FORCE AND MOTION

PART - III

11TH VOL- I

UNIT – 3 LAWS OF MOTION

INTRODUCTION

Each and every object in the universe interacts with every other object. The cool breeze interacts with the tree. The tree interacts with the Earth. In fact, all species interact with nature. But, what is the difference between a human's interaction with nature and that of an animal's. Human's interaction has one extra quality. We not only interact with nature but also try to understand and explain natural phenomena scientifically.

In the history of mankind, the most curiosity driven scientific question asked was about motion of objects–'How things move?' and 'Why things move?' Surprisingly, these simple questions have paved the way for development from early civilization to the modern technological era of the 21st century.

Objects move because something pushes or pulls them. For example, if a book is at rest, it will not move unless a force is applied on it. In other words, to move an object a force must be applied on it. About 2500 years ago, the famous philosopher, Aristotle, said that 'Force causes motion'. This statement is based on common sense. But any scientific answer cannot be based on common sense. It must be endorsed with quantitative experimental proof.

In the 15^{th} century, Galileo challenged Aristotle's idea by doing a series of experiments. He said force is not required to maintain motion.

Galileo demonstrated his own idea using the following simple experiment. When a ball rolls from the top of an inclined plane to its bottom, after reaching the ground it moves some distance and continues to move on to another inclined plane of same angle of inclination as shown in the Figure 3.1(a). By increasing the smoothness of both the inclined planes, the ball reach almost the same height(h) from where it was released (L1) in the second plane (L2) (Figure 3.1(b)). The motion of the ball is then observed by varying the angle of inclination of the second plane keeping the same smoothness. If the angle of inclination is reduced, the ball travels longer distance in the second plane to reach the same height (Figure 3.1 (c)). When the angle of inclination is made zero, the ball moves forever in the horizontal direction (Figure 3.1(d)). If the Aristotelian idea were true, the ball would not have moved in the second plane even if its smoothness is made maximum since no force acted on it in the horizontal direction. From this simple experiment, Galileo proved that force is not required to maintain motion. An object can be in motion even without a force acting on it.

In essence, Aristotle coupled the motion with force while Galileo decoupled the motion and force.

NEWTON'S LAWS

Newton analysed the views of Galileo, and other scientist like Kepler and Copernicus on motion and provided much deeper insights in the form of three laws.

Newton's First Law

Every object continues to be in the state of rest or of uniform motion (constant velocity) unless there is external force acting on it.

This inability of objects to move on its own or change its state of motion is called inertia. Inertia means resistance to change its state. Depending on the circumstances, there can be three types of inertia.

Inertia of rest:

When a stationary bus starts to move, the passengers experience a sudden backward push. Due to inertia, the body (of a passenger) will try to continue in the state of rest, while the bus moves forward. This appears as a backward push as shown in Figure 3.2. The inability of an object to change its state of rest is called inertia of rest.

Inertia of motion: When the bus is in motion, and if the brake is applied suddenly, passengers move forward and hit against the front seat. In this case, the bus comes to a stop, while the body (of a passenger) continues to move forward due to the property of inertia as shown in Figure 3.3. The inability of an object to change its state of uniform speed (constant speed) on its own is called inertia of motion.

Inertia of direction:

When a stone attached to a string is in whirling motion, and if the string is cut suddenly, the stone will not continue to move in circular motion but moves tangential to the circle as illustrated in Figure 3.4. This is because the body cannot change its direction of motion without any force acting on it. The inability of an object to change its direction of motion on its own is called inertia of direction.

When we say that an object is at rest or in motion with constant velocity, it has a meaning only if it is specified with respect to some reference frames. In physics, any motion has to be stated with respect to a reference frame. It is to be noted that Newton's fi rst law is valid only in certain special reference frames called inertial frames. In fact, Newton's first law defines an inertial frame.

Inertial Frames

If an object is free from all forces, then it moves with constant velocity or remains at rest when seen from inertial frames. Thus, there exists some special set of frames in which if an object experiences no force it moves with constant velocity or remains at rest. But how do we know whether an object is experiencing a force or not? All the objects in the Earth experience Earth's gravitational force. In the ideal case, if an object is in deep space (very far away from any other object), then Newton's first law will be certainly valid. Such deep space can be treated as an inertial frame. But practically it is not possible to reach such deep space and verify Newton's first law.

For all practical purposes, we can treat Earth as an inertial frame because an object on the table in the laboratory appears to be at rest always. This object never picks up acceleration in the horizontal direction since no force acts on it in the horizontal direction. So the laboratory can be taken as an inertial frame for all physics experiments and calculations. For making these conclusions, we analyse only the horizontal motion of the object as there is no horizontal force that acts on it. We should not analyse the motion in vertical direction as the two forces (gravitational force in the downward direction and normal force in upward direction) that act on it makes the net force is zero in vertical direction. Newton's first law deals with the motion of objects in the absence of any force and not the motion under zero net force. Suppose a train is moving with constant velocity with respect to an inertial frame, then an object at rest in the inertial frame (outside the train) appears to move with constant velocity with respect to the train (viewed from within the train). So the train can be treated as an inertial frame. All inertial frames are moving with constant velocity relative to each other. If an object appears to be at rest in one inertial frame, it may appear to move with constant velocity with respect to another inertial frame. For example, in Figure 3.5, the car is moving with uniform velocity v with respect to a person standing (at rest) on the ground. As the car is moving with constant velocity with respect to ground to the person is at rest on the ground, both frames (with respect to the car and to the ground) are inertial frames.

Suppose an object remains at rest on a smooth table kept inside the train, and if the train suddenly accelerates (which we may not sense), the object appears to accelerate backwards even without any force acting on it. It is a clear violation of Newton's first law as the object gets accelerated without being acted upon by a force. It implies that the train is not an inertial frame when it is accelerated. For example, Figure 3.6

shows that car 2 is a non-inertial frame since it moves with acceleration a with respect to the ground.

These kinds of accelerated frames are called non-inertial frames. A rotating frame is also a non inertial frame since rotation requires acceleration. In this sense, Earth is not really an inertial frame since it has self-rotation and orbital motion. But these rotational effects of Earth can be ignored for the motion involved in our day-to-day life. For example, when an object is thrown, or the time period of a simple pendulum is measured in the physics laboratory, the Earth's self rotation has very negligible effect on it. In this sense, Earth can be treated as an inertial frame. But at the same time, to analyse the motion of satellites and wind patterns around the Earth, we cannot treat Earth as an inertial frame since its self-rotation has a strong influence on wind patterns and satellite motion.

Newton's Second Law

This law states that

The force acting on an object is equal to the rate of change of its momentum

$$\vec{F} = \frac{d\vec{p}}{dt}$$

In simple words, whenever the momentum of the body changes, there must be a force acting on it. The momentum of the object is defined as $\vec{p} = m\vec{v}$. In most cases, the mass of the object remains constant during the motion. In such cases, the above equation gets modified into a simpler form

$$\vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}.$$

$$\vec{F} = m\vec{a}.$$

The above equation conveys the fact that if there is an acceleration a on the body, then there must be a force acting on it. This implies that if there is a change in velocity, then there must be a force acting on the body. The force and acceleration are always in the same direction. Newton's second law was a paradigm shift from Aristotle's idea of motion. According to Newton, the force need not cause the motion but only a change in motion. It is to be noted that Newton's second law is valid only in inertial frames. In non-inertial frames Newton's second law cannot be used in this form. It requires some modification.

In the SI system of units, the unit of force is measured in newtons and it is denoted by symbol 'N'.

One Newton is defined as the force which acts on 1 kg of mass to give an acceleration 1 m s^{-2} in the direction of the force.

Aristotle vs. Newton's approach on sliding object

Newton's second law gives the correct explanation for the experiment on the inclined plane that was discussed in section 3.1. In normal cases, where friction is not negligible, once the object reaches the bottom of the inclined plane (Figure 3.1), it travels some distance and stops. Note that it stops because there is a frictional force acting in the direction opposite to its velocity. It is this frictional force that reduces the velocity of the object to zero and brings it to rest. As per Aristotle's idea, as soon as the body reaches the bottom of the plane, it can travel only a small distance and stops because there is no force acting on the object. Essentially, he did not consider the frictional force acting on the object.

Newton's Third Law

Consider Figure 3.8(a) whenever an object 1 exerts a force on the object $2(\vec{F}_{21})$, then object 2 must also exert equal and opposite force on the object $1(\vec{F}_{12})$. These forces must lie along the line joining the two objects.

$$\vec{F}_{12} = -\vec{F}_{21}$$

Newton's third law assures that the forces occur as equal and opposite pairs. An isolated force or a single force cannot exist in nature. Newton's third law states that for every action there is an equal and opposite reaction. Here, action and reaction pair of forces do not act on the same body but on two different bodies. Any one of the forces can be called as an action force and the other the reaction force. Newton's third law is valid in both inertial and non-inertial frames.

These action-reaction forces are not cause and effect forces. It means that when the object 1 exerts force on the object 2, the object 2 exerts equal and opposite force on the body 1 at the same instant.

Discussion on Newton's Laws

Newton's laws are vector laws. The equation $\vec{F} = m\vec{a}$ is a vector equation and essentially it is equal to three scalar equations. In Cartesian coordinates, this equation can be written as $F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = ma_x\hat{i} + ma_y\hat{j} + ma_z\hat{k}$. By comparing both sides, the three scalar equations are

$F_x = ma_x$ The acceleration along the x direction depends only on the component of force acting along the x-direction.

The acceleration along the y direction depends only on the component of force acting along the y-direction.

$F_z = ma_z$ The acceleration along the z direction depends only on the component of force acting along the z-direction.

From the above equations, we can infer that the force acting along y direction cannot alter the acceleration along x direction. In the same way, F_z cannot affect a_y and a_x. This understanding is essential for solving problems.

The acceleration experienced by the body at time t depends on the force which acts on the body at that instant of time. It does not depend on the force which acted on the body before the time t. This can be expressed as

$$\vec{F}(t) = m\vec{a}(t)$$

Acceleration of the object does not depend on the previous history of the force. For example, when a spin bowler or a fast bowler throws the ball to the batsman, once the ball leaves the hand of the bowler, it experiences only gravitational force and air frictional force. The acceleration of the ball is independent of how the ball was bowled (with a lower or a higher speed).

In general, the direction of a force may be different from the direction of motion. Though in some cases, the object may move in the same direction as the direction of the force, it is not always true. A few examples are given below.

Case 1: Force and motion in the same direction

When an apple falls towards the Earth, the direction of motion (direction of velocity) of the apple and that of force are in the same downward direction

Case 2: Force and motion not in the same direction

The Moon experiences a force towards the Earth. But it actually moves in elliptical orbit. In this case, the direction of the force is different from the direction of motion

Case 3: Force and motion in opposite direction

If an object is thrown vertically upward, the direction of motion is upward, but gravitational force is downward as

Case 4: Zero net force, but there is motion

When a raindrop gets detached from the cloud it experiences both downward gravitational force and upward air drag force. As it descends towards the Earth, the upward air drag force increases and after a certain time, the upward air drag force cancels the downward gravity. From then on the raindrop moves at constant velocity till it touches the surface of the Earth. Hence the raindrop comes with zero net force, therefore with zero acceleration but with non-zero terminal velocity.

If multiple forces $\vec{F}_1, \vec{F}_2, \vec{F}_3 .... \vec{F}_n$ act on the same body, then the total force $\left(\vec{F}_{net}\right)$ is equivalent to the vectorial sum of the individual forces. Their net force provides the acceleration.

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n$$

Newton's second law for this case is

$$\vec{F}_{net} = m\vec{a}$$

In this case the direction of acceleration is in the direction of net force.

Newton's second law can also be written in the following form. Since the acceleration is the second derivative of position vector of the body $\left(\bar{a} = \frac{d^2\bar{r}}{dt^2}\right)$ the force on the body is

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}.$$

From this expression, we can infer that Newton's second law is basically a second order ordinary differential equation and whenever the second derivative of position vector is not zero, there must be a force acting on the body.

If no force acts on the body then Newton's second law, $m\frac{d\vec{v}}{dt} = 0$.

It implies that $\vec{v}$ constant. It is essentially Newton's first law. It implies that the second law is consistent with the first law. However, it should not be thought of as the reduction of second law to the first when no force acts on the object. Newton's first and second laws are independent laws. They can internally be consistent with each other but cannot be derived from each other.

Newton's second law is cause and effect relation. Force is the cause and acceleration is the effect. Conventionally, the effect should be written on the left and cause on the right hand side of the equation. So the correct way of writing Newton's second law is $m\bar{a} = \overline{F} \; or \; \frac{d\overline{p}}{dt} = \vec{F}$.

APPLICATION OF NEWTON'S LAWS

Free Body Diagram

Free body diagram is a simple tool to analyse the motion of the object using Newton's laws.

The following systematic steps are followed for developing the free body diagram:

1. Identify the forces acting on the object.
2. Represent the object as a point.
3. Draw the vectors representing the forces acting on the object.

When we draw the free body diagram for an object or a system, the forces exerted by the object should not be included in the free body diagram.

EXAMPLE

A book of mass m is at rest on the table. (1) What are the forces acting on the book? (2) What are the forces exerted by the book? (3) Draw the free body diagram for the book.

Solution

There are two forces acting on the book.

I. Gravitational force (mg) acting downwards on the book
II. Normal contact force (N) exerted by the surface of the table on the book. It acts upwards as shown in the figure.

According to Newton's third law, there are two reaction forces exerted by the book.

I. The book exerts an equal and opposite force (mg) on the Earth which acts upwards.
II. The book exerts a force which is equal and opposite to normal force on the surface of the table (N) acting downwards.

EXAMPLE

If two objects of masses 2.5 kg and 100 kg experience the same force 5 N, what is the acceleration experienced by each of them?

Solution

From Newton's second law (in magnitude form), F = ma

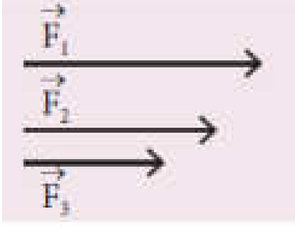
For the object of mass 2.5 kg, the acceleration is $a = \frac{F}{m} = \frac{5}{2.5} = 2\, m\, s^{-2}$

For the object of mass 100 kg, the acceleration is $a = \frac{F}{m} = \frac{5}{100} = 0.05\, m\, s^{-2}$

When an apple falls, it experiences Earth's gravitational force. According to Newton's third law, the apple exerts equal and opposite force on the Earth. Even though both the apple and Earth experience the same force, their acceleration is different. The mass of Earth is enormous compared to that of an apple. So an apple experiences larger acceleration and the Earth experiences almost negligible acceleration. Due to the negligible acceleration, Earth appears to be stationary when an apple falls.

EXAMPLE

Which is the greatest force among the three force $\dot{F}_1, \dot{F}_2, \dot{F}_3$ shown below



Solution

Force is a vector and magnitude of the vector is represented by the length of the vector. Here $\vec{F}_1$ has greater length compared to other two. So $\vec{F}_1$ is largest of the three.

