Two vectors $\vec{A}$ and $\vec{B}$ said to be equal when they have equal magnitude and same direction and represent the same physical quantity

Collinear vectors: Collinear vectors are those which act along the same line. The angle between them can be 0° or 180°.

Parallel Vectors:

If two vectors A and B act in the same direction along the same line or on parallel lines, then the angle between them is 0^0

Anti–parallel vectors:

Two vectors $\vec{A}$and $\vec{B}$ are said to be anti–parallel when they are in opposite directions along the same line or on parallel lines. Then the angle between them is 180^0

Unit vector:

A vector divided by its magnitude is a unit vector. The unit vector for $\vec{A}$ is denoted by $\hat{A}$ (read as A cap or A hat). It has a magnitude equal to unity or one.

$$\text{Since, } \hat{A} = \frac{\vec{A}}{A}, \text{ we can write } \vec{A} = A\hat{A}$$

Thus, we can say that the unit vector specifies only the direction of the vector quantity.

Orthogonal unit vectors: Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be three unit vectors which specify the directions along positive x–axis, positive y–axis and positive z–axis respectively. These three unit vectors are directed perpendicular to each other, the angle between any two of them is 90°. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are examples of orthogonal vectors. Two vectors which are perpendicular to each other are called orthogonal vectors.

Addition of Vectors

Since vectors have both magnitude and direction they cannot be added by the method of ordinary algebra. Thus, vectors can be added geometrically or analytically using certain rules called 'vector algebra'. In order to find the sum (resultant) of two vectors, which are inclined to each other, we use

1. Triangular law of addition method
2. Parallelogram law of vectors.

Triangular Law of addition method

Let us consider two vectors $\vec{A}$ and $\vec{B}$

To find the resultant of the two vectors we apply the triangular law of addition as follows:

Represent the vectors $\vec{A}$ and $\vec{B}$ by the two adjacent sides of a triangle taken in the same order. Then the resultant is given by the third side of the triangle.

To explain further, the head of the first vector $\vec{A}$ is connected to the tail of the second vector $\vec{B}$. Let θ be the angle between $\vec{A}$ and $\vec{B}$. Then $\vec{R}$ is the resultant vector connecting the tail of the first vector $\vec{A}$ to the head of the second vector $\vec{B}$. The magnitude of $\vec{R}$ (resultant) is given geometrically by the length of $\vec{R}$ (OQ)

and the direction of the resultant vector is the angle between $\vec{R}$ and $\vec{A}$. Thus we write
$\vec{R} = \vec{A} + \vec{B}$.

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$

Magnitude of resultant vector

The magnitude and angle of the resultant vector are determined as follows.

From consider the triangle ABN, which is obtained by extending the side OA to ON. ABN is a right angled triangle.

$$\cos\theta = \frac{AN}{B} \therefore AN = B\cos\theta \; and$$

$$\sin\theta = \frac{BN}{B} \quad \therefore BN = B\sin\theta$$

$$\text{For } \Delta OBN, \text{ we have } OB^2 = ON^2 + BN^2$$

$$\Rightarrow R^2 = (A + B\cos\theta)^2 + (B \ \sin\theta)^2$$

$$\Rightarrow R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\Rightarrow R^2 = A^2 + B^2\left(cos^2\theta + sin^2\theta\right) + 2AB\cos\theta$$

$$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

which is the magnitude of the resultant of $\vec{A}$ and $\vec{B}$

Direction of resultant vectors:

If θ is the angle between $\vec{A}$ and $\vec{B}$, then

$$\left|\vec{A}+\vec{B}\right| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

If $\vec{R}$ makes an angle α with $\vec{A}$, then in ΔOBN,

$$\tan\alpha = \frac{BN}{ON} = \frac{BN}{OA + AN}$$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$$

Two vectors

$\vec{A}$ and $\vec{B}$ of magnitude 5 units and 7 units respectively make an angle 60° with each other as shown below. Find the magnitude of the resultant vector and its direction with respect to the vector $\vec{A}$.

By following the law of triangular addition, the resultant vector is given by

$$\vec{R} = \vec{A} + \vec{B}$$

The magnitude of the resultant vector $\vec{R}$ is given by

$$R = \left|\vec{R}\right| = \sqrt{5^2 + 7^2 + 2 \times 5 \times 7 \cos 60^\circ}$$

$$R = \sqrt{25 + 49 + \frac{70 \times 1}{2}} = \sqrt{109}\,units$$

The angle α between $\vec{R}$ and $\vec{A}$ is given by

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

$$\tan\alpha = \frac{7\times\sin 60}{5+7\cos 60} = \frac{7\sqrt{3}}{10+7} = \frac{7\sqrt{3}}{17} \cong 0.713$$
$$\therefore \alpha \cong 35°$$

Subtraction of vectors

Since vectors have both magnitude and direction two vectors cannot be subtracted from each other by the method of ordinary algebra. Thus, this subtraction can be done either geometrically or analytically. We shall now discuss subtraction of two vectors geometrically.

For two non-zero vectors $\vec{A}$ and $\vec{B}$ which are inclined to each other at an angle θ, the difference $\vec{A}-\vec{B}$ is obtained as follows. First obtain $-\vec{B}$. The angle between $\vec{A}$ and $-\vec{B}$ is 180–θ.

The difference $\vec{A}-\vec{B}$ is the same as the resultant of $\vec{A}$ and $-\vec{B}$

We can $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$ write and using the equation.

$$\left|\vec{A}-\vec{B}\right| = \sqrt{A^2+B^2+2AB\cos(180-\theta)}$$

Since, $\cos(180-\theta) = -\cos\theta$ we, get

$$\Rightarrow \left|\vec{A}-\vec{B}\right| = \sqrt{A^2+B^2-2AB\cos\theta}$$

$$\tan\alpha_2 = \frac{B\sin(180°-\theta)}{A+B\cos(180°-\theta)}$$

But sin (180° − θ) = sin θ hence we get

$$\Rightarrow \tan\alpha_2 = \frac{B\sin\theta}{A-B\cos\theta}$$

Thus the difference $\vec{A}-\vec{B}$ is a vector with magnitude and direction given by equations.

Two vectors

$\vec{A}$ and $\vec{B}$ of magnitude 5 units and 7 units make an angle 60° with each other. Find the magnitude of the difference vector $\vec{A}-\vec{B}$ and its direction with respect to the vector $\vec{A}$.

$$\left|\vec{A}-\vec{B}\right| = \sqrt{5^2+7^2-2\times5\times7\cos60^\circ}$$
$$= \sqrt{25+49-35} = \sqrt{39}\ units$$

The angle that $\vec{A}-\vec{B}$ makes with the vector $\vec{A}$ is given by

$$\tan\alpha_2 = \frac{7\sin60}{5-7\cos60} = \frac{7\sqrt{3}}{10-7} = \frac{7}{\sqrt{3}} = 4.041$$
$$\alpha_2 = \tan^{-1}(4.041) \cong 76^\circ$$

COMPONENTS OF A VECTOR

In the Cartesian coordinate system any vector $\vec{A}$ can be resolved into three components along x, y and z directions.

Consider a 3–dimensional coordinate system. With respect to this a vector can be written in component form as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

.

Here A_x is the x–component of $\vec{A}$, A_y is the y–component of $\vec{A}$ and A_z is the z component of $\vec{A}$.

In a 2–dimensional Cartesian coordinate system.

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

If $\vec{A}$ makes an angle θ with x axis, and A_x and A_y are the components of $\vec{A}$ along x–axis and y–axis respectively.

$$A_x = A\cos\theta, \ A_y = A\sin\theta$$

where 'A' is the magnitude (length) of the vector $\vec{A}$, $A = \sqrt{A_x^2 + A_y^2}$

What are the unit vectors along the negative x–direction, negative y–direction, and negative z– direction?

The unit vectors along the negative directions can be shown as in the following figure.

1. The unit vector along the negative x direction $= -\hat{i}$
2. The unit vector along the negative y direction $= -\hat{j}$
3. The unit vector along the negative z direction $= -\hat{k}$

Vector addition using components

In the previous section we have learnt about addition and subtraction of two vectors using geometric methods. But once we choose a coordinate system, the addition and subtraction of vectors becomes much easier to perform.

The two vector $\vec{A}$ and $\vec{B}$ in a Cartesian coordinate system can be expressed as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Then the addition of two vectors is equivalent to adding their corresponding x, y and z components.

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

Similarly the subtraction of two vectors is equivalent to subtracting the corresponding x, y and z components.

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}$$

The above rules form an analytical way of adding and subtracting two vectors.

Two vectors $\vec{A}$ and $\vec{B}$ are given in the component form as $\vec{A} = 5\hat{i} + 7\hat{j} - 4\hat{k}$ and $\vec{B} = 6\hat{i} + 3\hat{j} + 2\hat{k}$. Find $\vec{A}+\vec{B}$, $\vec{B}+\vec{A}$, $\vec{A}-\vec{B}$, $\vec{B}-\vec{A}$

$$\vec{A}+\vec{B} = (5\hat{i} + 7\hat{j} - 4\hat{k}) + (6\hat{i} + 3\hat{j} + 2\hat{k})$$
$$= 11\hat{i} + 10\hat{j} - 2\hat{k}$$
$$\vec{B}+\vec{A} = (6\hat{i} + 3\hat{j} + 2\hat{k}) + (5\hat{i} + 7\hat{j} - 4\hat{k})$$
$$= (6+5)\hat{i} + (3+7)\hat{j} + (2-4)\hat{k}$$
$$= 11\hat{i} + 10\hat{j} - 2\hat{k}$$
$$\vec{A}-\vec{B} = (5\hat{i} + 7\hat{j} - 4\hat{k}) - (6\hat{i} + 3\hat{j} + 2\hat{k})$$
$$= -\hat{i} + 4\hat{j} - 6\hat{k}$$
$$\vec{B}-\vec{A} = \hat{i} - 4\hat{j} + 6\hat{k}$$

Note that the vectors $\vec{A}+\vec{B}$ and $\vec{B}+\vec{A}$ are same and the vectors $\vec{A}-\vec{B}$ and $\vec{B}-\vec{A}$ are opposite to each other.