EXAMPLE

Apply Newton's second law to a mango hanging from a tree. (Mass of the mango is 400 gm)

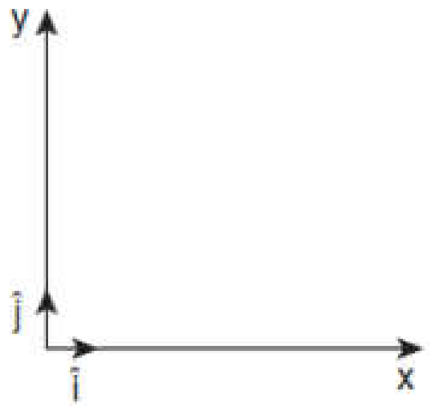
Solution

Note: Before applying Newton's laws, the following steps have to be followed:

1. Choose a suitable inertial coordinate system to analyse the problem. For most of the cases we can take Earth as an inertial coordinate system.
2. Identify the system to which Newton's laws need to be applied. The system can be a single object or more than one object.
3. Draw the free body diagram.
4. Once the forces acting on the system are identified, and the free body diagram is drawn, apply Newton's second law. In the left hand side of the equation, write the forces acting on the system in vector notation and equate it to the right hand side of equation which is the product of mass and acceleration. Here, acceleration should also be in vector notation.
5. If acceleration is given, the force can be calculated. If the force is given, acceleration can be calculated.

By following the above steps:

We fix the inertial coordinate system on the ground as shown in the figure.

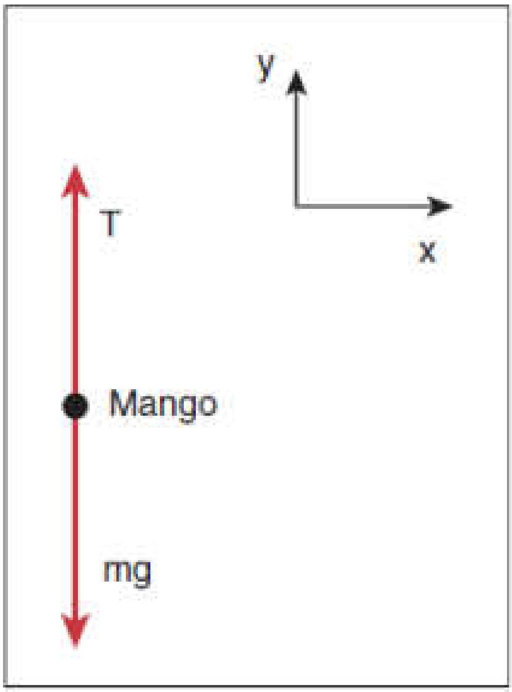


The forces acting on the mango are

1. Gravitational force exerted by the Earth on the mango acting downward along negative y axis

2. Tension (in the cord attached to the mango) acts upward along positive y axis.

The free body diagram for the mango is shown in the figure



$$\vec{F}_g = mg\left(-\hat{j}\right) = -mg\hat{j}$$

Here, mg is the magnitude of the gravitational force and $(-\hat{j})$ represents the unit vector in negative y direction

$$\vec{T} = T\hat{j}$$

Here T is the magnitude of the tension force and $\hat{j}$ represents the unit vector in positive y direction

$$\vec{F}_{net} = \vec{F}_g + \vec{T} = -mg\hat{j} + T\hat{j} = (T - mg)\hat{j}$$

From Newton's second law $\vec{F}_{net} = m\vec{a}$

Since the mango is at rest with respect to us (inertial coordinate system) the acceleration is zero $(\vec{a} = 0)$

So $\vec{F}_{net} = m\vec{a} = 0$

$$(T - mg)\hat{j} = 0$$

By comparing the components on both sides of the above equation, we get T −mg = 0

So the tension force acting on the mango is given by T=mg

Mass of the mango m = 400g and g = 9.8 m s^{-2}

Tension acting on the mango is T = 0.4 × 9.8 = 3.92 N

EXAMPLE

A person rides a bike with a constant velocity $\vec{v}$ with respect to ground and another biker accelerates with acceleration $\vec{a}$ with respect to ground. Who can apply Newton's second law with respect to a stationary observer on the ground?

Solution

Second biker cannot apply Newton's second law, because he is moving with acceleration $\vec{a}$ with respect to Earth (he is not in inertial frame). But the first biker can apply Newton's second law because he is moving at constant velocity with respect to Earth (he is in inertial frame).

EXAMPLE

The position vector of a particle is given by $\vec{r} = 3t\hat{i} + 5t^2\hat{j} + 7\hat{k}$. Find the direction in which the particle experiences net force?

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(3t)\hat{i} + \frac{d}{dt}(5t^2)\hat{j} + \frac{d}{dt}(7)\hat{k}$$

$$\frac{d\vec{r}}{dt} = 3\hat{i} + 10t\hat{j}$$

Acceleration of the particle

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = 10\hat{j}$$

Here, the particle has acceleration only along positive y direction. According to Newton's second law, net force must also act along positive y direction. In addition, the particle has constant velocity in positive x direction and no velocity in z direction. Hence, there are no net force along x or z direction.

EXAMPLE

Consider a bob attached to a string, hanging from a stand. It oscillates as shown in the figure.

Solution

1. Identify the forces that act on the bob?
2. What is the acceleration experienced by the bob?

Two forces act on the bob.

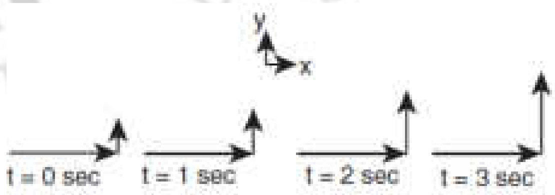
1. Gravitational force (mg) acting downwards
2. Tension (T) exerted by the string on the bob, whose position determines the direction of T as shown in figure.

The bob is moving in a circular arc as shown in the above figure. Hence it has centripetal acceleration. At a point A and C, the bob comes to rest momentarily and then its velocity increases when it moves towards point B. Hence, there is a tangential acceleration along the arc. The gravitational force can be resolved into two components (mg cosθ, mg sinθ) as shown below

EXAMPLE

The velocity of a particle moving in a plane is given by the following diagram. Find out the direction of force acting on the particle?

Solution



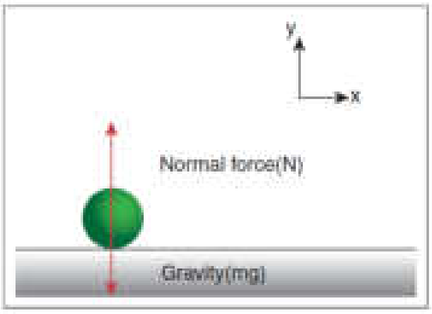
The velocity of the particle is $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ As shown in the figure, the particle is moving in the xy plane, there is no motion in the z direction. So velocity in the z direction is zero (v_z =0). The velocity of the particle has x component (v_x) and y component (v_y). From fi gure, as time increases from t = 0 sec to t = 3 sec, the length of the vector in y direction is changing (increasing). It means y component of velocity (v_y) is increasing with respect to time. According to Newton's second law, if velocity changes with respect to time then there must be acceleration. In this case, the particle has acceleration in the y direction since the y component of velocity changes. So the particle experiences force in the y direction. The length of the vector in x direction does not change. It means that the particle has constant velocity in the x direction. So no force or zero net force acts in the x direction.

EXAMPLE

Apply Newton's second law for an object at rest on Earth and analyse the result.

Solution

The object is at rest with respect to Earth (inertial coordinate system). There are two forces that act on the object.



1. Gravity acting downward (negative y-direction)
2. Normal force by the surface of the Earth acting upward (positive y-direction)

The free body diagram for this object is

$$\vec{F}_g = -mg\hat{j}$$
$$\vec{N} = N\hat{j}$$

Net force $\vec{F}_{net} = -mg\hat{j} + N\hat{j}$

But there is no acceleration on the ball. So $\vec{a} = 0.$. By applying Newton's second law $\left(\vec{F}_{net} = m\vec{a}\right)$

Since $\vec{a} = 0, \vec{F}_{net} = -mg\hat{j} + N\hat{j}$

$$(-mg + N)\hat{j} = 0$$

By comparing the components on both sides of the equation, we get

$$-mg + N = 0$$
$$N = mg$$

We can conclude that if the object is at rest, the magnitude of normal force is exactly equal to the magnitude of gravity.

EXAMPLE

A particle of mass 2 kg experiences two forces $\vec{F}_1 = 5\hat{i} + 8\hat{j} + 7\hat{k}$ and $\vec{F}_2 = 3\hat{i} - 4\hat{j} + 3\hat{k}$ What is the acceleration of the particle?

Solution

We use Newton's second law, $\vec{F}_{net} = m\vec{a}$ where $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$. From the above equations the acceleration is $\vec{a} = \frac{\vec{F}_{net}}{m}$ where

$$\vec{F}_{net} = (5+3)\hat{i} + (8-4)\hat{j} + (7+3)\hat{k}$$

$$\vec{F}_{net} = 8\hat{i} + 4\hat{j} + 10\hat{k}$$

$$\vec{a} = \left(\frac{8}{2}\right)\hat{i} + \left(\frac{4}{2}\right)\hat{j} + \left(\frac{10}{2}\right)\hat{k}$$

$$\vec{a} = 4\hat{i} + 2\hat{j} + 5\hat{k}$$

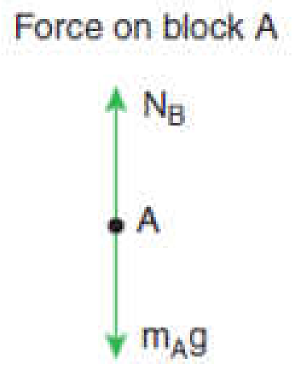
EXAMPLE

Identify the forces acting on blocks A, B and C shown in the figure.

Solution
Forces on block A:

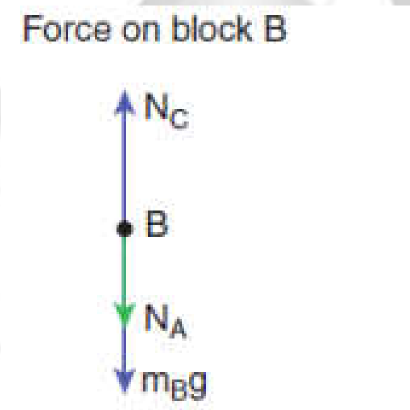
1. Downward gravitational force exerted by the Earth ($m_A g$)
2. Upward normal force (N_B) exerted by block B (N_B)

The free body diagram for block A is as shown in the following picture.



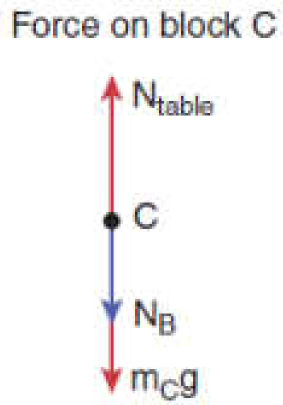
Forces on block B :

1. Downward gravitational force exerted by Earth (m_Bg)
2. Downward force exerted by block A (N_A)
3. Upward normal force exerted by block C (N_C)



Forces on block C:

1. Downward gravitational force exerted by Earth (m_Cg)
2. Downward force exerted by block B (N_B)
3. Upward force exerted by the table (N_{table})



EXAMPLE

Consider a horse attached to the cart which is initially at rest. If the horse starts walking forward, the cart also accelerates in the forward direction. If the horse pulls the cart with force F_h in forward direction, then according to Newton's third law, the cart also pulls the horse by equivalent opposite force $F_c=F_h$ in backward direction. Then total force on 'cart+horse' is zero. Why is it then the 'cart+horse' accelerates and moves forward?

Solution

This paradox arises due to wrong application of Newton's second and third laws. Before applying Newton's laws, we should decide 'what is the system?'. Once we identify the 'system', then it is possible to identify all the forces acting on the system. We should not consider the force exerted by the system. If there is an unbalanced force acting on the system, then it should have acceleration in the direction of the resultant force. By following these steps we will analyse the horse and cart motion.

If we decide on the cart+horse as a 'system', then we should not consider the force exerted by the horse on the cart or the force exerted by cart on the horse. Both are internal forces acting on each other. According to Newton's third law, total internal force acting on the system is zero and it cannot accelerate the system. The acceleration of the system is caused by some external force. In this case, the force exerted by the road on the system is the external force acting on the system. It is wrong to conclude that the total force acting on the system

(cart+horse) is zero without including all the forces acting on the system. The road is pushing the horse
and cart forward with acceleration. As there is an external force acting on the system, Newton's second law has to be applied and not Newton's third law.

The following figures illustrates this.

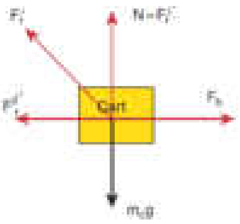
If we consider the horse as the 'system', then there are three forces acting on the horse.

1. Downward gravitational force (m_gh)
2. Force exerted by the road (F_r)
3. Backward force exerted by the cart (F_c)

The force exerted by the road can be resolved into parallel and perpendicular components. The perpendicular component balances the downward gravitational force. There is parallel component along the forward direction. It is greater than the backward force (F_c). So there is net force along the forward direction which causes the forward movement of the horse.

If we take the cart as the system, then there are three forces acting on the cart.

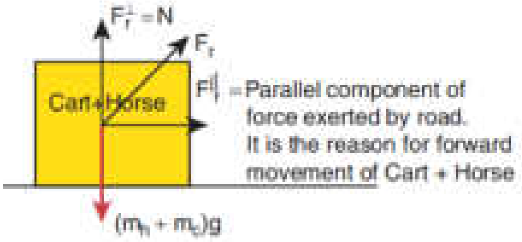
1. Downward gravitational force (m_cg)
2. Force exerted by the road (F_r)
3. Force exerted by the horse (F_h)



The force exerted by the road $\left(\dot{F_r}\right)$ can be resolved into parallel and perpendicular components. The perpendicular component cancels the downward gravity (m_cg). Parallel component acts backwards and the force exerted by the horse $(\dot{F}_h)$ acts forward. Force $(\dot{F}_h)$ is greater than the parallel component acting in the opposite direction. So there is an overall unbalanced force in the forward direction which causes the cart to accelerate forward.

If we take the cart+horse as a system, then there are two forces acting on the system.

1. Downward gravitational force $(m_h+m_c)g$
2. The force exerted by the road (F_r) on the system.



3. In this case the force exerted by the road (F_r) on the system (cart+horse) is resolved in to parallel and perpendicular components. The perpendicular component is the normal force which cancels the downward gravitational force $(m_h+m_c)g$. The parallel component of the force is not balanced, hence the system (cart+horse) accelerates and moves forward due to this force.

The acceleration is given by $a = \frac{d^2 y}{dt^2}$

$$a = \frac{dv}{dt}$$

v =velocity of the particle in y direction

$$v = \frac{dy}{dt} = u - gt$$

The momentum of the particle = mv = m (u-gt).

$$a=\frac{dv}{dt}=-g$$

The force acting on the object is given by F =ma = −mg

The negative sign implies that the force is acting on the negative y direction. This is exactly the force that acts on the object in projectile motion.

Particle Moving in an Inclined Plane

When an object of mass m slides on a frictionless surface inclined at an angleθ as shown in the Figure 3.12, the forces acting on it decides the

1. acceleration of the object
2. speed of the object when it reaches the bottom

The force acting on the object is

1. Downward gravitational force (mg)
2. Normal force perpendicular to inclined surface (N)

To draw the free body diagram, the block is assumed to be a point mass (Figure 3.13 (a)). Since the motion is on the inclined surface, we have to choose the coordinate system parallel to the inclined surface.

The gravitational force mg is resolved in to parallel component mg sinθ along the inclined plane and perpendicular component mg cosθ perpendicular to the inclined surface.

Note that the angle made by the gravitational force (mg) with the perpendicular to the surface is equal to the angle of inclination θ.

There is no motion(acceleration) along the y axis. Applying Newton's second law in the y direction

$$-mg\cos\theta\hat{j}+N\hat{j}=0\left(No\ acceleration\right)$$

By comparing the components on both sides, N −mg cosθ = 0

$$N = mg\cos\theta$$

The magnitude of normal force (N) exerted by the surface is equivalent to mg cosθ .

The object slides (with an acceleration) along the x direction. Applying Newton's\ second law in the x direction

$$mg\sin\theta\hat{i} = ma\hat{i}$$

By comparing the components on both sides, we can equate

$$mg\sin\theta = ma$$

The acceleration of the sliding object is

$$a = g\sin\theta$$

Note that the acceleration depends on the angle of inclination θ . If the angle θ is 90 degree, the block will move vertically with acceleration a = g.

Newton's kinematic equation is used to find the speed of the object when it reaches the bottom. The acceleration is constant throughout the motion.

$$v^2 = u^2 + 2as \text{ along the x direction}$$

The acceleration a is equal to g sinθ. The initial speed (u) is equal to zero as it starts from rest. Here s is the length of the inclined surface.

The speed (v) when it reaches the bottom is (using equation (3.3))

$$v = \sqrt{2sg\ sin\theta}$$

Two Bodies in Contact on a Horizontal Surface

Consider two blocks of masses m1 and m2 (m1> m2) kept in contact with each other on a smooth, horizontal frictionless surface as shown in Figure 3.14.

By the application of a horizontal force F, both the blocks are set into motion with acceleration 'a' simultaneously in the direction of the force F.

To find the acceleration a, Newton's second law has to be applied to the system (combined mass m = m1 + m2)

$$\vec{F} = m\vec{a}$$

If we choose the motion of the two masses along the positive x direction,

$$F\hat{i} = ma\hat{i}$$

By comparing components on both sides of the above equation

$$F = \mathrm{ma} \quad \text{where } \mathrm{m} = \mathrm{m}_1 + \mathrm{m}_2$$

The acceleration of the system is given by

$$\therefore a = \frac{F}{\mathrm{m}_1 + \mathrm{m}_2}$$

The force exerted by the block m1 on m_2 due to its motion is called force of contact $\left(\vec{f}_{21}\right)$. According to Newton's third law, the block m_2 will exert an equivalent opposite reaction force$\left(\vec{f}_{12}\right)$ on block m_1.

$$\therefore F\hat{i} - f_{12}\hat{i} = m_1 a\hat{i}$$

By comparing the components on both sides of the above equation, we get

$$F - f_{12} = m_1 a$$

$$f_{12} = F - m_1 a$$

Substituting the value of acceleration from equation

$$f_{12} = F - m_1\left(\frac{F}{m_1 + m_2}\right)$$

$$f_{12} = F\left[1 - \frac{m_1}{m_1 + m_2}\right]$$

$$f_{12} = \frac{Fm_2}{m_1 + m_2}$$

Equation (3.7) shows that the magnitude of contact force depends on mass m_2 which provides the reaction force. Note that this force is acting along the negative x direction.

In vector notation, the reaction force on mass m_1 is given by $\dot{f}_{12} = -\frac{Fm_2}{m_1 + m_2}$

For mass m_2 there is only one force acting on it in the x direction and it is denoted by $\dot{f}_{21}$. This force is exerted by mass m1. The free body diagram for mass m_2

Applying Newton's second law for mass m_2

$$f_{21}\hat{i} = m_2 a\hat{i}$$

By comparing the components on both sides of the above equation

$$f_{21} = m_2 a$$

Substituting for acceleration from equation (3.5) in equation (3.8), we get

$$f_{21} = \frac{Fm_2}{m_1 + m_2}$$

In this case the magnitude of the contact force is

$$f_{21} = \frac{Fm_2}{m_1 + m_2}$$

The direction of this force is along the positive x direction.

In vector notation, the force acting on mass m2exerted by mass $m_1$is $\vec{f}_{21} = \frac{Fm_2}{m_1 + m_2}$

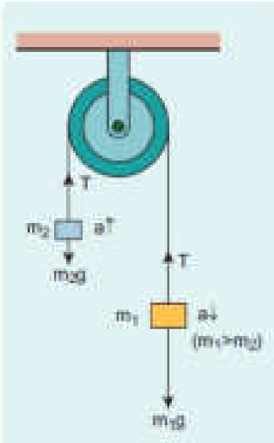
Note $\dot{f}_{12} = -\dot{f}_{21}$ which confirms Newton's third law.

Motion of Connected Bodies

When objects are connected by strings and a force F is applied either vertically or horizontally or along an inclined plane, it produces a tension T in the string, which affects the acceleration to an extent. Let us discuss various cases for the same.

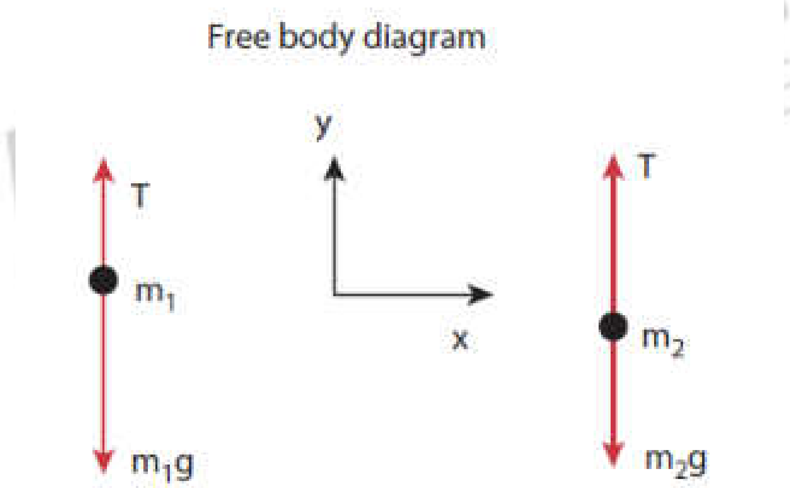
Case 1: Vertical motion

Consider two blocks of masses m_1 and m_2 ($m_1 > m_2$) connected by a light and inextensible string that passes over a pulley as shown in Figure



Let the tension in the string be T and acceleration a. When the system is released, both the blocks start moving, m_2 vertically upward and m_1 downward with same acceleration a. The gravitational force m1g on mass m1 is used in lifting the mass m_2.

The upward direction is chosen as y direction.



Applying Newton's second law for mass m2

$$T\hat{j} - m_2 g\hat{j} = m_2 a\hat{j}$$

The left hand side of the above equation is the total force that acts on m_2 and the right hand side is the product of mass and acceleration of m2 in y direction.

By comparing the components on both sides, we get

$$T - m_2 g = m_2 a$$

Similarly, applying Newton's second law for mass m_1

$$T\hat{j} - m_1 g\hat{j} = -m_1 a\hat{j}$$

As mass m_1 moves downward $(-\hat{j})$ its acceleration is along $(-\hat{j})$ By comparing the components on both sides, we get

$$T - m_1 g = -m_1 a$$

$$m_1 g - T = m_1 a$$

$$m_1 g - m_2 g = m_1 a + m_2 a$$

$$(m_1 - m_2) g = (m_1 + m_2) a$$

From equation (3.11), the acceleration of both the masses is

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

If both the masses are equal (m1=m2), from equation

$$a=0$$

This shows that if the masses are equal, there is no acceleration and the system as a whole will be at rest.

To find the tension acting on the string, substitute the acceleration from the equation (3.12) into the equation (3.9).

$$T - m_2 g = m_2\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

$$T = m_2 g + m_2\left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

By taking m2g common in the RHS of equation (3.13)

$$T = m_2 g\left(1 + \frac{m_1 - m_2}{m_1 + m_2}\right)$$

$$T = m_2 g\left(\frac{m_1 + m_2 + m_1 - m_2}{m_1 + m_2}\right)$$

$$T = \left(\frac{2m_1 m_2}{m_1 + m_2}\right)g$$

Equation (3.12) gives only magnitude of acceleration.

For mass m1, the acceleration vector is given by $\bar{a} = -\left(\frac{m_1 - m_2}{m_1 + m_2}\right)\hat{j}$

For mass m2, the acceleration vector is given b $\bar{a} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\hat{j}$

Case 2: Horizontal motion

In this case, mass m2 is kept on a horizontal table and mass m1 is hanging through a small pulley as shown in Figure 3.17. Assume that there is no friction on the surface.

As both the blocks are connected to the unstretchable string, if m1 moves with an acceleration a downward then m2 also moves with the same acceleration a horizontally.

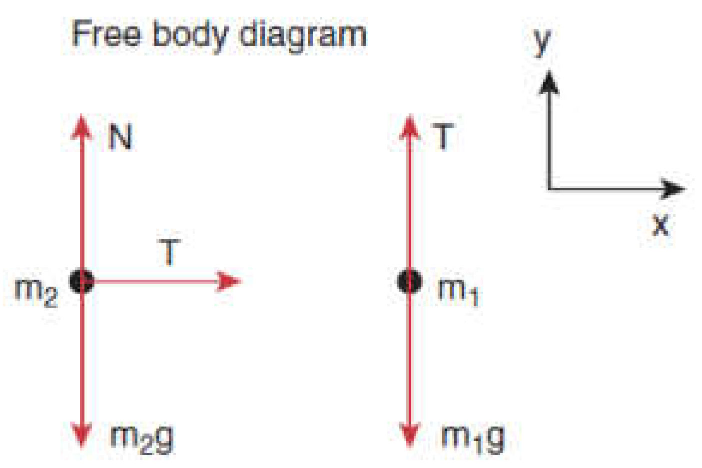
The forces acting on mass m_2 are

1. Downward gravitational force (m_2g)
2. Upward normal force (N) exerted by the surface
3. Horizontal tension (T) exerted by the string

The forces acting on mass m_1 are

1. Downward gravitational force (m_1g)
2. Tension (T) acting upwards

The free body diagrams for both the masses



Applying Newton's second law for m_1

$$T\hat{j} - m_1g\hat{j} = -m_1a\hat{j}$$

By comparing the components on both sides of the above equation

$$T - m_1g = -m_1a$$

Applying Newton's second law for m_2

$$T\hat{i} = m_2a\hat{i}$$

By comparing the components on both sides of above equation,

$$T = m_2 a$$

There is no acceleration along y direction for m_2.

$$N\hat{j} - m_2 g\hat{j} = 0$$

By comparing the components on both sides of the above equation

$$N - m_2 g = 0$$

$$N = m_2 g$$

By substituting equation (3.15) in equation (3.14), we can find the tension T

$$m_2 a - m_1 g = -m_1 a$$

$$m_2 a + m_1 a = m_1 g$$

$$a = \frac{m_1}{m_1 + m_2} g$$

Tension in the string can be obtained by substituting equation (3.17) in equation (3.15)

$$T = \frac{m_1 m_2}{m_1 + m_2} g$$

Comparing motion in both cases, it is clear that the tension in the string for horizontal motion is half of the tension for vertical motion for same set of masses and strings.

This result has an important application in industries. The ropes used in conveyor belts (horizontal motion) work for longer duration than those of cranes and lifts (vertical motion).

Concurrent Forces and Lami's Theorem

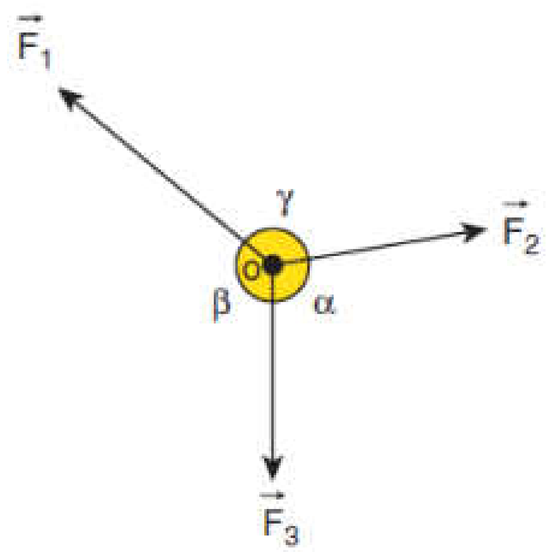
A collection of forces is said to be concurrent, if the lines of forces act at a common point. Figure 3.19 illustrates concurrent forces.

Concurrent forces need not be in the same plane. If they are in the same plane, they are concurrent as well as coplanar forces.

LAMI'S THEOREM

If a system of three concurrent and coplanar forces is in equilibrium, then Lami's theorem states that the magnitude of each force of the system is proportional to sine of the angle between the other two forces. The constant of proportionality is same for all three forces.

Let us consider three coplanar and concurrent forces $\vec{F}_1, \vec{F}_2$ *and* $\vec{F}_3$ which act at a common point O as shown in Figure 3.20. If the point is at equilibrium, then according to Lami's theorem



$$\left|\vec{F}_1\right| \propto \sin\alpha$$

$$\left|\vec{F}_2\right| \propto \sin\beta$$

$$\left|\vec{F}_3\right| \propto \sin\gamma$$

Therefore, $$\frac{\left|\vec{F}_1\right|}{\sin\alpha} = \frac{\left|\vec{F}_2\right|}{\sin\beta} = \frac{\left|\vec{F}_3\right|}{\sin\gamma}$$

Lami's theorem is useful to analyse the forces acting on objects which are in static equilibrium.

Application of Lami's Theorem

EXAMPLE

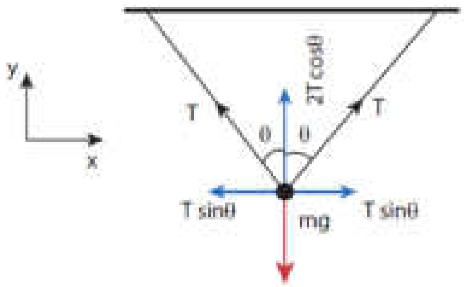
A baby is playing in a swing which is hanging with the help of two identical chains is at rest. Identify the forces acting on the baby. Apply Lami's theorem and find out the tension acting on the chain.

Solution

The baby and the chains are modeled as a particle hung by two strings as shown in the figure. There are three forces acting on the baby.

1. Downward gravitational force along negative y direction (mg)
2. Tension (T) along the two strings

These three forces are coplanar as well as concurrent as shown in the following figure.



$$\frac{T}{\sin(180-\theta)} = \frac{T}{\sin(180-\theta)} = \frac{mg}{\sin(2\theta)}$$

Since $\sin(180-\theta) = \sin\theta$ and $\sin(2\theta) = 2\sin\theta\cos\theta$

$$\frac{T}{\sin\theta} = \frac{mg}{2\sin\theta\cos\theta}$$

From this, the tension on each string is $T = \frac{mg}{2\cos\theta}$.

LAW OF CONSERVATION OF TOTAL LINEAR MOMENTUM

In nature, conservation laws play a very important role. The dynamics of motion of bodies can be analysed very effectively using conservation laws. There are three conservation laws in mechanics. Conservation of total energy, conservation of total linear momentum, and conservation of angular momentum. By combining Newton's second and third laws, we can derive the law of conservation of total linear momentum.

When two particles interact with each other, they exert equal and opposite forces on each other. The particle 1 exerts force $\vec{F}_{21}$ on particle 2 and particle 2 exerts an exactly equal and opposite force $\vec{F}_{12}$ on particle 1, according to Newton's third law.

$$\vec{F}_{21} = -\vec{F}_{12}$$

In terms of momentum of particles, the force on each particle (Newton's second law) can be written as

$$\vec{F}_{12} = \frac{d\vec{p}_1}{dt} \quad \text{and} \quad \vec{F}_{21} = \frac{d\vec{p}_2}{dt}.$$

Here $\vec{p}_1$ is the momentum of particle 1 which changes due to the force $\vec{F}_{12}$ exerted by particle 2. Further $\vec{p}_2$ is the momentum of particle 2. This changes due to $\vec{F}_{21}$ exerted by particle 1.