A vector $\vec{A}$ multiplied by a scalar λ results in another vector, λ $\vec{A}$. If λ is a positive number then λ $\vec{A}$ is also in the direction of $\vec{A}$. If λ is a negative number, λ $\vec{A}$ is in the opposite direction to the vector $\vec{A}$.

Given the vector $\vec{A} = 2\hat{i} + 3\hat{j}$, what is 3 $\vec{A}$?

$$3\vec{A} = 3\left(2\hat{i} + 3\hat{j}\right) = 6\hat{i} + 9\hat{j}$$

The vector $3\vec{A}$ is in the same direction as vector $\vec{A}$.

A vector $\vec{A}$ is given as in the following Figure. Find $4\vec{A}$ and $-4\vec{A}$

Solution

In physics, certain vector quantities can be defined as a scalar times another vector quantity.

For example

Force. $\vec{F} = m\vec{a}$ Here mass 'm' is a scalar, and $\vec{a}$ is the acceleration. Since 'm' is always a positive scalar, the direction of force is always in the direction of acceleration.

Linear momentum $\vec{p} = m\vec{v}$.Here $\vec{v}$ is the velocity. The direction of linear momentum is also in the direction of velocity.

Force $\vec{F} = q\vec{E}$ Here the electric charge 'q' is a scalar, and $\vec{E}$ is the electric field. Since charge can be positive or negative, the direction of force $\vec{F}$ is correspondingly either in the direction of $\vec{E}$ or opposite to the direction of $\vec{E}$.

Scalar Product of Two Vectors
Definition

The scalar product (or dot product) of two vectors is defined as the product of the magnitudes of both the vectors and the cosine of the angle between them.

Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having an angle θ between them, then their scalar product is defined as $\vec{A}.\vec{B}$ = AB cosθ. Here, A and B are magnitudes of $\vec{A}$ and $\vec{B}$.

Properties

The product quantity $\vec{A}.\vec{B}$ is always a scalar. It is positive if the angle between the vectors is acute (i.e., < 90°) and negative if the angle between them is obtuse (i.e. 90°<θ< 180°).

The scalar product is commutative,

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

The vectors obey distributive law i.e.

$$\vec{A}.\left(\vec{B}+\vec{C}\right)=\vec{A}.\vec{B}+\vec{A}.\vec{C}$$

The angle between the vectors

$$\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{AB}\right]$$

The scalar product of two vectors will be maximum when cos θ = 1, i.e. θ = 0°, i.e., when the vectors are parallel;

$$(\vec{A}\cdot\vec{B})_{max} = AB$$

The scalar product of two vectors will be minimum, when cos θ = –1, i.e. θ = 180° $(\vec{A}.\vec{B})_{max}$=AB when the vectors are anti-parallel.

If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other then their scalar product $\vec{A}.\vec{B} = 0$, because cos 90°=0. Then the vectors $\vec{A}$ and $\vec{B}$ are said to be mutually orthogonal.

The scalar product of a vector with itself is termed as self-dot product and is given by $(\vec{A})^2=\vec{A}.\vec{A}$=AA Cos$\theta$=$A^2$. Here angle θ = 0°

The magnitude or norm of the vector

$$\vec{A} \text{ is } \left|\vec{A}\right| = A = \sqrt{\vec{A}\cdot\vec{A}}$$

In case of a unit vector $\hat{n}$

$\hat{n}.\hat{n}=1\times1\times\cos0=1$. For example $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$

In the case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=1\cdot1\cos 90^\circ=0$$

In terms of components the scalar product of $\vec{A}$ and $\vec{B}$ can be written as

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

$$= A_xB_x + A_yB_y + A_zB_z, \text{ with all other terms zero.}$$

The magnitude of vector $|\vec{A}|$ is given by

$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Given two vectors $\vec{A} = 2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{B} = \hat{i} + 3\hat{j} + 6\hat{k}$ Find the product $\vec{A}.\vec{B}$ and the
magnitudes of $\vec{A}$ and $\vec{B}$. What is the angle between them?

Solution

$$\vec{A}.\vec{B} = 2 + 12 + 30 = 44$$

$$\text{Magnitude } A = \sqrt{4+16+25} = \sqrt{45} \text{ units}$$
$$\text{Magnitude } B = \sqrt{1+9+36} = \sqrt{46} \text{ units}$$

The angle between the two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$$

$$= \cos^{-1}\left(\frac{44}{\sqrt{45\times46}}\right) = \cos^{-1}\left(\frac{44}{45.49}\right)$$

$$= \cos^{-1}(0.967)$$

$$\therefore \theta \cong 15^\circ$$

Check whether the following vectors are orthogonal.

1. $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 4\hat{i} - 5\hat{j}$
2. $\vec{C} = 5\hat{i} + 2\hat{j}$ and $\vec{D} = 2\hat{i} - 5\hat{j}$

Solution

$$\vec{A}.\vec{B} = 8 - 15 = -7 \neq 0$$

Hence $\vec{A}$ and $\vec{B}$ are not orthogonal to each other

$$\vec{C}.\vec{D} = 10 - 10 = 0$$

Hence, $\vec{C}$ and $\vec{D}$ are orthogonal to each other.

It is also possible to geometrically show that the vectors $\vec{C}$ and $\vec{D}$ are orthogonal to each other. This is shown in the following Figure.

In physics, the work done by a force $\vec{F}$ to move an object through a small displacement $d\vec{r}$ is defined as,

$$W = \vec{F}.d\vec{r}$$
$$W = F\ dr \cos\theta$$

The work done is basically a scalar product between the force vector and the displacement vector. Apart from work done, there are other physical quantities which are also defined through scalar products.

The Vector Product of Two Vectors
Definition

The vector product or cross product of two vectors is defined as another vector having a magnitude equal to the product of the magnitudes of two vectors and the sine of the angle between them. The direction of the product vector is perpendicular to the plane containing the two vectors, in accordance with the right hand screw rule or right hand thumb rule

Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product is written as $\vec{A} \times \vec{B}$ which is a vector $\vec{C}$ defined by

$$\vec{C} = \vec{A} \times \vec{B} = (AB \sin \theta)\hat{n}$$

The direction $\hat{n}$ of $\vec{A} \times \vec{B}$ i.e., $\vec{C}$ is perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ and is in the sense of advancement of a right handed screw rotated $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus, if a right-handed screw whose axis is perpendicular to the plane formed by ⍰ A and B, is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A} \times \vec{B}$ i.e. $\vec{C}$

Properties of vector (cross) product.

The vector product of any two vectors is always another vector whose direction is perpendicular to the plane containing these two vectors, i.e., orthogonal to both the vectors $\vec{A}$ and $\vec{B}$ even though the vectors $\vec{A}$ and $\vec{B}$ may or may not be mutually orthogonal.

The vector product of two vectors is not commutative, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ But, $\vec{A} \times \vec{B} = -\left[\vec{B} \times \vec{A}\right]$.Here it is worthwhile to note that $\left|\vec{A} \times \vec{B}\right| = \left|\vec{B} \times \vec{A}\right| = AB \sin\theta$ in the case of the product vectors $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ the magnitudes are equal but directions are opposite to each other.

The vector product of two vectors will have maximum magnitude when $\sin\theta$ =1. θ =90°. when the vectors $\vec{A}$ and $\vec{B}$ are orthogonal to each other.

$$\left(\vec{A} \times \vec{B}\right)_{\max} = AB\hat{n}$$

The vector product of two non-zero vectors will be minimum when $|\sin\theta| = 0$, $\theta = 0^o$ or 180^o

$$\left(\vec{A} \times \vec{B}\right)_{\min} = 0$$

the vector product of two non–zero vectors vanishes, if the vectors are either parallel or antiparallel.

The self-cross product, i.e., product of a vector with itself is the null vector

$$\vec{A} \times \vec{A} = \text{AA} \sin \theta^{\circ} \hat{n} = \vec{0}$$

In physics the null vector $\vec{0}$ is simply denoted as zero.

The self–vector products of unit vectors are thus zero.

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

In the case of orthogonal unit vectors, $\hat{i}, \hat{j}, \hat{k}$ in accordance with the right hand screw rule:

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i} \text{ and } \hat{k} \times \hat{i} = \hat{j}$$

Also, since the cross product is not commutative,

$$\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}$$

In terms of components, the vector product of two vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\left(A_y B_z - A_z B_y\right) + \hat{j}\left(A_z B_x - A_x B_z\right) + \hat{k}\left(A_x B_y - A_y B_x\right)$$

Note that in the $\hat{j}^{th}$ component the order of multiplication is different than $\hat{i}^{th}$ and $\hat{k}^{th}$

If two vectors $\vec{A}$ and $\vec{B}$ form adjacent sides in a parallelogram, then the magnitude of $\vec{A} \times \vec{B}$ will give the area of the parallelogram as represented graphically

Triangle with $\vec{A}$ and $\vec{B}$ as sides is $\frac{1}{2}\left|\vec{A} \times \vec{B}\right|$.

A number of quantities used in Physics are defined through vector products. Particularly physical quantities representing rotational effects like torque, angular momentum, are defined through vector products.

Examples

1. Torque $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{F}$ is Force and $\vec{r}$ is position vector of a particle
2. Angular momentum $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{p}$ is the linear momentum
3. Linear Velocity $\vec{v} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is angular velocity

Two vectors are given as $\vec{r} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{F} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ Find the resultant vector $\vec{\tau} = \vec{r} \times \vec{F}$

Solution

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 3 & -2 & 4 \end{vmatrix}$$

$$\vec{\tau} = \left(12 - (-10)\right)\hat{i} + \left(15 - 8\right)\hat{j} + \left(-4 - 9\right)\right)\hat{k}$$

$$\vec{\tau} = 22\hat{i} + 7\hat{j} - 13\hat{k}$$

Properties of the components of vectors

If two vectors $\vec{A}$ and $\vec{B}$ are equal, then their individual components are also equal.

Let $\vec{A} = \vec{B}$

Then

$$A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$A_x = B_x, A_y = B_y, A_z = B_z$$

EXAMPLE

Compare the components for the following vector equations

- $\vec{F} = m\vec{a}$ Here m is positive number
- $\vec{P} = 0$

Solution

$\vec{F} = m\vec{a}$

$F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = m_x\hat{i} + m_y\hat{j} + m_z\hat{k}$

By comparing the components, we get

$$F_m = ma_x, F_y = ma_y, F_z = ma_z$$

This implies that one vector equation is equivalent to three scalar equations.

$$\vec{P} = 0$$

$$P_x\hat{i} + P_y\hat{j} + P_z\hat{k} = 0_x\hat{i} + 0_y\hat{j} + 0_z\hat{k}$$

By comparing the components, we get

$$P_x = 0, P_y = 0, P_z = 0$$

EXAMPLE

Determine the value of the T from the given vector equation.

$$5\hat{j} - T\hat{j} = 6\hat{j} = 3T\hat{j}$$

Solution

By comparing the components both sides, we can write

$$5 - 6 = 3T + T$$
$$-1 = 4T$$
$$T = -\frac{1}{4}$$

EXAMPLE

Compare the components of vector equation $\vec{F_1} + \vec{F}_2 + \vec{F}_3 = \vec{F}_4$

Solution

We can resolve all the vectors in x, y and z components with respect to Cartesian coordinate system.

Once we resolve the components we can separately equate the x components on both sides, y components on both sides, and z components on both the sides of the equation, we then get

$$\overrightarrow{F_{1x}} + \vec{F}_{2x} + \vec{F}_{3x} = \vec{F}_{4x}$$
$$\overrightarrow{F_{1y}} + \vec{F}_{2y} + \vec{F}_{3y} = \vec{F}_{4y}$$
$$\overrightarrow{F_{1z}} + \vec{F}_{2z} + \vec{F}_{3z} = \vec{F}_{4z}$$

POSITION VECTOR

It is a vector which denotes the position of a particle at any instant of time, with respect to some reference frame or coordinate system.

The position vector $\vec{r}$ of the particle at a point P is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where x, y and z are components of $\vec{r}$

Determine the position vectors for the following particles which are located at points P, Q, R, S.

Solution

The position vector for the point P is

$$\vec{r}_p = 3\hat{i}$$

The position vector for the point Q is

$$\vec{r}_Q = 5\hat{i} + 4\hat{j}$$

The position vector for the point R is

$$\vec{r}_R = -2\hat{i}$$

The position vector for the point S is

$$\vec{r}_s = 3\hat{i} - 6\hat{j}$$

Example:

A person initially at rest starts to walk 2 m towards north, then 1 m towards east, then 5 m towards south and then 3 m towards west. What is the position vector of the person at the end of the trip?

Solution

As shown in the Figure, the positive x axis is taken as east direction, positive y direction is taken as north.

After the trip, the person reaches the point P whose position vector given by

$$\vec{r} = 2\hat{i} - 3\hat{j}$$

DISTANCE AND DISPLACEMENT

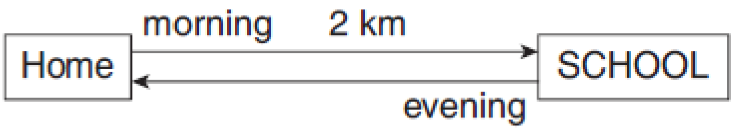
Distance is the actual path length travelled by an object in the given interval of time during the motion. It is a positive scalar quantity.

Displacement is the difference between the final and initial positions of the object in a given interval of time. It can also be defined as the shortest distance between these two positions of the object and its direction is from the initial to final position of the object, during the given interval of time. It is a vector quantity.

Example:

Assume your school is located 2 km away from your home. In the morning you are going to school and in the evening you come back home. In this entire trip what is the distance travelled and the displacement covered?

Solution



The displacement covered is zero. It is because your initial and final positions are the same. But the distance travelled is 4 km.

EXAMPLE

An athlete covers 3 rounds on a circular track of radius 50 m. Calculate the total distance and displacement travelled by him.

Solution

The total distance the athlete covered =3x circumference of track

Distance = $3 \times 2\pi \times 50$ m

= 300π m (or)

Distance ≈ 300 x 3.14 ≈ 942 m

The displacement is zero, since the athlete reaches the same point A after three rounds from where he started.

Displacement Vector in Cartesian Coordinate System

In terms of position vector, the displacement vector is given as follows. Let us consider a particle moving from a point P1 having position vector $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

to a point P_2 where its position vector is $\vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

The displacement vector is given by

$$\Delta\vec{r} = \vec{r_2} - \vec{r_1}$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

EXAMPLE

Calculate the displacement vector for a particle moving from a point P to Q as shown below. Calculate the magnitude of displacement.

Solution

The displacement vector $\Delta\vec{r} = \vec{r_2} - \vec{r_1}$ with

$$\vec{r}_1 = \hat{i} + \hat{j} \text{ and } \vec{r}_2 = 4\hat{i} + 2\hat{j}$$

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (4\hat{i} + 2\hat{j}) - (\hat{i} + \hat{j})$$

$$= (4-1)\hat{i} + (2-1)\hat{j}$$

$$\Delta\vec{r} = 3\hat{i} + \hat{j}$$

The magnitude of the displacement vector

$$\Delta r = \sqrt{3^2 + 1^2} = \sqrt{10}\,unit$$

DIFFERENTIAL CALCULUS

The Concept of a function

Any physical quantity is represented by a "function" in mathematics. Take the example of temperature T. We know that the temperature of the surroundings is changing throughout the day. It increases till noon and decreases in the evening. At any time "t" the temperature T has a unique value. Mathematically this variation can be represented by the notation 'T (t)' and it should be called "temperature as a function of time". It implies that if the value of 't' is given, then the function "T (t)" will give the value of the temperature at that time't'. Similarly, the position of a bus in motion along the x direction can be represented by x(t) and this is called 'x' as a function of time'. Here 'x' denotes the x coordinate.

Example

Consider a function f(x) = x^2. Sometimes it is also represented as y = x^2. Here y is called the dependent variable and x is called independent variable. It means as x changes, y also changes. Once a physical quantity is represented by a function, one can study the variation of the function over time or over the independent variable on which the quantity depends. Calculus is the branch of mathematics used to analyse the change of any quantity.

If a function is represented by y = f(x), then dy/dx represents the derivative of y with respect to x. Mathematically this represents the variation of y with respect to change in x, for various continuous values of x.

Mathematically the derivative dy/dx is defined as follows

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$\frac{dy}{dx}$ represents the limit that the quantity $\frac{\Delta y}{\Delta x}$ attains, as Δx tends to zero.

EXAMPLE

Consider the function y = x^2 . Calculate the derivative $\frac{dy}{dx}$ using the concept of limit.

Solution

Let us take two points given by

$x_1 = 2$ and $x_2 = 3$, then $y_1 = 4$ and $y_2 = 9$

Here $\Delta x = 1$ and $\Delta y = 5$

Then

$$\frac{\Delta y}{\Delta x}=\frac{9-4}{3-2}=5$$

If we take $x_1 = 2$ and $x_2 = 2.5$, then $y_1 = 4$ and $y_2 = (2.5)2 = 6.25$

Here $\Delta x = 0.5 = \frac{1}{2}$ and $\Delta y = 2.25$

Then

$$\frac{\Delta y}{\Delta x}=\frac{6.25-4}{0.5}=4.5$$

If we take $x_1 = 2$ and $x_2 = 2.25$, then $y_1 = 4$ and $y_2 = 5.0625$

Here $\Delta x = 0.25 = \frac{1}{4}$, $\Delta y = 1.0625$

$$\frac{\Delta y}{\Delta x}=\frac{5.0625-4}{0.25}=\frac{(5.0625-4)}{\frac{1}{4}}$$

$$=4(5.0625-4)=4.25$$

If we take $x_1 = 2$ and $x_2 = 2.1$, then $y_1 = 4$ and $y_2 = 4.41$

Here $\Delta x = 0.1 = \frac{1}{10}$ and

$$\frac{\Delta y}{\Delta x}=\frac{(4.41-4)}{\frac{1}{10}}=10(4.41-4)=4.1$$

x_1	x_2	Δx	y_1	y_2	$\frac{\Delta y}{\Delta x}$
2	2.25	0.25	4	5.0625	4.25
2	2.1	0.1	4	4.41	4.1
2	2.01	0.01	4	4.0401	4.01
2	2.001	0.001	4	4.004001	4.001
2	2.0001	0.0001	4	4.00040001	4.0001

From the above table, the following inferences can be made.

As Δx tends to zero $\frac{\Delta y}{\Delta x}$, x approaches the limit given by the number 4.

At a point x = 2, the derivative $\frac{dy}{dx}$=4.

It should also be mentioned here that $\Delta x \to 0$ does not mean that Δx = 0.

This is because, if we substitute Δx = 0, $\frac{\Delta y}{\Delta x}$ becomes indeterminate.

In general, we can obtain the derivative of the function $y = x^2$, as follows:

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

The table below shows the derivatives of some common functions used in physics

Function	Derivative
$y = x$	$dy/dx=1$
$y = x^2$	$dy/dx=2x$
$y = x^3$	$dy/dx =3x^2$
$y = x^n$	$dy/dx=nx^{n-1}$
$y = \sin x$	$dy/dx= \cos x$
$y = \cos x$	$dy/dx= -\sin x$
$y = \text{constant}$	$dy/dx= 0$
$y = AB$	$\frac{dy}{dx} = A\left(\frac{dB}{dx}\right)+\left(\frac{dA}{dx}\right)B$

In physics, velocity, speed and acceleration are all derivatives with respect to time't'. This will be dealt with in the next section.

Example:

Find the derivative with respect to t, of the function $x= A_0 + A_1t + A_2 t^2$ where A_0, A_1 and A_2 are constants.

Solution

Note that here the independent variable is 't' and the dependent variable is 'x'

The required derivative is $dx/dt = 0+ A_1+2A_2t$

The second derivative is $d^2x/d^2t = 2A_2$

INTEGRAL CALCULUS

Integration is basically an area finding process. For certain geometric shapes we can directly find the area. But for irregular shapes the process of integration is used. Consider for example the areas of a rectangle and an irregularly shaped curve.

The area of the rectangle is simply given by A = length $\times$ breadth = $(b-a)\,c$

But to find the area of the irregular shaped curve given by f(x), we divide the area into rectangular strips.

The area under the curve is approximately equal to sum of areas of each rectangular strip.