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0$$

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

It implies that $\vec{p}_1 + \vec{p}_2$ =constant vector (always).

$\vec{p}_1 + \vec{p}_2$ is the total linear momentum of the two particles $\left(\vec{P}_{tot} = \vec{p}_1 + \vec{p}_2\right)$. It is also called as total linear momentum of the system. Here, the two particles constitute the system. From this result, the law of conservation of linear momentum can be stated as follows.

If there are no external forces acting on the system, then the total linear momentum of the system ($\vec{P}_{tot}$)is always a constant vector. In other words, the total linear momentum of the system is conserved in time. Here the word 'conserve' means that $\vec{p}_1$ and $\vec{p}_2$ can vary, in such a way that $\vec{p}_1 + \vec{p}_2$ is a constant vector.

The forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are called the internal forces of the system, because they act only between the two particles. There is no external force acting on the two particles from outside. In such a case the total linear momentum of the system is a constant vector or is conserved.

EXAMPLE

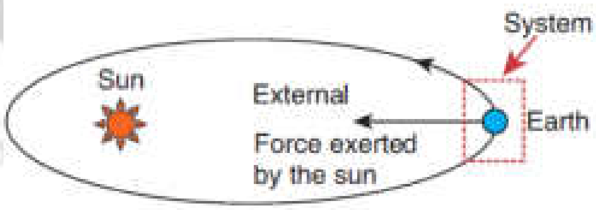
Identify the internal and external forces acting on the following systems.

1. Earth alone as a system
2. Earth and Sun as a system
3. Our body as a system while walking
4. Our body + Earth as a system

Solution

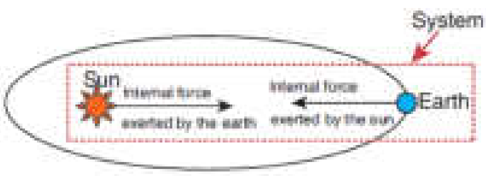
Earth alone as a system

Earth orbits the Sun due to gravitational attraction of the Sun. If we consider Earth as a system, then Sun's gravitational force is an external force. If we take the Moon into account, it also exerts an external force on Earth.



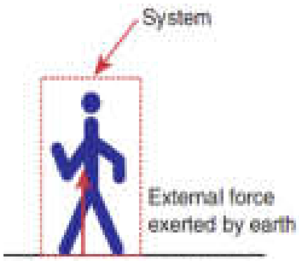
(Earth + Sun) as a system

In this case, there are two internal forces which form an action and reaction pair the gravitational force exerted by the Sun on Earth and gravitational force exerted by the Earth on the Sun.



Our body as a system

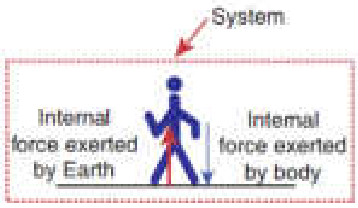
While walking, we exert a force on the Earth and Earth exerts an equal and opposite force on our body. If our body alone is considered as a system, then the force exerted by the Earth on our body is external.



Our body as a system

(Our body + Earth) as a system

In this case, there are two internal forces present in the system. One is the force exerted by our body on the Earth and the other is the equal and opposite force exerted by the Earth on our body.



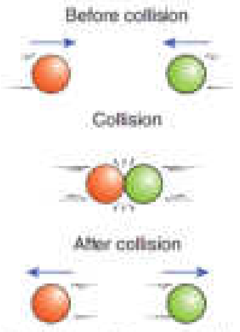
Our body + Earth as a system

Meaning of law of conservation of momentum

The Law of conservation of linear momentum is a vector law. It implies that both the magnitude and direction of total linear momentum are constant. In some cases, this total momentum can also be zero.

To analyse the motion of a particle, we can either use Newton's second law or the law of conservation of linear momentum. Newton's second law requires us to specify the forces involved in the process. This is difficult to specify in real situations. But conservation of linear momentum does not require any force involved in the process. It is covenient and hence important.

For example, when two particles collide, the forces exerted by these two particles on each other is difficult to specify. But it is easier to apply conservation of linear momentum during the collision process.



Collision of two particles

Examples

Consider the firing of a gun. Here the system is Gun+bullet. Initially the gun and bullet are at rest, hence the total linear momentum of the system is zero. Let $\vec{p}_1$ be the momentum of the bullet and $\vec{p}_2$ momentum of the gun before firing. Since initially both are at rest,

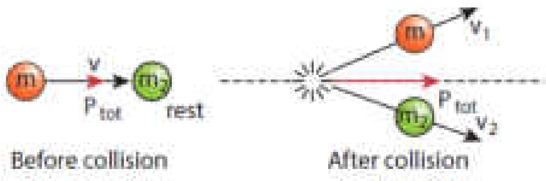
$$\vec{p}_1 = 0, \vec{p}_2 = 0.$$

Total momentum before fi ring the gun is zero, $\vec{p}_1 + \vec{p}_2 = 0.$

According to the law of conservation of linear momentum, total linear momemtum has to be zero after the fi ring also.

When the gun is fi red, a force is exerted by the gun on the bullet in forward direction. Now the momentum of the bullet changes from $\vec{p}_1$ to $\vec{p}_1$. To conserve the total linear momentum of the system, the momentum of the gun must also change from $\vec{p}_2$ to $\vec{p}_2$. Due to the conservation of linear momentum, $\vec{p}_1' + \vec{p}_2' = 0.$ It implies that $\vec{p}_1' = -\vec{p}_2'$, the momentum of the gun is exactly equal, but in the opposite direction to the momentum of the bullet. This is the reason after firing, the gun suddenly moves backward with the momentum $\left(-\vec{p}_2\right)$. It is called 'recoil momemtum'. This is an example of conservation of total linear momentum.

Consider two particles. One is at rest and the other moves towards the first particle (which is at rest). They collide and after collison move in some arbitrary directions. In this case, before collision, the total linear momentum of the system is equal to the initial linear momentum of the moving particle. According to conservation of momentum, the total linear momentum after collision also has to be in the forward direction. The following figure explains this.



A more accurate calculation is covered in section 4.4. It is to be noted that the total momentum vector before and after collison points in the same direction. This simply means that the total linear momentum is constant before and after the collision. At the time of collision, each particle exerts a force on the other. As the two particles are considered as a system, these forces are only internal, and the total linear momentum cannot be altered by internal forces.

Impulse

If a very large force acts on an object for a very short duration, then the force is called impulsive force or impulse.

If a force (F) acts on the object in a very short interval of time (Δt), from Newton's second law in magnitude form

$$Fdt = dp$$

Integrating over time from an initial time t_i to a final time t_f, we get

$$\int_{i}^{f} dp = \int_{t_i}^{t_f} F\,dt$$

$$p_f - p_i = \int_{t_i}^{t_f} F\,dt$$

p_i= initial momentum of the object at time t_i
p_f=final momentum of the object at time t_f

$p_f - p_i = \Delta p =$ change in momentum of the object during the time interval $t_f - t_i = \Delta t.$

The integral $\int_{t_i}^{t_f} F\,dt = J$ is called the impulse and it is equal to change in momentum of the object.

If the force is constant over the time interval, then

$$\int_{t_i}^{t_f} F\,dt = F\int_{t_i}^{t_f} dt = F\left(t_f - t_i\right) = F\Delta t$$

$$F\Delta t = \Delta p$$

For a constant force, the impulse is denoted as J = FΔt and it is also equal to change in momentum ('p) of the object over the time interval 't.

Impulse is a vector quantity and its unit is Ns.

The average force acted on the object over the short interval of time is defined by

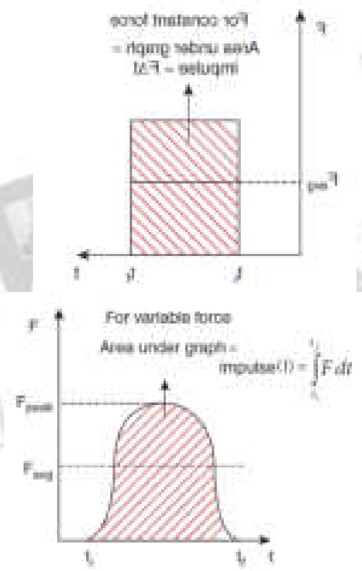
$$F_{avg} = \frac{\Delta p}{\Delta t}$$

From equation (3.25), the average force that act on the object is greater if 't is smaller. Whenever the momentum of the body changes very quickly, the average force becomes larger.

The impulse can also be written in terms of the average force. Since 'p is change in momentum of the object and is equal to impulse (J), we have

$$J = F_{avg}\Delta t$$

The graphical representation of constant force impulse and variable force impulse.



Illustration

When a cricket player catches the ball, he pulls his hands gradually in the direction of the ball’s motion. Why?

If he stops his hands soon after catching the ball, the ball comes to rest very quickly. It means that the momentum of the ball is brought to rest very quickly. So the average force acting on the body will be very large. Due to this large average force, the hands will get hurt. To avoid getting hurt, the player brings the ball to rest slowly.

When a car meets with an accident, its momentum reduces drastically in a very short time. This is very dangerous for the passengers inside the car since they will experience a large force. To prevent this fatal shock, cars are designed with air bags in such a way that when the car meets with an accident, the momentum of the passengers will reduce slowly so that the average force acting on them will be smaller.

The shock absorbers in two wheelers play the same role as airbags in the car. When there is a bump on the road, a sudden force is transferred to the vehicle. The shock absorber prolongs the period of transfer of force on to the body of the rider. Vehicles without shock absorbers will harm the body due to this reason.

Jumping on a concrete cemented floor is more dangerous than jumping on the sand. Sand brings the body to rest slowly than the concrete floor, so that the average force experienced by the body will be lesser.

EXAMPLE

An object of mass 10 kg moving with a speed of 15 m s^{-1} hits the wall and comes to rest within

1. 0.03 second
2. 10 second

Calculate the impulse and average force acting on the object in both the cases.

Solution

Initial momentum of the object p_i =10 × 15 =150 k gm s^{-1}

Final momentum of the object p_f =0

$$\Delta p = 150 - 0 = 150\ kg\ ms^{-1}$$

Impulse J = Δp = 150N s.
Impulse J = Δp = 150N s

$$\text{Average force } F_{avg} = \frac{\Delta p}{\Delta t} = \frac{150}{0.03} = 5000\,N$$

$$\text{Average force } F_{avg} = \frac{150}{10} = 15\,N$$

We see that, impulse is the same in both cases, but the average force is different.

FRICTION

Introduction

If a very gentle force in the horizontal direction is given to an object at rest on the table, it does not move. It is because of the opposing force exerted by the surface on the object which resists its motion. This force is called the frictional force which always opposes the relative motion between an object and the surface where it is placed. If the force applied is increased, the object moves after a certain limit.

Relative motion: when a force parallel to the surface is applied on the object, the force tries to move the object with respect to the surface. This 'relative motion' is opposed by the surface by exerting a frictional force on the object in a direction opposite to applied force. Frictional force always acts on the object parallel to the surface on which the object is placed. There are two kinds of friction namely 1) Static friction and 2) Kinetic friction.

Static Friction $\left(\vec{f}_s\right)$

Static friction is the force which opposes the initiation of motion of an object on the surface. When the object is at rest on the surface, only two forces act on it. They are the downward gravitational force and upward normal force. The resultant of these two forces on the object is zero. As a result the object is at rest as shown in Figure 3.23

some external force F_{ext} is applied on the object parallel to the surface on which the object is at rest, the surface exerts exactly an equal and opposite force on the object to resist its motion and tries to keep the object at rest. It implies that external force and frictional force are exactly equal and opposite. Therefore, no motion parallel to the surface takes place. But if the external force is increased, after a particular limit, the surface cannot provide sufficient opposing frictional force to balance the external force on the object. Then the object starts to slide. This is the maximal static friction that can be exerted by the surface. Experimentally, it is found that the magnitude of static frictional force f_s satisfies the following empirical relation.

$$0 \leq f_s \leq \mu_s N,$$

where μs is the coefficient of static friction. It depends on the nature of the surfaces in contact. N is normal force exerted by the surface on the body and sometimes it is equal to mg. But it need not be equal to mg always.

Equation (3.27) implies that the force of static friction can take any value from zero to $\mu_s N$.

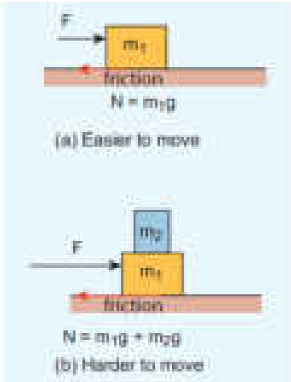
If the object is at rest and no external force is applied on the object, the static friction acting on the object is zero (f_s=0).

If the object is at rest, and there is an external force applied parallel to the surface, then the force of static friction acting on the object is exactly equal to the external force applied on the object $(f_s = F_{ext})$. But still the static friction f_s is less than $\mu_s N$.

When object begins to slide, the static friction (f_s) acting on the object attains maximum,

The static and kinetic frictions (which we discuss later) depend on the normal force acting on the object. If the object is pressed hard on the surface then the normal force acting on the object will increase. As a consequence it is more difficult to move the object. This is shown in

Figure 3.23 (a) and (b). The static friction does not depend upon the area of contact.



EXAMPLE

Consider an object of mass 2 kg resting on the floor. The coefficient of static friction between the object and the floor is μ_s = 0.8. What force must be applied on the object to move it?

Solution

Since the object is at rest, the gravitational force experienced by an object is balanced by normal force exerted by floor.

$$N = mg$$

The maximum static frictional force $f_s^{max} = \mu_s N = \mu_s mg$

$$f_s^{max} = 0.8 \times 2 \times 9.8 = 15.68\,N$$

Therefore to move the object the external force should be greater than maximum static friction.

$$F_{ext} > 15.68\,N$$

EXAMPLE

Consider an object of mass 50 kg at rest on the floor. A Force of 5 N is applied on the object but it does not move. What is the frictional force that acts on the object?

Solution

When the object is at rest, the external force and the static frictional force are equal and opposite

The magnitudes of these two forces are equal, $f_s = F_{ext}$

Therefore, the static frictional force acting on the object is

$$f_s = 5N .$$

The direction of this frictional force is opposite to the direction of F_{ext}.

EXAMPLE

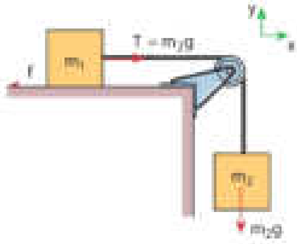
Two bodies of masses 7 kg and 5 kg are connected by a light string passing over a smooth pulley at the edge of the table as shown in the figure. The coefficient of static friction between the surfaces (body and table) is 0.9. Will the mass m_1 = 7 kg on the surface move? If not what value of m_2 should be used so that mass 7 kg begins to slide on the table?

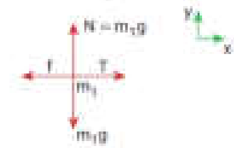
Solution

As shown in the figure, there are four forces acting on the mass m_1

1. Downward gravitational force along the negative y-axis (m_1g)
2. Upward normal force along the positive y axis (N)
3. Tension force due to mass m2 along the positive x axis
4. Frictional force along the negative x axis

Since the mass m_1 has no vertical motion, m_1g = N





To determine whether the mass m_1 moves on the surface, calculate the maximum static friction exerted by the table on the mass m1. If the tension on the mass m_1 is equal to or greater than this maximum static friction, the object will move.

$$f_s^{max} = \mu_s N = \mu_s m_1 g$$

$$f_s^{max} = 0.9X7X9.8 = 61.74\ N$$

$$T = m_2 g = 5X9.8 = 49\ N$$

$$T < f_s^{max}$$

The tension acting on the mass m1 is less than the maximum static friction. So the mass m1 will not move.

To move the mass m_1, T> f_s^{max} where T = m_2g

$$m_2 = \frac{\mu_s m_1 g}{g} = \mu_s m_1$$

$$m_2 = 0.9X7 = 6.3\ kg$$

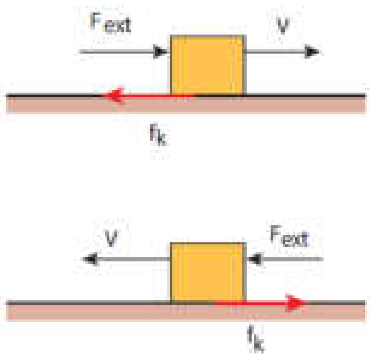
If the mass m_2 is greater than 6.3 kg then the mass m1 will begin to slide. Note that if there is no friction on the surface, the mass m_1 will move for m_2 even for just 1 kg.

The values of coefficient of static friction for pairs of materials are presented in Table 3.1. Note that the ice and ice pair have very low coefficient of static friction. This means a block of ice can move easily over another block of ice.

Material	Coefficient of Static Friction
Glass and glass	1.0
Ice and ice	0.10
Steel and steel	0.75
Wood and wood	0.35
Rubber tyre and dry concrete road	1.0
Rubber tyre and wet road	0.7

Kinetic Friction

If the external force acting on the object is greater than maximum static friction, the objects begin to slide. When an object slides, the surface exerts a frictional force called kinetic friction $\dot{f}_k$ (also called sliding friction or dynamic friction). To move an object at constant velocity we must apply a force which is equal in magnitude and opposite to the direction of kinetic friction.



Experimentally it was found that the magnitude of kinetic friction satisfies the relation

$$f_k = \mu_k N$$

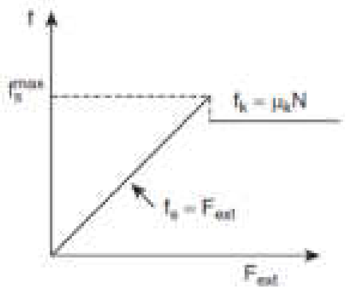
where μ_k is the coefficient of kinetic friction and N the normal force exerted by the surface on the object,

$$\mu_k < \mu_s$$

This implies that starting of a motion is more difficult than maintaining it. The salient features of static and kinetic friction

Static friction	Kinetic friction
It opposes the starting of motion	It opposes the relative motion of the object with respect to the surface
Independent of surface of contact	Independent of surface of contact
μ_s depends on the nature of materials in mutual contact	μ_k depends on nature of materials and temperature of the surface
Depends on the magnitude of applied force	Independent of magnitude of applied force
It can take values from zero to $\mu_s N$	It can never be zero and always equals to $\mu_k N$ whatever be the speed (true <10 ms^{-1})
$f_s^{max} > f_k$	It is less than maximal value of static friction
$\mu_s > \mu_k$	Coefficient of kinetic friction is less than coefficient of static friction

The variation of both static and kinetic frictional forces with external applied force



The Figure 3.25 shows that static friction increases linearly with external applied force till it reaches the maximum. If the object begins to move then the kinetic friction is slightly lesser than the maximum static friction. Note that the kinetic friction is constant and it is independent of applied force.

To Move an Object - Push or pull? Which is easier?

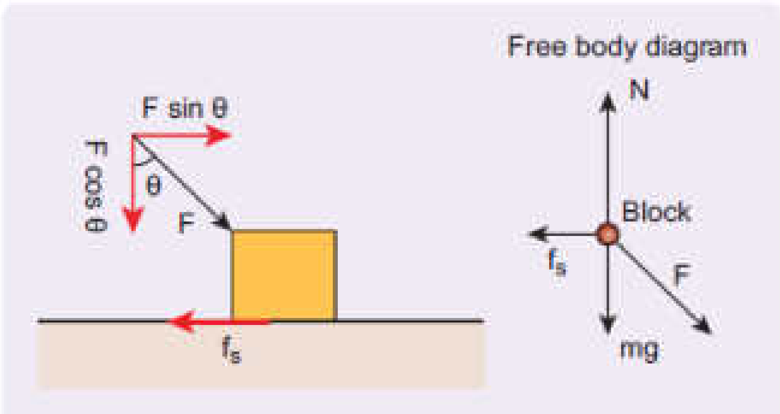
When a body is pushed at an arbitrary angle θ $\left(0 \; to \; \frac{\pi}{2}\right)$ the applied force F can be resolved into two components as F sinθ parallel to the surface and F cosθ perpendicular to the surface as shown in Figure 3.26. Th e total downward force acting on the body is mg + Fcosθ. It implies that the normal force acting on the body increases. Since there is no acceleration along the vertical direction the normal force N is equal to

$$N_{push} = mg + F\cos\theta$$

As a result the maximal static friction also increases and is equal to

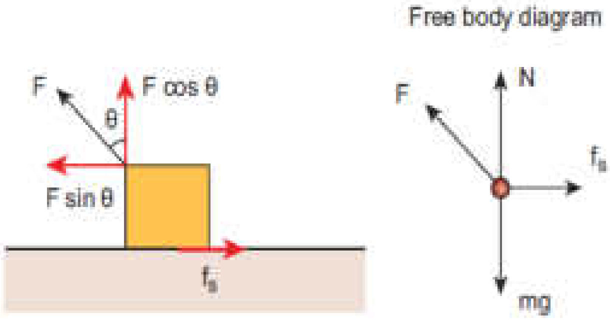
$$f_s^{max} = \mu_s N_{push} = \mu_s\left(mg + F\cos\theta\right)$$

Equation (3.30) shows that a greater force needs to be applied to push the object into motion.



When an object is pulled at an angle θ, the applied force is resolved into two components as shown in Figure 3.27 The total downward force acting on the object is

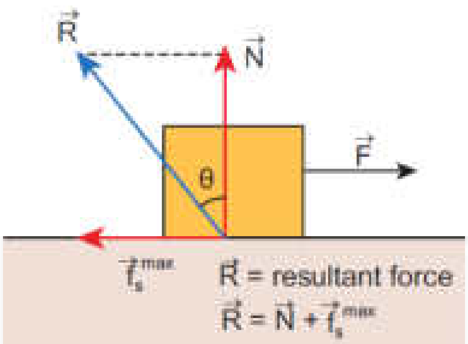
$$N_{pull} = mg - F\cos\theta$$



Equation (3.31) shows that the normal force is less than N_{push}. From equations (3.29) and (3.31), it is easier to pull an object than to push to make it move.

Angle of Friction

The angle of friction is defined as the angle between the normal force (N) and the resultant force (R) of normal force and maximum friction force $f_s^{\max}$



In Figure 3.28 the resultant force is

$$R = \sqrt{\left(f_s^{max}\right)^2 + N^2}$$

$$\tan\theta = \frac{f_s^{max}}{N}$$

But from the frictional relation, the object begins to slide when $f_s^{\max} = \mu_s N$

$$\text{or when } \frac{f_s^{max}}{N} = \mu_s$$

From equations (3.32) and (3.33) the coefficient of static friction is

$$\mu_s = \tan\theta$$

Angle of Repose

Consider an inclined plane on which an object is placed, as shown in Figure 3.30. Let the angle which this plane makes with the horizontal be θ. For small angles of θ , the object may not slide down. As θ is increased, for a particular value of θ, the object begins to slide down. This value is called angle of repose. Hence, the angle of repose is the angle of inclined plane with the horizontal such that an object placed on it begins to slide.

Let us consider the various forces in action here. The gravitational force mg is resolved into components parallel (mg $\sin\theta$) and perpendicular (mg $\cos\theta$) to the inclined plane.

The component of force parallel to the inclined plane (mg $\sin\theta$) tries to move the object down.

The component of force perpendicular to the inclined plane (mg $\cos\theta$) is balanced by the Normal force (N).

$$N = mg\cos\theta$$

When the object just begins to move, the static friction attains its maximum value

$$f_s = f_s^{max} = \mu_s N = \mu_s mg\cos\theta$$

This friction also satisfies the relation

$$f_s^{max} = mg\sin\theta$$

Equating the right hand side of equations (3.35) and (3.36), we get

$$\mu_s = sin\theta / cos\theta$$

From the definition of angle of friction, we also know that in which θ is the angle of friction.

$$\tan\theta = \mu_s,$$

Thus the angle of repose is the same as angle of friction. But the difference is that the angle of repose refers to inclined surfaces and the angle of friction is applicable to any type of surface

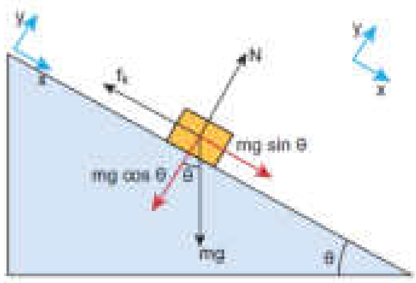
EXAMPLE

A block of mass m slides down the plane inclined at an angle 60° with an acceleration $\frac{g}{2}$. Find the coefficient of kinetic friction?

Solution

Kinetic friction comes to play as the block is moving on the surface.

The forces acting on the mass are the normal force perpendicular to surface, downward gravitational force and kinetic friction f_k along the surface



$$mg\sin\theta - f_k = \text{ma}$$

But a = g/2

$$mg\sin 60^\circ - f_k = \text{mg}/2$$

$$\frac{\sqrt{3}}{2}\text{mg} - f_k = \text{mg}/2$$

$$f_k = mg\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

$$f_k = \left(\frac{\sqrt{3}-1}{2}\right)\text{mg}$$

There is no motion along the y-direction as normal force is exactly balanced by the mg cos θ.

$$\text{mg}\cos\theta = \text{N} = \text{mg}/2$$

$$f_k = \mu_k \text{N} = \mu_k \text{mg}/2$$

$$\mu_k = \frac{\left(\frac{\sqrt{3}-1}{2}\right)mg}{\frac{mg}{2}}$$

$$\mu_k = \sqrt{3} - 1$$

Application of Angle of Repose

Antlions make sand traps in such a way that when an insect enters the edge of the trap, it starts to slide towards the bottom where the antilon hide itself. The angle of inclination of sand trap is made to be equal to angle of repose.

Children are fond of playing on sliding board (Figure 3.31). Sliding will be easier when the angle of inclination of the board is greater than the angle of repose. At the same time if inclination angle is much larger than the angle of repose, the slider will reach the bottom at greater speed and get hurt.

Rolling Friction

The invention of the wheel plays a crucial role in human civilization. One of the important applications is suitcases with rolling on coasters. Rolling wheels makes it easier than carrying luggage. When an object moves on a surface, essentially it is sliding on it. But wheels move on the surface through rolling motion. In rolling motion when a wheel moves on a surface, the point of contact with surface is always at rest. Since the point of contact is at rest, there is no relative motion between the wheel and surface. Hence the frictional force is very less. At the same time if an object moves without a wheel, there is a relative motion between the object and the surface. As a result frictional force is larger. This makes it difficult to move the object. The Figure 3.32 shows the difference between rolling and kinetic friction.

Ideally in pure rolling, motion of the point of contact with the surface should be at rest, but in practice it is not so. Due to the elastic nature of the surface at the point of contact there will be some deformation on the object at this point on the wheel or surface as shown in Figure 3.33. Due to this deformation, there will be minimal friction between wheel and surface. It is called 'rolling friction'. In fact, 'rolling friction' is much smaller than kinetic friction.

Methods to Reduce Friction

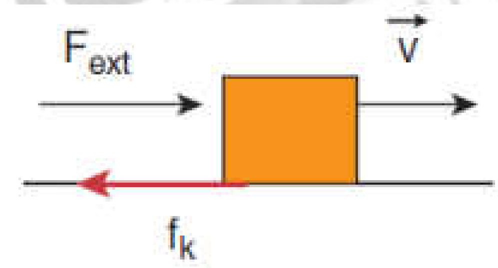
Frictional force has both positive and negative effects. In some cases it is absolutely necessary. Walking is possible because of frictional force. Vehicles (bicycle, car) can move because of the frictional force between the tyre and the road. In the braking system, kinetic friction plays a major role. As we have already seen, the frictional force comes into effect whenever there is relative motion between two surfaces. In big machines used in industries, relative motion between different parts of the machine produce unwanted heat which reduces its efficiency. To reduce this kinetic friction lubricants are used as shown in Figure 3.34.

Ball bearings provides another effective way to reduce the kinetic friction (Figure 3.35) in machines. If ball bearings are fixed between two surfaces, during the relative motion only the rolling friction comes to effect and not kinetic friction. As we have seen earlier, the rolling friction is much smaller than kinetic friction; hence the machines are protected from wear and tear over the years.

During the time of Newton and Galileo, frictional force was considered as one of the natural forces like gravitational force. But in the twentieth century, the understanding on atoms, electron and protons has changed the perspective. The frictional force is actually the electromagnetic force between the atoms on the two surfaces. Even well polished surfaces have irregularities on the surface at the microscopic level as seen in the Figure 3.36.

EXAMPLE

Consider an object moving on a horizontal surface with a constant velocity. Some external force is applied on the object to keep the object moving with a constant velocity. What is the net force acting on the object?



Solution

If an object moves with constant velocity, then it has no acceleration. According to Newton's second law there is no net force acting on the object. The external force is balanced by the kinetic friction.

DYNAMICS OF CIRCULAR MOTION

In the previous sections we have studied how to analyse linear motion using Newton's laws. It is also important to know how to apply Newton's laws to circular motion, since circular motion is one of the very common types of motion that we come across in our daily life. A particle can be in linear motion with or without any external force. But when circular motion occurs there must necessarily be some force acting on the object. There is no Newton's first law for circular motion. In other words without a force, circular motion cannot occur in nature. A force can change the velocity of a particle in three different ways.

1. The magnitude of the velocity can be changed without changing the direction of the velocity. In this case the particle will move in the same direction but with acceleration.

Examples

Particle falling down vertically, bike moving in a straight road with acceleration

2. The direction of motion alone can be changed without changing the magnitude (speed). If this happens continuously then we call it 'uniform circular motion
3. Both the direction and magnitude (speed) of velocity can be changed. If this happens non circular motion occurs. For example oscillation of a swing or simple pendulum, elliptical motion of planets around the Sun.

In this section we will deal with uniform circular motion and non-circular motion.

Centripetal force

If a particle is in uniform circular motion, there must be centripetal acceleration towards the center of the circle. If there is acceleration then there must be some force acting on it with respect to an inertial frame. This force is called centripetal force.

As we have seen in chapter 2, the centripetal acceleration of a particle in the circular motion is given by $a = \frac{v^2}{r}$ and it acts towards center of the circle. According to Newton's second law, the centripetal force is given by

$$F_{cp} = ma_{cp} = \frac{mv^2}{r}$$

The word Centripetal force means center seeking force. In vector notation

$$\vec{F}_{cp} = -\frac{mv^2}{r}\hat{r}$$

For uniform circular motion

$$\vec{F}_{cp} = -m\omega^2 r\hat{r}$$

The direction –rˆ points towards the center of the circle which is the direction of centripetal force as shown in Figure 3.38.