This is given by $A \approx f(a)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x.$

Where f (a) is the value of the function f (x) at x = a, f (x_1) is the value of f (x) for x = x_1 and so on.

As we increase the number of strips, the area evaluated becomes more accurate. If the area under the curve is divided into N strips, the area under the curve is given by

$$A = \sum_{n=1}^{N} f(x_n)\Delta x_n$$

As the number of strips goes to infinity, N $\rightarrow\infty$, the sum becomes an integral,

$$A = \int_a^b f(x)dx$$

(Note: As N $\rightarrow\infty$, Δx$\rightarrow$0)

The integration will give the total area under the curve f (x).

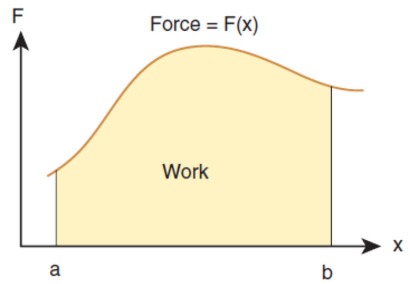
Examples

In physics the work done by a force F(x) on an object to move it from point a to point b in one dimension is given by

$$W = \int_a^b F(x)dx$$

(No scalar products is required here, since motion here is in one dimension)

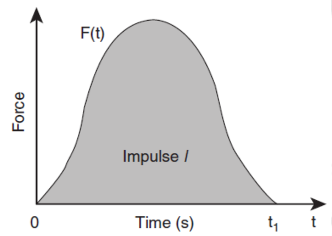
1. The work done is the area under the force displacement graph



2. The impulse given by the force in an interval of time is calculated between the interval from time t = 0 to time t = t_1 as

$$\text{Impulse I} = \int_0^{t_1} F dt$$

The impulse is the area under the force function F (t) - t graph



Consider a particle located initially at point P having position vector $\vec{r_1}$. In a time interval Δt the particle is moved to the point Q having position vector $\vec{r_2}$. The displacement vector is $\Delta\vec{r} = \vec{r_2} - \vec{r_1}$.

The average velocity is defined as ratio of the displacement vector to the corresponding time interval

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$$

It is a vector quantity. The direction of average velocity is in the direction of the displacement vector $(\Delta\vec{r})$.

Average speed

The average speed is defined as the ratio of total path length travelled by the particle in a time interval.

Average speed = total path length / total time

EXAMPLE

Consider an object travelling in a semicircular path from point O to point P in 5 second, as is shown in the Figure. Calculate the average velocity and average speed.

Solution

$$\text{Average velocity}\vec{v}_{avg} = \frac{\vec{r}_p - \vec{r}_o}{\Delta t}$$

$$\text{Here}\Delta t = 5s$$

$$\vec{r}_o = 0, \vec{r}_p = 10\hat{i}$$

$$\vec{v}_{avg} = \frac{10\hat{i}}{5\sec} = 2\hat{i}cms^{-1}$$

The average velocity is in the positive x direction.
The average speed = total path length / time taken (the path is semi-circular)

$$= \frac{5\pi cm}{5\ s} = \pi cm\, s^{-1} \approx 3.14\ cm\, s^{-1}$$

Note that the average speed is greater than the magnitude of the average velocity.

Instantaneous velocity or velocity

The instantaneous velocity at an instant t or simply 'velocity' at an instant t is defined as limiting value of the average velocity as Δt →0, evaluated at time t.

In other words, velocity is equal to rate of change of position vector with respect to time. Velocity is a vector quantity.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

In component form, this velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Here $\frac{dx}{dt} = v_x = x -$ component of velocity

$\frac{dy}{dt} = v_y = y -$ component of velocity

$\frac{dz}{dt} = v_z = z -$ component of velocity

The magnitude of velocity v is called speed and is given by

$$\mathrm{v} = \sqrt{\mathrm{v}_x^2 + \mathrm{v}_y^2 + \mathrm{v}_z^2}$$

Speed is always a positive scalar. The unit of speed is also meter per second.

EXAMPLE

The position vector of a particle is given $\vec{r} = 2t\hat{i} + 3t^2\,\hat{j} - 5\hat{k}$.

a. Calculate the velocity and speed of the particle at any instant t
b. Calculate the velocity and speed of the particle at time t = 2 s

Solution

The velocity $\vec{\mathrm{v}} = \frac{d\vec{r}}{dt} = 2\hat{i} + 6t\hat{j}$

The speed v(t) = $\sqrt{2^2 + (6t)^2}\ ms^{-1}$

The velocity of the particle at t = 2 s

$$\vec{v}(2\sec) = 2\hat{i} + 12\hat{j}$$

The speed of the particle at t = 2 s

$$v(2s) = \sqrt{2^2 + 12^2} = \sqrt{4 + 144}$$
$$= \sqrt{148} \approx 12.16\, m\, s^{-1}$$

Note that the particle has velocity components along x and y direction. Along the z direction the position has constant value (−5) which is independent of time. Hence there is no z-component for the velocity.

EXAMPLE

The velocity of three particles A, B, C are given below. Which particle travels at the greatest speed?

$$\overrightarrow{v_A} = 3\hat{i} - 5\hat{j} + 2\hat{k}$$
$$\overrightarrow{v_B} = \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\overrightarrow{v_C} = 5\hat{i} + 3\hat{j} + 4\hat{k}$$

Solution

We know that speed is the magnitude of the velocity vector. Hence,

$$\text{Speed of A} = \left|\overrightarrow{v_A}\right| = \sqrt{(3)^2 + (-5)^2 + (2)^2}$$
$$= \sqrt{9 + 25 + 4} = \sqrt{38}\, m\, s^{-1}$$
$$\text{Speed of B} = \left|\overrightarrow{v_B}\right| = \sqrt{(1)^2 + (2)^2 + (3)^2}$$
$$= \sqrt{1 + 4 + 9} = \sqrt{14}\, m\, s^{-1}$$
$$\text{Speed of C} = \left|\overrightarrow{v_C}\right| = \sqrt{(5)^2 + (3)^2 + (4)^2}$$
$$= \sqrt{25 + 9 + 16} = \sqrt{50}\, m\, s^{-1}$$

The particle C has the greatest speed.

$$\sqrt{50} > \sqrt{38} > \sqrt{14}$$

EXAMPLE

Two cars are travelling with respective velocities $\vec{v}_1 = 10\,m\,s^{-1}$ along east and $\vec{v}_2 = 10\,m\,s^{-1}$ along west. What are the speeds of the cars?

Solution

Both cars have the same magnitude of velocity. This implies that both cars travel at the same speed even though they have velocities in different directions. Speed will not give the direction of motion.

Momentum The linear momentum or simply momentum of a particle is defined as product of mass with velocity. It is denoted as $\vec{p}$. Momentum is also a vector quantity.

Solution

We use p = mv
For the mass of 10 g, m = 0.01 kg

$$p = 0.01 \times 10 = 0.1\,kg\,m\,s^{-1}$$

For the mass of 1 kg

$$p = 1 \times 10 = 10\,kg\,m\,s^{-1}$$

Thus even though both the masses have the same speed, the momentum of the heavier mass is 100 times greater than that of the lighter mass.

MOTION ALONG ONE DIMENSION
Average velocity

If a particle moves in one dimension, say for example along the x direction, then

$$\text{The average velocity} = \frac{\Delta \text{x}}{\Delta \text{t}} = \frac{x_2 - x_1}{t_2 - t_1}$$

The average velocity is also a vector quantity. But in one dimension we have only two directions (positive and negative x direction), hence we use positive and negative signs to denote the direction.

The instantaneous velocity or velocity is defined as $v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

Graphically the slope of the position-time graph will give the velocity of the particle. At the same time, if velocity time graph is given, the distance and displacement are determined by calculating the area under the curve. This is explained below.

We know that velocity is given by $\frac{dx}{dt}$ =v

Therefore, we can write dx =vdt

By integrating both sides, we get $\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt.$

As already seen, integration is equivalent to area under the given curve. So the term $\int_{t_1}^{t_2} v\,dt$ represents the area under the curve v as a function of time.

Since the left hand side of the integration represents the displacement travelled by the particle from time t_1 to t_2, the area under the velocity time graph will give the displacement of the particle. If the area is negative, it means that displacement is negative, so the particle has travelled in the negative direction.

EXAMPLE

A particle moves along the x-axis in such a way that its coordinates x varies with time 't' according to the equation x = 2 - 5t + 6t^2. What is the initial velocity of the particle?

Solution

x t t

Velocity, $v = \frac{dx}{dt} = \frac{d}{dt}(2 - 5t + 6t^2)$

or $v = -5 + 12t$

For initial velocity, $t = 0$

Initial velocity $= -5ms^{-1}$

The negative sign implies that at t = 0 the velocity of the particle is along negative x direction.

Average speed

The average speed is defined as the ratio of the total path length traveled by the particle in a time interval, to the time interval

Average speed = total path length / total time period

Relative Velocity in One and Two Dimensional Motion

When two objects A and B are moving with different velocities, then the velocity of one object A with respect to another object B is called relative velocity of object A with respect to B.

Case 1

Consider two objects A and B moving with uniform velocities V_A and V_B, as shown, along straight tracks in the same direction $\vec{V_A}, \vec{V_B}$ with respect to ground.

The relative velocity of object A with respect to object B is $\vec{V_{AB}} = \vec{V_A} - \vec{V_B}$

The relative velocity of object B with respect to object A is $\vec{V_{BA}} = \vec{V_B} - \vec{V_A}$

Thus, if two objects are moving in the same direction, the magnitude of relative velocity of one object with respect to another is equal to the difference in magnitude of two velocities.

EXAMPLE

Suppose two cars A and B are moving with uniform velocities with respect to ground along parallel tracks and in the same direction. Let the velocities of A and B be 35 km h^{-1} due east and 40 km h^{-1} due east respectively. What is the relative velocity of car B with respect to A?

Solution

The relative velocity of B with respect to A, $\overrightarrow{V_{BA}} = \overrightarrow{V_B} - \overrightarrow{V_A}$ =5 km h^{-1} due east

Similarly, the relative velocity of A with respect to B i.e., $\overrightarrow{V_{AB}} = \overrightarrow{V_A} - \overrightarrow{V_B}$ =5 km h^{-1} due west.