It should be noted that 'centripetal force' is not other forces like gravitational force or spring force. It can be said as 'force towards center'. The origin of the centripetal force can be gravitational force, tension in the string, frictional force, Coulomb force etc. Any of these forces can act as a centripetal force.

1. In the case of whirling motion of a stone tied to a string, the centripetal force on the particle is provided by the tensional force on the string. In circular motion in an amusement park, the centripetal force is provided by the tension in the iron ropes.
2. In motion of satellites around the Earth, the centripetal force is given by Earth's gravitational force on the satellites. Newton's second law for satellite motion is

$$F = \text{earth's gravitational force} = \frac{mv^2}{r}$$

Where r- distance of the planet from the center of the Earth.

3. When a car is moving on a circular track the centripetal force is given by the frictional force between the road and the tyres Newton's second law for this case is

$$\text{Frictional force} = \frac{mv^2}{r}$$

m-mass of the car

v-speed of the car

r-radius of curvature of track

Even when the car moves on a curved track, the car experiences the centripetal force which is provided by frictional force between the surface and the tyre of the car. This is shown in the Figure 3.41.

4. When the planets orbit around the Sun, they experience centripetal force towards the center of the Sun. Here gravitational force of the Sun acts as centripetal force on the planets as shown in Figure 3.42

Newton's second law for this motion Gravitational force of Sun on the planet $= \frac{mv^2}{r}$

EXAMPLE

If a stone of mass 0.25 kg tied to a string executes uniform circular motion with a speed of 2 m s^{-1} of radius 3 m, what is the magnitude of tensional force acting on the stone?

Solution

$$F_{cp} = \frac{\frac{1}{4}\times(2)^2}{3} = 0.333\ \text{N}.$$

EXAMPLE

The Moon orbits the Earth once in 27.3 days in an almost circular orbit. Calculate the centripetal acceleration experienced by the Moon? (Radius of the Earth is 6.4×10^6 m)

Solution

The centripetal acceleration is given by $a = \frac{v^2}{r}$. This expression explicitly depends on Moon's speed which is non trivial. We can work with the formula

$$\omega^2 R_m = a_m$$

a_m is centripetal acceleration of the Moon due to Earth's gravity.

ω is angular velocity.

R_m is the distance between Earth and the Moon, which is 60 times the radius of the Earth

$$R_m = 60R = 60 \times 6.4 \times 10^6 = 384 \times 10^6\ m$$

As we know the angular velocity $\omega = \frac{2\pi}{T}$ and T = 27.3 days = 27.3 × 24 × 60 × 60 second = 2.358 × 106 sec

By substituting these values in the formula for acceleration

$$a_m = \frac{\left(4\pi^2\right)\left(384 \times 10^6\right)}{\left(2.358 \times 10^6\right)^2} = 0.00272\ \text{m s}^{-2}$$

The centripetal acceleration of Moon towards the Earth is 0.00272 m s^{-2}

Vehicle on a levelled circular road

When a vehicle travels in a curved path, there must be a centripetal force acting on it. This centripetal force is provided by the frictional force between tyre and surface of the road. Consider a vehicle of mass 'm' moving at a speed 'v' in the circular track of radius 'r'. There are three forces acting on the vehicle when it moves as shown in the Figure 3.43

1. Gravitational force (mg) acting downwards
2. Normal force (mg) acting upwards
3. Frictional force (F_s) acting horizontally inwards along the road

Suppose the road is horizontal then the normal force and gravitational force are exactly equal and opposite. The centripetal force is provided by the force of static friction Fs between the tyre and surface of the road which acts towards the center of the circular track,

$$\frac{mv^2}{r} = F_s$$

As we have already seen in the previous section, the static friction can increase from zero to a maximum value

$$F_s \le \mu_s mg.$$

There are two conditions possible:

$$\text{If } \frac{mv^2}{r} \le \mu_s mg, \text{ or } \mu_s \ge \frac{v^2}{rg} \text{ or } \sqrt{\mu_s rg} \ge v$$

The static friction would be able to provide necessary centripetal force to bend the car on the road. So the coefficient of static friction between the tyre and the surface of the road determines what maximum speed the car can have for safe turn.

$$\text{If } \frac{mv^2}{r} > \mu_s mg, \text{ or } \mu_s < \frac{v^2}{rg} \text{ (skid)}$$

If the static friction is not able to provide enough centripetal force to turn, the vehicle will start to skid.

EXAMPLE

Consider a circular leveled road of radius 10 m having coefficient of static friction 0.81. Three cars (A, B and C) are travelling with speed 7 m s^{-1}, 8 m s^{-1} and 10 ms^{-1} respectively. Which car will skid when it moves in the circular level road? (g =10 m s^{-2})

Solution

From the safe turn condition the speed of the vehicle (v) must be less than or equal to $\sqrt{\mu_s rg}$

$$v \leq \sqrt{\mu_s rg}$$

$$\sqrt{\mu_s rg} = \sqrt{0.81 \times 10 \times 10} = 9 ms^{-1}$$

For Car C, $\sqrt{\mu_s rg}$ is less than v

The speed of car A, B and C are 7 m s^{-1}, 8 m s−1 and 10 m s^{-1} respectively. The cars A and B will have safe turns. But the car C has speed 10 m s^{-1} while it turns which exceeds the safe turning speed. Hence, the car C will skid.

Banking of Tracks

In a leveled circular road, skidding mainly depends on the coefficient of static friction ms The coefficient of static friction depends on the nature of the surface which has a maximum limiting value. To avoid this problem, usually the outer edge of the road is slightly raised compared to inner edge as shown in the Figure 3.44. This is called banking of roads or tracks. This introduces an inclination, and the angle is called banking angle.

Let the surface of the road make angle θ with horizontal surface. Then the normal force makes the same angle θ with the vertical. When the car takes a turn, there are two forces acting on the car:

1. Gravitational force mg (downwards)
2. Normal force N (perpendicular to surface)

We can resolve the normal force into two components. N $\cos\theta$ and N $\sin\theta$ as shown in Figure 3.46. The component N $\cos\theta$ balances the downward gravitational force 'mg' and component N $\sin\theta$ will provide the necessary centripetal acceleration. By using Newton second law

$$N\cos\theta = mg$$

$$N\sin\theta = \frac{mv^2}{r}$$

By dividing the equations we get $\tan\theta = \frac{v^2}{rg}$

$$v = \sqrt{rg\tan\theta}$$

The banking angle θ and radius of curvature of the road or track determines the safe speed of the car at the turning. If the speed of car exceeds this safe speed, then it starts to skid outward but frictional force comes into effect and provides an additional centripetal force to prevent the outward skidding. At the same time, if the speed of the car is little lesser than safe speed, it starts to skid inward and frictional force comes into effect, which reduces centripetal force to prevent inward skidding. However if the speed of the vehicle is sufficiently greater than the correct speed, then frictional force cannot stop the car from skidding.

EXAMPLE

Consider a circular road of radius 20 meter banked at an angle of 15 degree. With what speed a car has to move on the turn so that it will have safe turn?

Solution

$$v = \sqrt{(rg\tan\theta)} = \sqrt{20\times 9.8\times \tan 15^\circ}$$

$$= \sqrt{20\times 9.8\times 0.26} = 7.1\ \mathrm{m\,s^{-1}}$$

Centrifugal Force

Circular motion can be analysed from two different frames of reference. One is the inertial frame (which is either at rest or in uniform motion) where Newton's laws are obeyed. The other is the rotating frame of reference which is a non-inertial frame of reference as it is accelerating. When we examine the circular motion from these frames of reference the situations are entirely different. To use Newton's first and second laws in the rotational frame of reference, we need to include a pseudo force called 'centrifugal force'. This 'centrifugal force' appears to act on the object with respect to rotating frames. To understand the concept of centrifugal force, we can take a specific case and discuss as done below.

Consider the case of a whirling motion of a stone tied to a string. Assume that the stone has angular velocity ω in the inertial frame (at rest). If the motion of the stone is observed from a frame which is also rotating along with the stone with same angular velocity ω then, the stone appears to be at rest. This implies that in addition to the inward centripetal force $-m\omega^2 r$ there must be an equal and opposite force that acts on the stone outward with value $+m\omega^2 r$. So the total force acting on the stone in a rotating frame is equal to zero ($-m\omega^2 r + m\omega^2 r = 0$). This outward force $+m\omega^2 r$ is called the centrifugal force. The word 'centrifugal' means 'flee from center'. Note that the 'centrifugal force' appears to act on the particle, only when we analyse the motion from a rotating frame. With respect to an inertial frame there is only centripetal force which is given by the tension in the string. For this reason centrifugal force is called as a 'pseudo force'. A pseudo force has no origin. It arises due to the non inertial nature of the frame considered. When circular motion problems are solved from a rotating frame of reference, while drawing free body diagram of a particle, the centrifugal force should necessarily be included as shown in the Figure 3.45.

Effects of Centrifugal Force

Although centrifugal force is a pseudo force, its effects are real. When a car takes a turn in a curved road, person inside the car feels an outward force which pushes the person away. This outward force is also called centrifugal force. If there is sufficient friction between the person and the seat, it will prevent the person from moving outwards. When a car moving in a straight line suddenly takes a turn, the objects not fixed to the car try to continue in linear motion due to their inertia of direction. While observing this motion from an inertial frame, it appears as a straight line as shown in Figure 3.46. But, when it is observed from the rotating frame it appears to move outwards.

A person standing on a rotating platform feels an outward centrifugal force and is likely to be pushed away from the platform. Many a time the frictional force between the platform and the person is not sufficient to overcome outward push. To avoid this, usually the outer edge of the platform is little inclined upwards which exerts a normal force on the person which prevents the person from falling as illustrated in Figures 3.47.

Centrifugal Force due to Rotation of the Earth

Even though Earth is treated as an inertial frame, it is actually not so. Earth spins about its own axis with an angular velocity ω. Any object on the surface of Earth (rotational frame) experiences a centrifugal force. The centrifugal force appears to act exactly in opposite direction from the axis of rotation. It is shown in the Figure 3.48.

The centrifugal force on a man standing on the surface of the Earth is $F_c = m\omega^2 r$

where r is perpendicular distance of the man from the axis of rotation. By using right angle triangle as shown in the Figure 3.48, the distance $r = R\cos\theta$

Here R = radius of the Earth

and θ = latitude of the Earth where the man is standing.

EXAMPLE

Calculate the centrifugal force experienced by a man of 60 kg standing at Chennai? (Given: Latitude of Chennai is 13°

Solution

The centrifugal force is given by $F_c = m \omega^2 R \cos\theta$

The angular velocity (ω) of Earth $= \frac{2\pi}{r}$ where T is time period of the Earth (24 hours)

$$\omega = \frac{2\pi}{24 \times 60 \times 60} = \frac{2\pi}{86400}$$
$$= 7.268 \times 10^{-5} \, rad sec^{-1}$$

The radius of the Earth R = 6400 Km = 6400×10^3 m

Latitude of Chennai = 13°

$$F_{cf} = 60 \times \left(7.268 \times 10^{-5}\right)^2 \times 6400 \times 10^3$$
$$\times \cos\left(13°\right) = 1.9678 \, N$$

A 60 kg man experiences centrifugal force of approximately 2 Newton. But due to Earth's gravity a man of 60 kg experiences a force = mg = 60 × 9.8 = 588N. This force is very much larger than the centrifugal force.

Centripetal Force Versus Centrifugal Force

Salient features of centripetal and centrifugal forces are compared in Table 3.4.

Centripetal force	Centrifugal force
It is a real force which is exerted on the body by the external agencies like gravitational force, tension in the string, normal force etc.	It is a pseudo force or fictitious force which cannot arise from gravitational force, tension force, normal force etc.
Acts in both inertial and non-inertial frames	Acts only in rotating frames (non-inertial frame)
It acts towards the axis of rotation or center of the circle in circular motion	It acts outwards from the axis of rotation or radially outwards from the center of the circular motion
$\left\vert F_{cp}\right\vert = m\omega^2 r = \frac{mv^2}{r}$	$\left\vert F_{cf}\right\vert = m\omega^2 r = \frac{mv^2}{r}$
Real force and has real effects	Pseudo force but has real effects
Origin of centripetal force is interaction between two objects.	Origin of centrifugal force is inertia. It does not arise from interaction. In an inertial frame the object's inertial motion appears as centrifugal force in the rotating frame.
In inertial frames centripetal force has to be included when free body diagrams are drawn.	In inertial frames there is no centrifugal force. In rotating frames, both centripetal and centrifugal force have to be included when free body diagrams are drawn.

UNIT- 3 MOTION OF SYSTEM OF PARTICLES AND RIGID BODIES

INTRODUCTION

Most of the objects that we come across in our day to day life consist of large number of particles. In the previous Units, we studied the motion of bodies without considering their size and shape. So far we have treated even the bulk bodies as only point objects. In this section, we will give importance to the size and shape of the bodies. These bodies are actually made up of a large number of particles. When such a body moves, we consider it as the motion of collection of particles as a whole. We define the concept of center of mass to deal with such a system of particles.

The forces acting on these bulk bodies are classified into internal and external forces. Internal forces are the forces acting among the particles within a system that constitute the body. External forces are the forces acting on the particles of a system from outside. In this unit, we deal with such system of particles which make different rigid bodies. A rigid body is the one which maintains its definite and fixed shape even when an external force acts on it. This means that, the interatomic distances do not change in a rigid body when an external force is applied. However, in real life situation, we have bodies which are not ideally rigid, because the shape and size of the body change when forces act on them. For the rigid bodies we study here, we assume that such deformations are negligible. The deformations produced on non-rigid bodies are studied separately in Unit 7 under elasticity of solids.

CENTER OF MASS

When a rigid body moves, all particles that constitute the body need not take the same path. Depending on the type of motion, different particles of the body may take different paths. For example, when a wheel rolls on a surface, the path of the center point of the wheel and the paths of other points of the wheel are different. In this Unit, we study about the translation, rotation and the combination of these motions of rigid bodies in detail.

Center of Mass of a Rigid Body

When a bulk object (say a bat) is thrown at an angle in air as shown in Figure 5.1; do all the points of the body take a parabolic path? Actually, only one point takes the parabolic path and all the other points take different paths.

The one point that takes the parabolic path is a very special point called center of mass (CM) of the body. Its motion is like the motion of a single point that is thrown. The center of mass of a body is defined as a point where the entire mass of the body appears to be concentrated. Therefore, this point can represent the entire body.

For bodies of regular shape and uniform mass distribution, the center of mass is at the geometric center of the body. As examples, for a circle and sphere, the center of mass is at their centers; for square and rectangle, at the point their diagonals meet; for cube and cuboid, it is at the point where their body diagonals meet. For other bodies, the center of mass has to be determined using some methods. The center of mass could be well within the body and in some cases outside the body as well.

Center of Mass for Distributed Point Masses

A point mass is a hypothetical point particle which has nonzero mass and no size or shape. To find the center of mass for a collection of n point masses, say, m1, m2, m3 . . . mn we have to first choose an origin and an appropriate coordinate system as shown in Figure 5.2. Let, x1, x2, x3 . . . xn be the X-coordinates of the positions of these point masses in the X direction from the origin.

The equation for the x coordinate of the center of mass is,

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i}$$

where, $\sum m_i$ is the total mass M of all the particles, $(\sum m_i = m)$.

$$x_{CM} = \frac{\sum m_i x_i}{M}$$

Similarly, we can also find y and z coordinates of the center of mass for these distributed point masses as indicated in Figure (5.2).

$$y_{CM} = \frac{\sum m_i y_i}{M}$$

$$z_{CM} = \frac{\sum m_i z_i}{M}$$

Hence, the position of center of mass of these point masses in a Cartesian coordinate system is (x_{CM}, y_{CM}, z_{CM}). In general, the position of center of mass can be written in a vector form as,

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M}$$

where, $\dot{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$ is the position vector of the center of mass and $\dot{r} = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ is the position vector of the distributed point mass; where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along X, Y and Z-axes respectively.