To a passenger in the car A, the car B will appear to be moving east with a velocity 5 km h^{-1}. To a passenger in train B, the train A will appear to move westwards with a velocity of 5 km h^{-1}

Case 2

Consider two objects A and B moving with uniform velocities V_A and V_B along the same straight tracks but opposite in direction

$$\underset{\overrightarrow{V_A}}{\longrightarrow} \quad \underset{\overrightarrow{V_B}}{\longleftarrow}$$

The relative velocity of object A with respect to object B is

$$\vec{V}_{AB} = \vec{V}_A - (-\overrightarrow{V_B}) = \vec{V}_A + \vec{V}_B$$

The relative velocity of object B with respect to object A is

$$\vec{V}_{BA} = -\overrightarrow{V_B} - \vec{V}_A = -(\vec{V}_A + \vec{V}_B)$$

Thus, if two objects are moving in opposite directions, the magnitude of relative velocity of one object with respect to other is equal to the sum of magnitude of their velocities.

Case 3

Consider the velocities $\vec{V}_A$ and $\vec{V}_B$ at an angle θ between their directions.

The relative velocity of A with respect to B, $\overrightarrow{V_{AB}} = \overrightarrow{V_A} - \overrightarrow{V_B}$

Then, the magnitude and direction of $\overrightarrow{V_{AB}}$ is given by $V_{AB} = \sqrt{V_A^2 + V_B^2 - 2V_A V_B \cos\theta}$

and $\tan\beta = \dfrac{V_B \sin\theta}{V_A - V_B \cos\theta}$ (Here β is angle between $\vec{V}_{AB}$ and $\overrightarrow{V_B}$)

1. When θ = 0°, the bodies move along parallel straight lines in the same direction,
   We have $V_{AB} = (V_A - V_B)$ in the direction of $\vec{V}_A$. Obviously $V_{BA} = (V_B + V_A)$ in the
   the
   direction of $\overrightarrow{V_B}$.

2. When θ = 180°, the bodies move along parallel straight lines in opposite directions,
   We have $V_{AB} = (V_A + V_B)$ in the direction of $\vec{V}_A$.Similarly, $V_{BA} = (V_B - V_A)$ in the direction of $\overrightarrow{V_B}$.

3. If the two bodies are moving at right angles to each other, then θ = 90°. The magnitude of the relative velocity of A with respect to $B = v_{AB} = \sqrt{v_A^2 + v_B^2}$.

4. Consider a person moving horizontally with velocity $\vec{V}_M$. Let rain fall vertically with velocity $\vec{V}_R$. An umbrella is held to avoid the rain. Then the relative velocity of the rain with respect to the person is,

$$\vec{V}_{RM} = \vec{V}_R . - \vec{V}_M .$$

   which has magnitude

$$V_{RM} = \sqrt{V_R^2 + V_M^2}$$

   and direction $\theta = \tan^{-1}\left(\dfrac{V_M}{V_R}\right)$

In order to save himself from the rain, he should hold an umbrella at an angle θ with the vertical.

EXAMPLE

Suppose two trains A and B are moving with uniform velocities along parallel tracks but in opposite directions. Let the velocity of train A be 40 km h−1 due east and that of train B be 40 km h^{-1} due west. Calculate the relative velocities of the trains

Solution

Relative velocity of A with respect to B, v_{AB} = 80 km h^{-1} due east

Thus to a passenger in train B, the train A will appear to move east with a velocity of 80 km h^{-1}.

The relative velocity of B with respect to A, V_{BA} = 80 km h_{-1} due west

To a passenger in train A, the train B will appear to move westwards with a velocity of 80 km h^{-1}.

EXAMPLE

Consider two trains A and B moving along parallel tracks with the same velocity in the same direction. Let the velocity of each train be 50 km h^{-1} due east. Calculate the relative velocities of the trains.

Solution

Relative velocity of B with respect to A, $v_{BA} = v_B - v_A$

= 50 km h^{-1} + (−50) km h^{-1}
= 0 km h^{-1}

Similarly, relative velocity of A with respect to B i.e., v_{AB} is also zero.
Thus each train will appear to be at rest with respect to the other.

EXAMPLE

How long will a boy sitting near the window of a train travelling at 36 km h^{-1} see a train passing by in the opposite direction with a speed of 18 km h^{-1}. The length of the slow moving train is 90 m.

Solution

The relative velocity of the slow-moving train with respect to the boy is = (36 + 18) km h^{-1} = 54 km h^{-1} = = $54 \times \frac{5}{18} m s^{-1} = 15 m s^{-1}$

Since the boy will watch the full length of the other train, to find the time taken to watch the full train:

$$15 = \frac{90}{t} \, or \, t = \frac{90}{15} = 6 s$$

EXAMPLE

A swimmer's speed in the direction of flow of a river is 12 km h^{-1}. Against the direction of flow of the river the swimmer's speed is 6 km h^{-1}. Calculate the swimmer's speed in still water and the velocity of the river flow.

Solution

Let v_s and vr, represent the velocities of the swimmer and river respectively with respect to ground.

$$v_s + v_r = 12 \qquad (1)$$
$$\text{and } v_s - v_r = 6 \qquad (2)$$

Adding the both equations (1) and (2) $2v_s = 12 + 6 = 18$ km h^{-1} or vs = 9 km h^{-1}.

From Equation (1),

$$9 + v_r = 12 \text{ or}$$
$$v_r = 3 \text{ km h}^{-1}$$

When the river flow and swimmer move in the same direction, the net velocity of swimmer is 12 km h^{-1}.

Accelerated Motion

During non-uniform motion of an object, the velocity of the object changes from instant to instant i.e., the velocity of the object is no more constant but changes with time. Such a motion is said to be an accelerated motion.

1. In accelerated motion, if the change in velocity of an object per unit time is same (constant) then the object is said to be moving with uniformly accelerated motion.
2. On the other hand, if the change in velocity per unit time is different at different times, then the object is said to be moving with non-uniform accelerated motion.

Average acceleration

If an object changes its velocity from $\vec{v}_1$ $\vec{v}_2$ to in a time interval Δt = $t_1 - t_2$, then the average acceleration is defined as the ratio of change in velocity over the time interval Δt = $t_1 - t_2$.

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

Average acceleration is a vector quantity in the same direction as the vector Δv .

Instantaneous acceleration

Usually, the average acceleration will give the change in velocity only over the entire time interval. It will not give value of the acceleration at any instant time t.

Instantaneous acceleration or acceleration of a particle at time 't' is given by the ratio of change in velocity over Δt, as Δt approaches zero.

$$\text{Acceleration } \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

In other words, the acceleration of the particle at an instant t is equal to rate of change of velocity.

Acceleration is a vector quantity. Its SI unit is ms^{-2} and its dimensional formula is $M^0L^1T^{-2}$

Acceleration is positive if its velocity is increasing, and is negative if the velocity is decreasing. The negative acceleration is called retardation or deceleration.

In terms of components, we can write

$$\vec{a}_x = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = \frac{d\vec{v}}{dt}$$

Thus $a_x = \frac{dv_x}{dt}, a_y = \frac{dv_y}{dt}, a_z = \frac{dv_z}{dt}$ are the components of instantaneous acceleration.

Since each component of velocity is the derivative of the corresponding coordinate, we can express the components a_x, a_y, and a_z, as

$$a_x = \frac{d^2x}{dt^2}, a_y = \frac{d^2y}{dt^2}, a_z = \frac{d^2z}{dt^2}$$

Then the acceleration vector $\vec{a}$ itself is

$$\vec{a}_x = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} = \frac{d^2\vec{r}}{dt^2}$$

Thus acceleration is the second derivative of position vector with respect to time.

Graphically the acceleration is the slope in the velocity-time graph. At the same time if the acceleration-time graph is given, then the velocity can be found from the area under the acceleration-time graph.

From $\frac{dv}{dt} = \text{a, we have dv} = \text{adt; hence}$

$$v = \int_{t_1}^{t_2} a\,dt$$

For an initial time t_1 and final time t_2

EXAMPLE

A velocity–time graph is given for a particle moving in x direction, as below

1. Describe the motion qualitatively in the interval 0 to 55 s .
2. Find the distance and displacement travelled from 0 s to 40 s .
3. Find the acceleration at t = 5 s and at t 20 s

Solution

From O to A: (0 s to 10 s)

At t = 0 s the particle has zero velocity. At t > 0, particle has positive velocity and moves in the positive x direction. From 0 s to 10 s the slope $(\frac{dv}{dt})$ is positive, implying the particle is accelerating. Thus the velocity increases during this time interval.

From A to B: (10 s to 15 s)

From 10 s to 15 s the velocity stays constant at 60 m s-1. The acceleration is 0 during this period. But the particle continues to travel in the positive x-direction.

From B to C : (15 s to 30 s)

From the 15 s to 30 s the slope is negative, implying the velocity is decreasing. But the particle is moving in the positive x direction. At t = 30 s the velocity becomes zero, and the particle comes to rest momentarily at t = 30 s .

From C to D: (30 s to 40 s)

From 30 s to 40 s the velocity is negative. It implies that the particle starts to move in the negative x direction. The magnitude of velocity increases to a maximum 40 m s^{-1}.

From D to E: (40 s to 55 s)

From 40 s to 55 s the velocity is still negative, but starts increasing from –40 m s^{-1} At t = 55 s the velocity of the particle is zero and particle comes to rest.

The total area under the curve from 0 s to 40 s will give the displacement. Here the area from O to C represents motion along positive x–direction and the area under the graph from C to D represents the particle's motion along negative x–direction.

The displacement travelled by the particle from 0 s to 10 s $= \frac{1}{2} \times 10 \times 60 =$ 300m

The displacement travelled from 10 s to 15 s = 60 × 5 = 300 m

The displacement travelled from 15 s to 30 s $= \frac{1}{2} \times 15 \times 60 = 450$m

The displacement travelled from 30 s to 40 s $= \frac{1}{2} \times 10 \times (-40) = -200$m.

Here the negative sign implies that the particle travels 200 m in the negative x direction.

The total displacement from 0 s to 40 s is given by

$$300 \text{ m} + 300 \text{ m} + 450 \text{ m} - 200 \text{ m} = +850 \text{ m}.$$

Thus the particle's net displacement is along the positive x-direction.

The total distance travelled by the particle from 0 s to 40 s = 300 + 300 + 450 + 200 = 1250 m.

The acceleration is given by the slope in the velocity-time graph. In the first 10 seconds the velocity has constant slope (constant acceleration). It implies that the acceleration a is from v1 = 0 to v2 = 60 m s^{-1}.

Hence $a = \frac{v_2 - v_1}{t_2 - t_1}$ gives

$$a = \frac{60-0}{10-0} = 6\,m\,s^{-2}$$

Next, the particle has constant negative slope from 15 s to 30 s . In this case $v_2=0$ and $v_1=60$ m s^{-1}. Thus the acceleration at t = 20 s is given by $a = \frac{0-60}{30-15} = -4m\,s^{-2}$ Here the negative sign implies that the particle has negative acceleration.

EXAMPLE 2.32

If the position vector of the particle is given by $\vec{r} = 3t^2\hat{i} + 5t\hat{j} + 4\hat{k}$ Find the

a. The velocity of the particle at t = 3 s
b. Speed of the particle at t = 3 s
c. acceleration of the particle at time t = 3 s

Solution

a. The velocity $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ We obtain, $\vec{v}(t) = 6t\hat{i} + 5\hat{j}$ The velocity has only two components $v_x=6t$, depending on time t and $v_y=5$ which is independent of time.
   The velocity at $t = 3s$ $is\,\vec{v}(3) = 18\hat{i} + 5\hat{j}$

b. The speed at t = 3 s is $v = \sqrt{18^2 + 5^2} = \sqrt{349} \approx 18.68m\,s^{-1}$

c. The acceleration $\vec{a}$ is, $\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6\hat{i}$ The acceleration has only the x-component. Note that acceleration here is independent of t, which means $\vec{a}$ is constant. Even at t = 3 s it has same value $\vec{a} = 6\hat{i}$. The velocity is non-uniform, but the acceleration is uniform (constant) in this case.

EXAMPLE

An object is thrown vertically downward. What is the acceleration experienced by the object?

Solution

We know that when the object falls towards the Earth, it experiences acceleration due to gravity g = 9.8 m s^{-2} downward. We can choose the coordinate system as shown in the figure.

The acceleration is along the negative y direction.

$$\vec{a} = g(-j) = -g\hat{j}$$

Equations of Uniformly Accelerated Motion by Calculus Method

Consider an object moving in a straight line with uniform or constant acceleration 'a'.

Let u be the velocity of the object at time t = 0, and v be velocity of the body at a later time t.

Velocity - time relation

a. The acceleration of the body at any instant is given by the first derivative of the velocity with respect to time,

$$a = \frac{dv}{dt} \, or \, dv = a\, dt$$

Integrating both sides with the condition that as time changes from 0 to t, the velocity changes from u to v. For the constant acceleration,

$$\int_u^v dv = \int_0^t a\, dt = a\int_0^t dt \Rightarrow [v]_u^v = a[t]_0^t$$

$$v - u = at \quad (or) \quad v = u + at \qquad \rightarrow (2.7)$$

Displacement – time relation

b. The velocity of the body is given by the first derivative of the displacement with respect to time.

$$v = \frac{ds}{dt} \ or \ ds = vdt$$

and since $v = u + at$,

We get $ds = (u + at)dt$

Assume that initially at time t = 0, the particle started from the origin. At later time t, the particle displacement is s. Further assuming that acceleration is time independent, we have

$$\int_0^s ds = \int_0^t u\,dt + \int_o^t at\,dt \ (or) \ s = ut + \frac{1}{2}at^2 \ (2.8)$$

Velocity – displacement relation

c. The acceleration is given by the first derivative of velocity with respect to time.

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

[since ds/dt = v] where s is displacement traversed.

This is rewritten as $a = \frac{1}{2}\frac{dv^2}{ds}$

$$\text{or} \ ds = \frac{1}{2a}d(v^2)$$

Integrating the above equation, using the fact when the velocity changes from u^2 to v^2, displacement changes from 0 to s, we get

$$\int_0^s ds = \int_u^v \frac{1}{2a} d\left(v^2\right)$$

$$\therefore s = \frac{1}{2a}\left(v^2 - u^2\right)$$

$$\therefore v^2 = u^2 + 2as$$

We can also derive the displacement s in terms of initial velocity u and final velocity v.

From the equation (2.7) we can write,

$$at = v - u$$

Substitute this in equation (2.8), we get

$$s = ut + \frac{1}{2}\left(v - u\right)t$$

$$s = \frac{\left(u + v\right)t}{2}$$

The equations (2.7), (2.8), (2.9) and (2.10) are called kinematic equations of motion, and have a wide variety of practical applications.

Kinematic equations

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{(u+v)t}{2}$$

It is to be noted that all these kinematic equations are valid only if the motion is in a straight line with constant acceleration. For circular motion and oscillatory motion these equations are not applicable.

Equations of motion under gravity

A practical example of a straight line motion with constant acceleration is the motion of an object near the surface of the Earth. We know that near the surface of the Earth, the acceleration due to gravity 'g' is constant. All straight line motions under this acceleration can be well understood using the kinematic equations given earlier

Case (1): A body falling from a height h

Consider an object of mass m falling from a height h. Assume there is no air resistance. For convenience, let us choose the downward direction as positive y-axis as shown in the Figure 2.37. The object experiences acceleration 'g' due to gravity which is constant near the surface of the Earth. We can use kinematic equations to explain its motion. We have

The acceleration $\vec{a} = g\hat{j}$

By comparing the components, we get

$$a_x = 0, a_z = 0, a_y = g$$

Let us take for simplicity, $a_y = a = g$

If the particle is thrown with initial velocity 'u' downward which is in negative y axis, then velocity and position at of the particle any time t is given by

$$v = u + gt$$

$$y = ut + \frac{1}{2}gt^2$$

The square of the speed of the particle when it is at a distance y from the hill-top, is

$$v^2 = u^2 + 2gy$$

Suppose the particle starts from rest.
Then u = 0

Then the velocity v, the position of the particle and v2 at any time t are given by (for a point y from the hill-top)

$$v = gt$$

$$y = \frac{1}{2}gt^2$$

$$v^2 = 2gy$$

The time (t = T) taken by the particle to reach the ground (for which y = h), is given by using equation

$$h = \frac{1}{2}gT^2$$

$$T = \sqrt{\frac{2h}{g}}$$

The equation (2.18) implies that greater the height(h), particle takes more time(T) to reach the ground. For lesser height(h), it takes lesser time to reach the ground.

The speed of the particle when it reaches the ground (y = h) can be found using equation (2.16), we get

$$v_{ground} = \sqrt{2gh}$$

The above equation implies that the body falling from greater height(h) will have higher velocity when it reaches the ground.

The motion of a body falling towards the Earth from a small altitude (h << R), purely under the force of gravity is called free fall. (Here R is radius of the Earth)

EXAMPLE

An iron ball and a feather are both falling from a height of 10 m.

a. What are the time taken by the iron ball and feather to reach the ground?
b. What are the velocities of iron ball and feather when they reach the ground? (Ignore air resistance and take g = 10 m s^{-2})

Solution

Since kinematic equations are independent of mass of the object, according to equation (2.8) the time taken by both iron ball and feather to reach the ground are the same. This is given by

$$T = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times10}{10}} = \sqrt{2}\,s \approx 1.414\,s$$

Thus, both feather and iron ball reach ground at the same time.

By following equation (2.19) both iron ball and feather reach the Earth with the same speed. It is given by

$$v = \sqrt{2gh} = \sqrt{2\times10\times10}$$
$$= \sqrt{200}\,m\,s^{-1} \approx 14.14\,m\,s^{-1}$$

EXAMPLE

Is it possible to measure the depth of a well using kinematic equations?

Consider a well without water, of some depth d. Take a small object (for example lemon) and a stopwatch. When you drop the lemon, start the stop watch. As soon as the lemon touches the bottom of the well, stop the watch. Note the time taken by the lemon to reach the bottom and denote the time as t.

Since the initial velocity of lemon u=0 and the acceleration due to gravity g is constant over the well, we can usethe equations of motion for constant acceleration.

$$s = ut + \frac{1}{2}at^2$$

Since u=0, s=d,a=g (Since we choose the y axis downwards), Then

$$d = \frac{1}{2}gt^2$$

Substituting g=9.8 m s^{-2} we get the depth of the well.

To estimate the error in our calculation we can use another method to measure the depth of the well. Take a long rope and hang the rope inside the well till it touches the bottom. Measure the length of the rope which is the correct depth of the well ($d_{correct}$). Then

$$error = d_{correct} - d$$

$$relative\ error = \frac{d_{correct} - d}{d_{correct}}$$

$$percentage\ of\ relative\ error = \frac{d_{correct} - d}{d_{correct}} \times 100$$

What would be the reason for an error, if any?

Repeat the experiment for different masses and compare the result with $d_{correct}$ every time.

Case (ii): A body thrown vertically upwards

Consider an object of mass m thrown vertically upwards with an initial velocity u. Let us neglect the air friction. In this case we choose the vertical direction as positive y axis as shown in the Figure 2.38, then the acceleration a = −g (neglect air friction) and g points towards the negative y axis. The kinematic equations for this motion are,

The velocity and position of the object at any time t are,

$$v = u - gt$$

$$s = ut - \frac{1}{2}gt^2$$

The velocity of the object at any position y (from the point where the object is thrown) is

$$v^2 = u^2 - 2gy$$

EXAMPLE

A train was moving at the rate of 54 km h^{-1} when brakes were applied. It came to rest within a distance of 225 m. Calculate the retardation produced in the train.

Solution

The final velocity of the particle v = 0 The initial velocity of the particle

$$u = 54 \times \frac{5}{18} m\,s^{-1} = 15\ m\,s^{-1}$$

$$S = 225\ m$$

Retardation is always against the velocity of the particle.

$$v^2 = u^2 - 2aS$$

$$0 = (10)^2 - 2a\,(225)$$

$$-450\ a = 100$$

$$a = -\frac{1}{2} ms^{-2} = 0.5\ \ ms^{-2}$$

$$\text{Hence, retardation} = 0.5\ m\ s^{-2}$$

PROJECTILE MOTION

Introduction

When an object is thrown in the air with some initial velocity (NOT just upwards), and then allowed to move under the action of gravity alone, the object is known as a projectile. The path followed by the particle is called its trajectory.