Center of Mass of Two Point Masses

With the equations for center of mass, let us find the center of mass of two point masses m_1 and m_2, which are at positions x_1 and x_2 respectively on the X-axis. For this case, we can express the position of center of mass in the following three ways based on the choice of the coordinate system.

When the masses are on positive X-axis:

The origin is taken arbitrarily so that the masses m_1 and m_2 are at positions x_1 and x_2 on the positive X-axis as shown in Figure 5.3(a). The center of mass will also be on the positive X-axis at x_{CM} as given by the equation,

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

When the origin coincides with any one of the masses:

The calculation could be minimised if the origin of the coordinate system is made to coincide with any one of the masses as shown in Figure 5.3(b). When the origin coincides with the point mass m_1, its position x_1 is zero, (i.e. $x_1 = 0$). Then,

$$x_{CM} = \frac{m_1(0) + m_2 x_2}{m_1 + m_2}$$

The equation further simplifies as,

$$x_{CM} = \frac{m_2 x_2}{m_1 + m_2}$$

When the origin coincides with the center of mass itself:

If the origin of the coordinate system is made to coincide with the center of mass, then, $x_{CM} = 0$ and the mass m_1 is found to be on the negative X-axis as shown in Figure 5.3(c). Hence, its position x_1 is negative, (i.e. $-x_1$).

$$0=\frac{m_1(-x_1)+m_2x_2}{m_1+m_2}$$

$$0=m_1(-x_1)+m_2x_2$$

$$m_1x_1=m_2x_2$$

The equation given above is known as principle of moments.

EXAMPLE

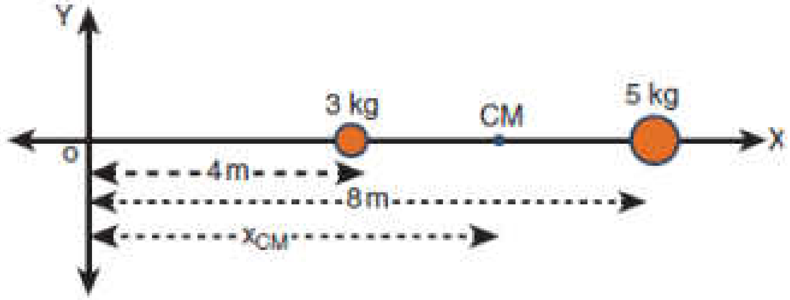
Two point masses 3 kg and 5 kg are at 4 m and 8 m from the origin on X-axis. Locate the position of center of mass of the two point masses (i) from the origin and (ii) from 3 kg mass.

Solution

Let us take, m_1 = 3 kg and m_2= 5 kg

To find center of mass from the origin:

The point masses are at positions, x_1 = 4 m, x_2 = 8 m from the origin along X axis.



The center of mass x_{CM} can be obtained using equation

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

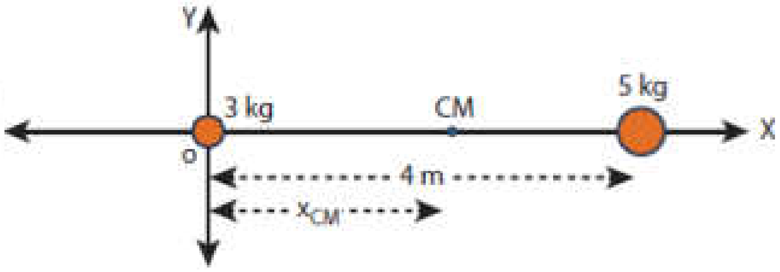
$$x_{CM} = \frac{(3\times4)+(5\times8)}{3+5}$$

$$x_{CM} = \frac{12+40}{8} = \frac{52}{8} = 6.5\,m$$

The center of mass is located 6.5 m from the origin on X-axis.

To find the center of mass from 3 kg mass:

The origin is shifted to 3 kg mass along X-axis. The position of 3 kg point mass is zero (x_1 = 0) and the position of 5 kg point mass is 4 m from the shifted origin (x_2 = 4 m).



$$x_{CM} = \frac{(3\times0)+(5\times4)}{3+5}$$

$$x_{CM} = \frac{0+20}{8} = \frac{20}{8} = 2.5\,m$$

The center of mass is located 2.5 m from 3 kg point mass, (and 1.5 m from the 5 kg point mass) on X-axis.

When we compare case (i) with case (ii), the x_{CM} = 2.5 m from 3 kg mass could also be obtained by subtracting 4 m (the position of 3 kg mass) from 6.5 m, where the center of mass was located in case (i)

EXAMPLE

From a uniform disc of radius R, a small disc of radius $\frac{R}{2}$ is cut and removed as shown in the diagram. Find the center of mass of the remaining portion of the disc.

Solution

Let us consider the mass of the uncut full disc be M. Its center of mass would be at the geometric center of the disc on which the origin coincides.

Let the mass of the small disc cut and removed be m and its center of mass is at a position $\frac{R}{2}$ to the right of the origin as shown in the figure.

Hence, the remaining portion of the disc should have its center of mass to the left of the origin; say, at a distance x. We can write from the principle of moments,

$$(M-m)x = (m)\frac{R}{2}$$

$$x = \left(\frac{m}{(M-m)}\right)\frac{R}{2}$$

If σ is the surface mass density (i.e. mass per unit surface area), $\sigma = \frac{M}{\pi R^2}$ then, the mass m of small disc is,

m = surface mass density × surface area

$$m = \sigma \times \pi\left(\frac{R}{2}\right)^2$$

$$m = \left(\frac{M}{\pi R^2}\right)\pi\left(\frac{R}{2}\right)^2 = \frac{M}{\pi R^2}\pi\frac{R^2}{4} = \frac{M}{4}$$

substituting m in the expression for x

$$x=\frac{\frac{M}{4}}{\left(M-\frac{M}{4}\right)}\times\frac{R}{2}=\frac{\frac{M}{4}}{\left(\frac{3M}{4}\right)}\times\frac{R}{2}$$

$$x=\frac{R}{6}$$

The center of mass of the remaining portion is at a distance $\frac{R}{6}$ to the left from the center of the disc.

EXAMPLE

The position vectors of two point masses 10 kg and 5 kg are $\left(-3\hat{i}+2\hat{j}+4\hat{k}\right)$ m and $\left(3\hat{i}+6\hat{j}+5\hat{k}\right)$ m respectively. Locate the position of center of mass.

Solution

$$m_1 = 10kg$$

$$m_2 = 5kg$$

$$\vec{r}_1 = (-3\hat{i}+2\hat{j}+4\hat{k})m$$

$$\vec{r}_2 = (3\hat{i}+6\hat{j}+5\hat{k})m$$

$$\vec{r}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$$

$$\therefore \vec{r} = \frac{10(-3\hat{i}+2\hat{j}+4\hat{k})+5(3\hat{i}+6\hat{j}+5\hat{k})}{10+5}$$

$$= \frac{-30\hat{i}+20\hat{j}+40\hat{k}+15\hat{i}+30\hat{j}+25\hat{k}}{15}$$

$$= \frac{-15\hat{i}+50\hat{j}+65\hat{k}}{15}$$

$$\vec{r} = \left(-\hat{i}+\frac{10}{3}\hat{j}+\frac{13}{3}\hat{k}\right)\ m$$

The center of mass is located at position $\vec{r}$

Center of mass for uniform distribution of mass

If the mass is uniformly distributed in a bulk object, then a small mass (Δm) of the body can be treated as a point mass and the summations can be done to obtain the expressions for the coordinates of center of mass.

$$x_{CM} = \frac{\sum(\Delta m_i)x_i}{\sum \Delta m_i}$$

$$y_{CM} = \frac{\sum(\Delta m_i)y_i}{\sum \Delta m_i}$$

$$z_{CM} = \frac{\sum(\Delta m_i)z_i}{\sum \Delta m_i}$$

On the other hand, if the small mass taken is infinitesimally * small (dm) then, the summations can be replaced by integrations as given below.

$$x_{cm} = \frac{\int x dm}{\int dm}$$

$$y_{cm} = \frac{\int y dm}{\int dm}$$

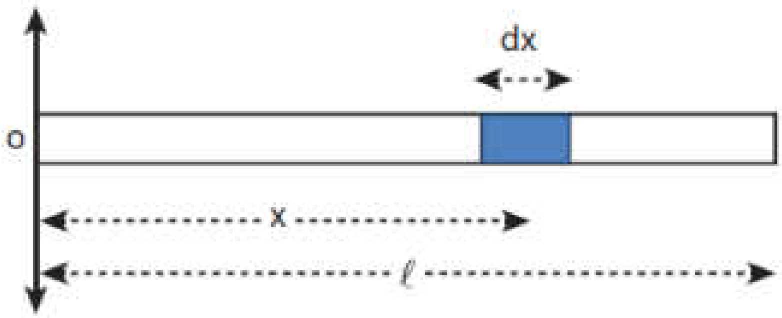
$$z_{cm} = \frac{\int z dm}{\int dm}$$

EXAMPLE

Locate the center of mass of a uniform rod of mass M and length ℓ.

Solution

Consider a uniform rod of mass M and length ℓ whose one end coincides with the origin as shown in Figure. The rod is kept along the x axis. To find the center of mass



of this rod, we choose an infinitesimally small mass dm of elemental length dx at a distance x from the origin.

λ is the linear mass density (i.e. mass per unit length) of the rod $\lambda = \frac{M}{\ell}$

The mass of small element (dm) is, $dm = \frac{M}{\ell} dx$

Now, we can write the center of mass equation for this mass distribution as,

$$x_{CM} = \frac{\int x\,dm}{\int dm}$$

$$x_{CM} = \frac{\int_0^{\ell} x\left(\frac{M}{\ell}dx\right)}{M} = \frac{1}{\ell}\int_0^{\ell} x\,dx$$

$$= \frac{1}{\ell}\left[\frac{x^2}{2}\right]_0^{l} = \frac{1}{\ell}\left(\frac{\ell^2}{2}\right)$$

$$x_{CM} = \frac{\ell}{2}$$

As the position $\frac{\ell}{2}$ is the geometric center of the rod, it is concluded that the center of mass of the uniform rod is located at its geometric center itself.

Motion of Center of Mass

When a rigid body moves, its center of mass will also move along with the body. For kinematic quantities like velocity (v_{CM}) and acceleration (a_{CM}) of the center of mass, we can differentiate the expression for position of center of mass with respect to time once and twice respectively. For simplicity, let us take the motion along X direction only.

$$\vec{v}_{CM} = \frac{d\vec{x}_{CM}}{dt} = \frac{\sum m_i\left(\frac{d\vec{x}_i}{dt}\right)}{\sum m_i}$$

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i}$$

$$\vec{a}_{CM} = \frac{d}{dt}\left(\frac{d\vec{x}_{CM}}{dt}\right) = \left(\frac{d\vec{v}_{CM}}{dt}\right) = \frac{\sum m_i\left(\frac{d\vec{v}_i}{dt}\right)}{\sum m_i}$$

$$\vec{a}_{CM} = \frac{\sum m_i \vec{a}_i}{\sum m_i}$$

In the absence of external force, i.e. $\vec{F}_{ext} = 0$ the individual rigid bodies of a system can move or shift only due to the internal forces. This will not affect the position of the center of mass. This means that the center of mass will be in a state of rest or uniform motion. Hence, $\vec{v}_{CM}$ will be zero when center of mass is at rest and constant when center of mass has uniform motion $\left(\vec{v}_{CM} = 0 \; or \; \vec{v}_{CM} = \text{constant}\right)$. There will be no acceleration of center of mass, $\left(\vec{a}_{CM} = 0\right)$.

From equation

$$0 = \frac{\sum m_i \vec{v}_i}{\sum m_i} \quad \text{(or)} \quad \text{constant,}$$

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i}; \quad \vec{a}_{CM} = 0$$

Here, the individual particles may still move with their respective velocities and accelerations due to internal forces. In the presence of external force, (i.e. $\vec{F}_{ext} \neq 0$), the center of mass of the system will accelerate as given by the following equation.

$$\vec{F}_{ext} = \left(\sum m_i\right)\vec{a}_{CM}; \quad \vec{F}_{ext} = M\vec{a}_{CM}; \quad \vec{a}_{CM} = \frac{\vec{F}_{ext}}{M}$$

EXAMPLE

A man of mass 50 kg is standing at one end of a boat of mass 300 kg floating on still water. He walks towards the other end of the boat with a constant velocity of 2 m s-1 with respect to a stationary observer on land. What will be the velocity of the boat, (a) with respect to the stationary observer on land? (b) with respect to the man walking in the boat?

[Given: There is friction between the man and the boat and no friction between the
boat and water.]

Solution

Mass of the man (m_1) is, m_1= 50 kg
Mass of the boat (m_2) is, m_2 = 300 kg

With respect to a stationary observer:

The man moves with a velocity, $v_1 = 2$ m s^{-1} and the boat moves with a velocity v_2 (which is to be found)

To determine the velocity of the boat with respect to a stationary observer on land:

As there is no external force acting on the system, the man and boat move due to the friction, which is an internal force in the boat-man system. Hence, the velocity of the center of mass is zero ($v_{CM} = 0$).

$$0 = \frac{\sum m_i v_i}{\sum m_i} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$0 = m_1 v_1 + m_2 v_2$$

$$-m_2 v_2 = m_1 v_1$$

$$v_2 = -\frac{m_1}{m_2} v_1$$

$$v_2 = -\frac{50}{300} \times 2 = -\frac{100}{300}$$

$$v_2 = -0.33\,\text{m s}^{-1}$$

The negative sign in the answer implies that the boat moves in a direction opposite to that of the walking man on the boat to a stationary observer on land.

To determine the velocity of the boat with respect to the walking man:

We can find the relative velocity as,

$$v_{21} = v_2 - v_1$$

where, v_{21} is the relative velocity of the boat with respect to the walking man.

$$v_{21} = (-0.33) - (2)$$

$$v_{21} = -2.33\,\text{ms}^{-1}$$

The negative sign in the answer implies that the boat appears to move in the opposite direction to the man walking in the boat.

Center of mass in explosions:

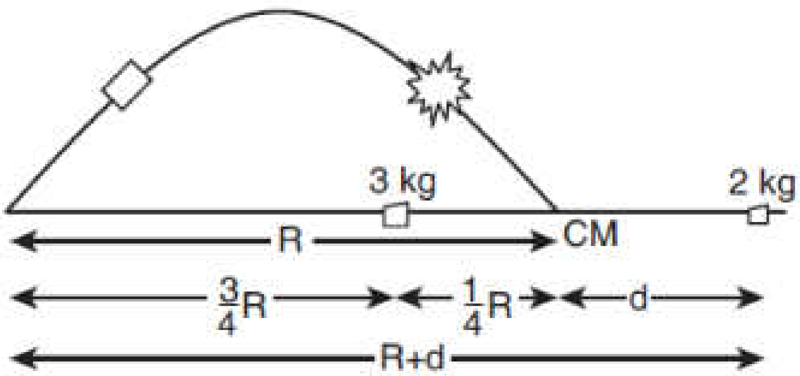
Many a times rigid bodies are broken in to fragments. If an explosion is caused by the internal forces in a body which is at rest or in motion, the state of the center of mass is not affected. It continues to be in the same state of rest or motion. But, the kinematic quantities of the fragments get affected. If the explosion is caused by an external agency, then the kinematic quantities of the center of mass as well as the fragments get affected.