Examples of projectile are

1. An object dropped from window of a moving train.
2. A bullet fired from a rifle.
3. A ball thrown in any direction.
4. A javelin or shot put thrown by an athlete.
5. A jet of water issuing from a hole near the bottom of a water tank.

It is found that a projectile moves under the combined effect of two velocities.

i. A uniform velocity in the horizontal direction, which will not change provided there is no air resistance.

ii. A uniformly changing velocity (i.e., increasing or decreasing) in the vertical direction.

There are two types of projectile motion:

i. Projectile given an initial velocity in the horizontal direction (horizontal projection)

ii. Projectile given an initial velocity at an angle to the horizontal (angular projection)

To study the motion of a projectile, let us assume that,

i. Air resistance is neglected.

ii. The effect due to rotation of Earth and curvature of Earth is negligible.

iii. The acceleration due to gravity is constant in magnitude and direction at all points of the motion of the projectile

Projectile in horizontal projection

Consider a projectile, say a ball, thrown horizontally with an initial velocity $\vec{u}$ from the top of a tower of height h

As the ball moves, it covers a horizontal distance due to its uniform horizontal velocity u, and a vertical downward distance because of constant acceleration due to gravity g. Thus, under the combined effect the ball moves along the path OPA. The motion is in a 2-dimensional plane. Let the ball take time t to reach the ground at point A, Then the horizontal distance travelled by the ball is x (t) = x, and the vertical distance travelled is y (t) = y

We can apply the kinematic equations along the x direction and y direction separately. Since this is two-dimensional motion, the velocity will have both horizontal component u_x and vertical component u_y.

Motion along horizontal direction

The particle has zero acceleration along x direction. So, the initial velocity u_x remains constant throughout the motion.

The distance traveled by the projectile at a time t is given by the equation $x = u_x t + \frac{1}{2}at^2$.

Since a = 0 along x direction, we have

$$x = u_x t$$

Motion along downward direction

Here u_y = 0 (initial velocity has no downward component), a = g (we choose the +ve y-axis in downward direction), and distance y at time t

$$\therefore \text{From equation, } y = u_y t + \frac{1}{2}at^2, \text{ we get}$$

$$y = \frac{1}{2}gt^2 \qquad (2.24)$$

Substituting the value of t from equation

$$y = \frac{1}{2}g\frac{x^2}{u_x^{\ 2}} = \left(\frac{g}{2u_x^{\ 2}}\right)x^2$$

$$y = Kx^2$$

$$\text{where } K = \frac{g}{2u_x^2} \text{ is constant}$$

The equation of a parabola. Thus, the path followed by the projectile is a parabola.

Time of Flight:

The time taken for the projectile to complete its trajectory or time taken by the projectile to hit the ground is called time of flight.

Consider the example of a tower and projectile. Let h be the height of a tower. Let T be the time taken by the projectile to hit the ground, after being thrown horizontally from the tower.

We know that $s_y = u_y t + \frac{1}{2}at^2$ for vertical motion. Here $s_y = h$, $t = T$, $u_y = 0$ (i.e., no initial vertical velocity). Then

$$h = \frac{1}{2}gT^2 \quad \text{or} \quad T = \sqrt{\frac{2h}{g}}$$

Thus, the time of flight for projectile motion depends on the height of the tower, but is independent of the horizontal velocity of projection. If one ball falls vertically and another ball is projected horizontally with some velocity, both the balls will reach the bottom at the same time.

Horizontal range:

The horizontal distance covered by the projectile from the foot of the tower to the point where the projectile hits the ground is called horizontal range. For horizontal motion, we have

$$s_x = u_x t + \frac{1}{2}at^2.$$

Here, s_x = R (range), u_x = u, a = 0 (no horizontal acceleration) T is time of flight. Then horizontal range = uT.

Since the time of flight $T = \sqrt{\frac{2h}{g}}$ we substitute this and we get the horizontal range of the particle as $R = u\sqrt{\frac{2h}{g}}$.

The above equation implies that the range R is directly proportional to the initial velocity u and inversely proportional to acceleration due to gravity g.

Resultant Velocity (Velocity of projectile at any time):

At any instant t, the projectile has velocity components along both x-axis and y-axis. The resultant of these two components gives the velocity of the projectile at that instant t,

The velocity component at any t along horizontal (x-axis) is $v_x = u_x + a_x t$
Since, u_x = u, a_x = 0 , we get

$$v_x = u$$

The component of velocity along vertical direction (y-axis) is $v_y = u_y + a_y t$
Since, u_y = 0, a_y = g, we get

$$v_y = gt$$

Hence the velocity of the particle at any instant is

$$\vec{v} = u\hat{i} + gt\hat{j}$$

The speed of the particle at any instant t is given by

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2}$$
$$v = \sqrt{u^2 + g^2t^2}$$

Speed of the projectile when it hits the ground:

When the projectile hits the ground after initially thrown horizontally from the top of tower of height h, the time of flight is

$$t = \sqrt{\frac{2h}{g}}$$

The horizontal component velocity of the projectile remains the same i.e v_x= u

The vertical component velocity of the projectile at time T is

$$v_y = gT = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}$$

The speed of the particle when it reaches the ground is

$$v = \sqrt{u^2 + 2gh}$$

Projectile under an angular projection

This projectile motion takes place when the initial velocity is not horizontal, but at some angle with the vertical,

(Oblique projectile)
Examples:

- Water ejected out of a hose pipe held obliquely.
- Cannon fired in a battle ground.

Consider an object thrown with initial velocity $\vec{u}$ at an angle θ with the horizontal.

$$\vec{u} = u_x\hat{i} + u_y\hat{j}$$

where u_x= u cosθ is the horizontal component and u_y=usinθ the vertical component of velocity.

Since the acceleration due to gravity is in the direction opposite to the direction of vertical component u_y, this component will gradually reduce to zero at the maximum height of the projectile. At this maximum height, the same gravitational force will push the projectile to move downward and fall to the ground. There is no acceleration along the x direction throughout the motion. So, the horizontal component of the velocity ($u_x = u \cos\theta$) remains the same till the object reaches the ground.

Hence after the time t, the velocity along horizontal motion $v_x = u_x + a_x t = u_x = u \cos \theta$

The horizontal distance travelled by projectile in time t is $S_x = u_x t + \frac{1}{2} a_x t^2$

Here, $s_x = x$, $u_x = u \cos\theta$, $a_x = 0$

$$\text{Thus, } x = u\cos\theta.t \text{ or } t = \frac{x}{u\cos\theta}$$

Next, for the vertical motion $v_y = u_y + a_y t$

Here $u_y = u \sin\theta$, $a_y = -g$ (acceleration due to gravity acts opposite to the motion). Thus

$$\text{Thus, } v_y = usin\theta - gt$$

The vertical distance travelled by the projectile in the same time t is $S_y = u_y t + \frac{1}{2} a_y t^2$

Here, $s_y = y$, $u_y = u \sin\theta$, $a_x = -g$. Then

$$y = u \sin\theta\, t - \frac{1}{2} g t^2$$

Substitute the value of t from equation

$$y = u\sin\theta \frac{x}{u\cos\theta} - \frac{1}{2} g \frac{x^2}{u^2\cos^2\theta}$$

$$y = x \tan \theta - \frac{1}{2} g \frac{x^2}{u^2 \cos^2 \theta} \qquad (2.31)$$

Thus the path followed by the projectile is an inverted parabola.

Maximum height (h_{max})

The maximum vertical distance travelled by the projectile during its journey is called maximum height. This is determined as follows:

For the vertical part of the motion,

$$v_y^2 = u_y^2 + 2a_y s$$

Here, u_y = u sinθ, a = −g, s = h_{max}, and at the maximum height v_y = 0
Hence,

$$(0)^2 = u^2 \sin^2\theta = 2gh_{\text{max}}$$

$$\text{Or } h_{\text{max}} = \frac{u^2 \sin^2\theta}{2g}$$

Time of flight (T_f)

The total time taken by the projectile from the point of projection till it hits the horizontal plane is called time of flight.

This time of flight is the time taken by the projectile to go from point O to B via point A

We know that $S_y = u_y t + \frac{1}{2} a_y t^2$

Here, s_y = y = 0 (net displacement in y-direction is zero), u_y = u sinθ, a_y = −g, t = T_f Then

$$0 = u\sin\theta\, T_f - \frac{1}{2} g T_f^2$$

$$T_f = 2u\frac{\sin\theta}{g}$$

Horizontal range (R)

The maximum horizontal distance between the point of projection and the point on the horizontal plane where the projectile hits the ground is called horizontal range (R). This is found easily since the horizontal component of initial velocity remains the same. We can write

Range R = Horizontal component of velocity x time of flight $= u\cos\theta \times T_f$

$$R = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{2u^2 \sin\theta\cos\theta}{g}$$

$$\therefore R = \frac{u^2 \sin 2\theta}{g} \qquad (2.33)$$

The horizontal range directly depends on the initial speed (u) and the sine of angle of projection (θ). It inversely depends on acceleration due to gravity 'g'

For a given initial speed u, the maximum possible range is reached when sin2θ is maximum, sin2θ = 1. This implies 2θ = $\pi/2$

$$\theta = \frac{\pi}{4}$$

This means that if the particle is projected at 45 degrees with respect to horizontal, it attains maximum range, given by.

$$R_{max} = \frac{u^2}{g}$$

EXAMPLE

Suppose an object is thrown with initial speed 10 m s^{-1} at an angle $\pi/4$ with the horizontal, what is the range covered? Suppose the same object is thrown similarly in the Moon, will there be any change in the range? If yes, what is the change? (The acceleration due to gravity in the Moon $g_{moon} = \frac{1}{6}g$)

Solution

In projectile motion, the range of particle is given by,

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\theta = \pi/4 \quad u = v_0 = 10 m\ s^{-1}$$

$$\therefore R_{earth} = \frac{(10)^2 \sin \pi/2}{9.8} = 100/9.8$$

$$R_{earth} = 10.20\ m \text{ (Approximately 10 m)}$$

If the same object is thrown in the Moon, the range will increase because in the Moon, the acceleration due to gravity is smaller than g on Earth,

$$g_{moon} = \frac{g}{6}$$

$$R_{moon} = \frac{u^2 \sin 2\theta}{g_{moon}} = \frac{v_0^2 \sin 2\theta}{g/6}$$

$$\therefore R_{moon} = 6R_{earth}$$

$$R_{moon} = 6 \times 10.20 = 61.20\ m$$

(Approximately 60 m)

The range attained on the Moon is approximately six times that on Earth.

EXAMPLE

In the cricket game, a batsman strikes the ball such that it moves with the speed 30 m s^{-1} at an angle 300 with the horizontal as shown in the figure. The boundary line of the cricket ground is located at a distance of 75 m from the batsman? Will the ball go for a six? (Neglect the air resistance and take acceleration due to gravity g = 10 m s^{-2}).

Solution

The motion of the cricket ball in air is essentially a projectile motion. As we have already seen, the range (horizontal distance) of the projectile motion is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

The initial speed $u = 30\,m\,s^{-1}$

The projection angle $\theta = 30°$

The horizontal distance travelled by the cricket ball

$$R = \frac{(30)^2 \times sin 60°}{10} = \frac{900 \times \frac{\sqrt{3}}{2}}{10} = 77.94\,m$$

This distance is greater than the distance of the boundary line. Hence the ball will cross this line and go for a six.

Introduction to Degrees and Radians

In measuring angles, there are several possible units used, but the most common units are degrees and radians. Radians are used in measuring area, volume, and circumference of circles and surface area of spheres.

Radian describes the planar angle subtended by a circular arc at the center of a circle. It is defined as the length of the arc divided by the radius of the arc. One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle.

Degree is the unit of measurement which is used to determine the size of an angle. When an angle goes all the way around in a circle, the total angle covered is equivalent to 360°. Thus, a circle has 360°. In terms of radians, the full circle has 2π radian.

Hence we write $360° = 2\pi$ *radians*

$$\text{or 1 radians} = \frac{180}{\pi} \text{ degrees}$$

which means 1 rad = 57.295^0

EXAMPLE

Calculate the angle θ subtended by the two adjacent wooden spokes of a bullock cart wheel is shown in the figure. Express the angle in both radian and degree.

Solution

The full wheel subtends 2? radians at the center of the wheel. The wheel is divided into 12 parts (arcs).

So one part subtends an angle $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ radian at the center

Since, π rad =180°, $\frac{\pi}{6}$ radian is equal to 30 degree.

The angle subtended by two adjacent wooden spokes is 30 degree at the center.

Angular displacement

Consider a particle revolving around a point O in a circle of radius r (Figure 2.45). Let the position of the particle at time t = 0 be A and after time t, its position is B.

Then,

The angle described by the particle about the axis of rotation (or center O) in a given time is called angular displacement.
angular displacement = $\angle AOB = \theta$
The unit of angular displacement is radian.

The angular displacement (θ) in radian is related to arc length S (AB) and radius r as

$$\theta = \frac{S}{r}, \quad \text{or} \quad S = r\theta$$

Angular velocity (ϖ)

The rate of change of angular displacement is called angular velocity.

If θ is the angular displacement in time t, then the angular velocity ω is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The unit of angular velocity is radian per second (rad s^{-1}). The direction of angular velocity is along the axis of rotation following the right hand rule.

Angular acceleration (α)

The rate of change of angular velocity is called angular acceleration.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

The angular acceleration is also a vector quantity which need not be in the same direction as angular velocity.

Tangential acceleration

Consider an object moving along a circle of radius r. In a time Δt, the object travels an arc distance Δs as shown in Figure 2.47. The corresponding angle subtended is Δθ

The Δs can be written in terms of Δθ as,

$$\Delta s = r\Delta\theta$$

In a time Δt, we have

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

In the limit Δt →0, the above equation becomes

$$\frac{ds}{dt} = r\omega$$

Here $\frac{ds}{dt}$ linear speed (v) which is tangential to the circle and ω is angular speed. So equation (2.37) becomes

$$v = r\omega$$

which gives the relation between linear speed and angular speed.

Equation (2.38) is true only for circular motion. In general the relation between linear and angular velocity is given by

$$\vec{v} = \vec{\omega} \times \vec{r}$$

For circular motion equation (2.39) reduces to equation (2.38) since ϖ and $\vec{r}$ are perpendicular to each other.

Differentiating the equation (2.38) with respect to time, we get (since r is constant)

$$\frac{dv}{dt} = \frac{rd\omega}{dt} = r\alpha$$

Here $\frac{dv}{dt}$ is the tangential acceleration and is denoted as a_t $\frac{d\omega}{dt}$ is the angular
acceleration α. Then eqn. (2.39) becomes

$$a_t = r\alpha$$

The tangential acceleration a_t experienced by an object is circular motion

Circular Motion

When a point object is moving on a circular path with a constant speed, it covers equal distances on the circumference of the circle in equal intervals of time. Then the object is said to be in uniform circular motion.

In uniform circular motion, the velocity is always changing but speed remains the same. Physically it implies that magnitude of velocity vector remains constant and only the direction changes continuously.

If the velocity changes in both speed and direction during the circular motion, we get non uniform circular motion.

Centripetal acceleration

As seen already, in uniform circular motion the velocity vector turns continuously without changing its magnitude (speed),

Note that the length of the velocity vector (blue) is not changed during the motion, implying that the speed remains constant. Even though the velocity is tangential at every point in the circle, the acceleration is acting towards the center of the circle. This is called centripetal acceleration. It always points towards the center of the circle.

The centripetal acceleration is derived from a simple geometrical relationship between position and velocity vectors

Let the directions of position and velocity vectors shift through the same angle θ in a small interval of time Δt, as shown in Figure 2.52. For uniform circular motion, $r = |\vec{r_1}| = |\vec{r_2}|$ and $v = |\vec{v_1}| = |\vec{v_2}|$. If the particle moves from position vector $\vec{r_1}$ to $\vec{r_2}$ the displacement is given by $\Delta\vec{r} = \vec{r_2} - \vec{r_1}$ and the change in velocity from $\vec{v_1}$ to $\vec{v_2}$
is given by $\Delta\vec{v} = \vec{v_2} - \vec{v_1}$ The magnitudes of the displacement Δr and of Δv satisfy the following relation

$$\frac{\Delta r}{r} = -\frac{\Delta v}{v} = \theta$$

Here the negative sign implies that Δv pointsradially inward, towards the center of the circle.

$$\Delta v = -v\left(\frac{\Delta r}{r}\right)$$

$$\text{Then,} \quad a = \frac{\Delta v}{\Delta t} = \frac{v}{r}\left(\frac{\Delta r}{\Delta t}\right) = -\frac{v^2}{r}$$

For uniform circular motion v =ωr, where ω is the angular velocity of the particle about the center. Then the centripetal acceleration can be written as

$$a = -\omega^2 r$$

Non uniform circular motion

If the speed of the object in circular motion is not constant, then we have non-uniform circular motion. For example, when the bob attached to a string moves in vertical circle, the speed of the bob is not the same at all time. Whenever the speed is not same in circular motion, the particle will have both centripetal and tangential acceleration.

The resultant acceleration is obtained by vector sum of centripetal and tangential acceleration.

Since centripetal acceleration is $\frac{v^2}{r}$ the magnitude of this resultant acceleration is given by $a_R = \sqrt{a_t^2 + \left(\frac{v^2}{r}\right)^2}$

This resultant acceleration makes an angle θ with the radius vector

This angle is given by $\tan\theta = \frac{a_t}{\left(\frac{v^2}{r}\right)}$

EXAMPLE

A particle moves in a circle of radius 10 m. Its linear speed is given by v =3t where t is in second and v is in m s^{-1}.

1. Find the centripetal and tangential acceleration at t = 2 s.
2. Calculate the angle between the resultant acceleration and the radius vector.

Solution

The linear speed at t = 2 s

$$v = 3t = 6\,m\,s^{-1}$$

The centripetal acceleration at t = 2 s is

$$a_c = \frac{v^2}{r} = \frac{(6)^2}{10} = 3.6\,m\,s^{-2}$$

The tangential acceleration is $a_t = \frac{dv}{dt} = 3\,m\,s^{-2}$

The angle between the radius vector with resultant acceleration is given by

$$\tan\theta = \frac{a_t}{a_c} = \frac{3}{3.6} = 0.833$$

$$\theta = \tan^{-1}(0.833) = 0.69 \text{ radian}$$

$$\text{In terms of degree } \theta = 0.69 \times 57.17° \approx 40°$$

Kinematic Equations of circular motion

If an object is in circular motion with constant angular acceleration α, we can derive kinematic equations for this motion, analogous to those for linear motion.

Let us consider a particle executing circular motion with initial angular velocity ω_0. After a time interval t it attains a final angular velocity ω. During this time, it covers an angular displacement θ. Because of the change in angular velocity there is an angular acceleration α .

The kinematic equations for circular motion are easily written by following the kinematic equations for linear motion in section 2.4.3

The linear displacement (s) is replaced by the angular displacement(θ).

The velocity (v) is replaced by angular velocity (ω).

The acceleration (a) is replaced by angular acceleration (α).

The initial velocity (u) is replaced by the initial angular velocity (ω_0).

By following this convention, kinematic equations for circular motion are as in the table given below.

Kinematic equations for linear motion	Kinematic equations for angular motion
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = \frac{(v+u)t}{2}$	$\theta = \frac{(\omega_0 + \omega)t}{2}$

EXAMPLE

A particle is in circular motion with an acceleration α =0.2rad s^{-2}.

1. What is the angular displacement made by the particle after 5 s?
2. What is the angular velocity at t = 5 s?. Assume the initial angular velocity is zero.

Solution

Since the initial angular velocity is zero ($\omega_0 = 0$).

The angular displacement made by the particle is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \frac{1}{2} \times 2 \times 10^{-1} \times 25 = 2.5\ rad$$

In terms of degree

$$\theta = 2.5 \times 57.17° \approx 143°$$