EXAMPLE

A projectile of mass 5 kg, in its course of motion explodes on its own into two fragments. One fragment of mass 3 kg falls at three fourth of the range R of the projectile. Where will the other fragment fall?

Solution

It is an explosion of its own without any external influence. After the explosion, the center of mass of the projectile will continue to complete the parabolic path even though the fragments are not following the same parabolic path. After the fragments have fallen on the ground, the center of mass rests at a distance R (the range) from the point of projection as shown in the diagram.



If the origin is fixed to the final position of the center of mass, the principle of moments holds good.

$$m_1x_1 = m_2x_2$$

where, m_1 = 3 kg, m_2 = 2 kg, $x_1 = \frac{1}{4}$R. The value of x_2 = d

$$3\times\frac{1}{4}R=2\times d;$$

$$d=\frac{3}{8}R$$

The distance between the point of launching and the position of 2 kg mass is R+d.

$$R+d=R+\frac{3}{8}R=\frac{11}{8}R=1.375R$$

The other fragment falls at a distance of 1.375R from the point of launching. (Here R is the range of the projectile.)

TORQUE AND ANGULAR MOMENTUM

When a net force acts on a body, it produces linear motion in the direction of the applied force. If the body is fixed to a point or an axis, such a force rotates the body depending on the point of application of the force on the body. This ability of the force to produce rotational motion in a body is called torque or moment of force. Examples for such motion are plenty in day to day life. To mention a few; the opening and closing of a door about the hinges and turning of a nut using a wrench.

The extent of the rotation depends on the magnitude of the force, its direction and the distance between the fixed point and the point of application. When torque produces rotational motion in a body, its angular momentum changes with respect to time. In this Section we will learn about the torque and its effect on rigid bodies.

Definition of Torque

Torque is defined as the moment of the external applied force about a point or axis of rotation. The expression for torque is,

$$\vec{\tau}=\vec{r}\times\vec{F}$$

where, r is the position vector of the point where the force $\dot{F}$ is acting on the body as shown in Figure 5.4.

Here, the product of r and $\dot{F}$ is called the vector product or cross product. The vector product of two vectors results in another vector that is perpendicular to both the vectors (refer Section 2.5.2). Hence, torque (τ) is a vector quantity.

Tor que has a magnitude (rFsinθ) and direction perpendicular to r and $\dot{F}$. Its unit is N m.

$$\vec{\tau} = \left(rF\sin\theta\right)\hat{n}$$

Here, θ is the angle between r and $\dot{F}$, and ˆn is the unit vector in the direction of τ. Torque (τ) is sometimes called as a pseudo vector as it needs the other two vectors r and $\dot{F}$ for its existence.

The direction of torque is found using right hand rule. This rule says that if fingers of right hand are kept along the position vector with palm facing the direction of the force and when the fingers are curled the thumb points to the direction of the torque. This is shown in Figure 5.5.

The direction of torque helps us to find the type of rotation caused by the torque. For example, if the direction of torque is out of the paper, then the rotation produced by the torque is anticlockwise. On the other hand, if the direction of the torque is into the paper, then the rotation is clockwise as shown in Figure

In many cases, the direction and magnitude of the torque are found separately. For direction, we use the vector rule or right hand rule. For magnitude, we use scalar form as,

$$\tau = r\ F\ \sin\theta$$

The expression for the magnitude of torque can be written in two different ways by associating sin θ either with r or F in the following manner.

$$\tau = r(F\ \sin\theta) = r \times (F\perp)$$

$$\tau = (r\ \sin\theta)F = (r\perp) \times F$$

Here, (F sinθ) is the component of $\vec{F}$ perpendicular to $\vec{r}$. Similarly, (r sinθ) is the component of $\vec{r}$ perpendicular to $\vec{F}$.

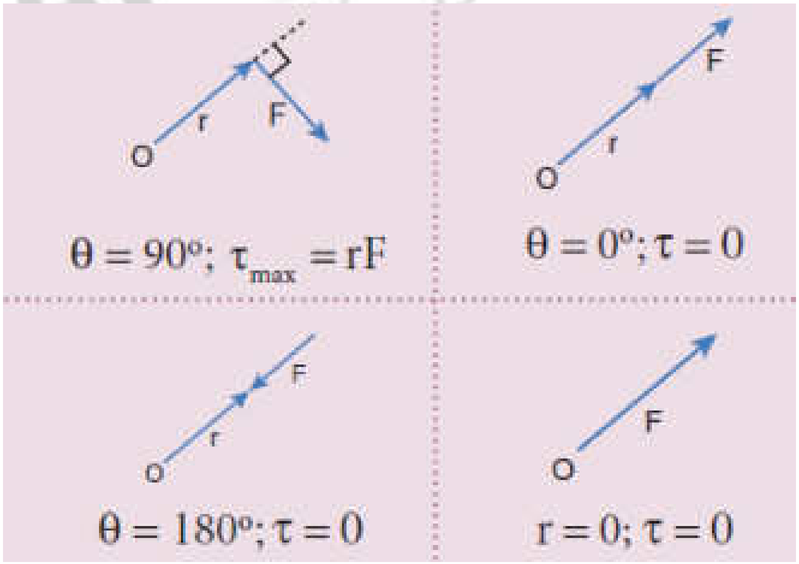
Based on the angle θ between $\vec{r}$ and $\vec{F}$ the torque takes different values.

The torque is maximum when, $\vec{r}$ and $\vec{F}$ are perpendicular to each other. That is when θ = 90° and sin 90° = 1, Hence, τ_{max} = rF.

The torque is zero when $\vec{r}$ and $\vec{F}$ are parallel or antiparallel. If parallel, then θ =0° and sin 0° = 0. If antiparallel, then θ = 180° and sin 180° = 0. Hence, τ = 0.

The torque is zero if the force acts at the reference point. i.e. as $\vec{r}$ =0, τ = 0.

The Value of τ for different cases.



EXAMPLE

If the force applied is perpendicular to the handle of the spanner as shown in the diagram, find the (i) torque exerted by the force about the center of the nut, (ii) direction of torque and (iii) type of rotation caused by the torque about the nut.

Solution

Arm length of the spanner, r = 15 cm = 15×10^{-2}m

Force, F = 2.5 N
Angle between r and F, $\theta = 90^{o}$

Torque, $\tau = rF\sin\theta$

$$\tau = 15\times10^{-2}\times2.5\times\sin(90^{\circ})$$

$$[\text{here, } \sin 90^{\circ} = 1]$$

$$\tau = 37.5\times10^{-2}\ \text{N m}$$

As per the right hand rule, the direction of torque is out of the page.

The type of rotation caused by the torque is anticlockwise.

EXAMPLE

A force of $(4\hat{i}-3\hat{j}+5\hat{k})$ N is applied at a point whose position vector is $(7\hat{i}+4\hat{j}-2\hat{k})$ m. Find the torque of force about the origin.

Solution

$$\vec{r} = 7\hat{i}+4\hat{j}-2\hat{k}$$

$$\vec{F} = 4\hat{i}-3\hat{j}+5\hat{k}$$

Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 4 & -2 \\ 4 & -3 & 5 \end{vmatrix}$$

$$\vec{\tau} = \hat{i}(20-6) - \hat{j}(35+8) + \hat{k}(-21-16)$$

$$\vec{\tau} = \left(14\hat{i} - 43\hat{j} - 37\hat{k}\right) \text{ N m}$$

EXAMPLE

A crane has an arm length of 20 m inclined at 30^{o} with the vertical. It carries a container of mass of 2 ton suspended from the top end of the arm. Find the torque produced by the gravitational force on the container about the point where the arm is fixed to the crane. [Given: 1 ton = 1000 kg; neglect the weight of the arm. g = 10 m s^{-2}]

Solution

The force F at the point of suspension is due to the weight of the hanging mass.

$$F = mg = 2 \times 1000 \times 10 = 20000 \text{ N};$$
$$\text{The arm length, } r = 20 \text{ m}$$

We can solve this problem by three different methods.

Method – I

The angle (θ) between the arm length (r) and the force (F) is, $\theta = 150^{o}$
The torque (τ) about the fixed point of the arm is,

$$\tau = rF\sin\theta$$

$$\tau = 20 \times 20000 \times \sin(150^\circ)$$

$$= 400000 \times \sin(90^\circ + 60^\circ)$$

$$[\text{here, } \sin(90^\circ + \theta) = \cos\theta]$$

$$= 400000 \times \cos(60^\circ)$$

$$= 400000 \times \frac{1}{2} \quad \left[\cos 60^\circ = \frac{1}{2}\right]$$

$$= 200000\,\text{N m}$$

$$\tau = 2 \times 10^5\,\text{N m}$$

Method – II

Let us take the force and perpendicular distance from the point where the arm is fixed to the crane.

$$\tau = (r\perp)F$$

$$\tau = r\cos\phi\, mg$$

$$\tau = 20 \times \cos 60^\circ \times 20000$$

$$= 20 \times \frac{1}{2} \times 20000$$

$$= 200000\,\text{Nm}$$

$$\tau = 2 \times 10^5\,\text{Nm}$$

Method – III

Let us take the distance from the fixed point and perpendicular force.

$$\tau = r(F\perp)$$
$$\tau = r\,mg\ \cos\phi$$
$$\tau = 20\times 20000\times \cos 60^\circ$$
$$= 20\times 20000\times \frac{1}{2}$$
$$= 200000\ \text{Nm}$$
$$\tau = 2\times 10^5\ \text{Nm}$$

All the three methods, give the same answer.

Torque about an Axis

In the earlier sections, we have dealt with the torque about a point. In this section we will deal with the torque about an axis. Let us consider a rigid body capable of rotating about an axis AB as shown in Figure 5.8. Let the force F act at a point P on the rigid body. The force F may not be on the plane ABP. We can take the origin O at any random point on the axis AB.

The torque of the force $\vec{F}$ about O is, $\tau = \vec{r}\times\vec{F}$. The component of the torque along the axis is the torque of $\vec{F}$ about the axis. To find it, we should first find the vector $\tau = \vec{r}\times\vec{F}$ and then find the angle φ between τ and AB. (Remember here, $\vec{F}$ is not on the plane ABP). The torque about AB is the parallel component of the torque along AB, which is $|\vec{r}\times\vec{F}|$ cos φ. And the torque perpendicular to the axis AB is $|\vec{r}\times\vec{F}|$sin φ.

The torque about the axis will rotate the object about it and the torque perpendicular to the axis will turn the axis of rotation. When both exist simultaneously on a rigid body, the body will have a precession. One can witness the precessional motion in a spinning top when it is about to come to rest as shown in Figure 5.9.

Study of precession is beyond the scope of the higher secondary physics course. Hence, it is assumed that there are constraints to cancel the effect of the perpendicular components of the torques, so that the fixed position of the axis is maintained. Therefore, perpendicular components of the torque need not be taken into account.

Hereafter, for the calculation of torques on rigid bodies we will:

1. Consider only those forces that lie on planes perpendicular to the axis (and do not intersect the axis).
2. Consider position vectors which are perpendicular to the axis

EXAMPLE

Three mutually perpendicular beams AB, OC, GH are fixed to form a structure which is fixed to the ground firmly as shown in the Figure. One string is tied to the point C and its free end D is pulled with a force F. Find the magnitude and direction of the torque produced by the force,

Solution

1. Torque about point D is zero. (as F passes through D).
   Torque about point C is zero. (as F passes through C).
   Torque about point O is $(\vec{OC}) \times \vec{F}$ and direction is along GH.
   Torque about point B is $(\vec{BD}) \times \vec{F}$ and direction is along GH.

   (The $\perp$ of $\overrightarrow{BD}$ with respect to $\vec{F}$ is $\overrightarrow{OC}$).

2. Torque about axis CD is zero (as F is parallel to CD).
   Torque about axis OC is zero (as F intersects OC).
   Torque about axis AB is zero (as F is parallel to AB).
   Torque about axis GH is $(\vec{OC}) \times \vec{F}$ and direction is along GH.

The torque of a force about an axis is independent of the choice of the origin as long as it is chosen on that axis itself. This can be shown as below.

Let O be the origin on the axis AB, which is the rotational axis of a rigid body. F is the force acting at the point P. Now, choose another point O' anywhere on the axis as shown in Figure 5.10

The torque of F about O′ is,

$$\overline{O'P}\times\vec{F}=\left(\overline{O'O}+\overline{OP}\right)\times\vec{F}$$
$$=\left(\overline{O'O}\times\vec{F}\right)+\left(\overline{OP}\times\vec{F}\right)$$

As $O'\dot{O}\times F$ is perpendicular to $O'\dot{O}$, this term will not have a component along AB. Thus, the component of $O'\dot{P}\times\dot{F}$ is equal to that of $OP\times\dot{F}$.

Torque and Angular Acceleration

Let us consider a rigid body rotating about a fixed axis. A point mass m in the body will execute a circular motion about a fixed axis as shown in Figure 5.11. A tangential force $\dot{F}$ acting on the point mass produces the necessary torque for this rotation. This force $\dot{F}$ is perpendicular to the position vector r of the point mass

The torque produced by the force on the point mass m about the axis can be written as,

$$\tau = \mathrm{r\,F}\sin 90 = \mathrm{r\,F} \quad [\because\ \sin 90 = 1]$$
$$\tau = \mathrm{r\,ma} \quad [\because\ (\mathrm{F} = \mathrm{ma})]$$
$$\tau = \mathrm{r\,m\,r}\alpha = \mathrm{mr}^2\alpha \quad [\because\ (\mathrm{a} = \mathrm{r}\alpha)]$$

$$\tau = \left(\mathrm{mr}^2\right)\alpha$$

Hence, the torque of the force acting on the point mass produces an angular acceleration (α) in the point mass about the axis of rotation.

In vector notation,

$$\vec{\tau} = \left(\mathrm{mr}^2\right)\vec{\alpha}$$

The directions of τ and α are along the axis of rotation. If the direction of τ is in the direction of α, it produces angular acceleration. On the other hand if, τ is opposite to α, angular deceleration or retardation is produced on the point mass.

The term mr2 in equations 5.14 and 5.15 is called moment of inertia (I) of the point mass. A rigid body is made up of many such point masses. Hence, the moment of inertia of a rigid body is the sum of moments of inertia of all such individual point masses that constitute the body $\left(I = \sum m_i r_i^2\right)$. Hence, torque for the rigid body can be written as,

$$\vec{\tau} = \left(\sum m_i r_i^2\right)\vec{\alpha}$$
$$\vec{\tau} = I\vec{\alpha}$$

We will learn more about the moment of inertia and its significance for bodies with different shapes in section 5.4.

Angular Momentum

The angular momentum in rotational motion is equivalent to linear momentum in translational motion. The angular momentum of a point mass is defined as the moment of its linear momentum. In other words, the angular momentum L of a point mass having a linear momentum p at a position r with respect to a point or axis is mathematically written as,

$$\vec{L} = \vec{r} \times \vec{p}$$

The magnitude of angular momentum could be written as,

$$L = r\,p\sin\theta$$

where, θ is the angle between r and $\dot{p}$. L is perpendicular to the plane containing r and $\dot{p}$. As we have written in the case of torque, here also we can associate sin θ with either r or $\dot{p}$.

$$L = r(p\sin\theta) = r(p\perp)$$

$$L = (r\sin\theta)p = (r\perp)p$$

where, $p\perp$ is the component of linear momentum p perpendicular to r, and $r\perp$ is the component of position r perpendicular to p.

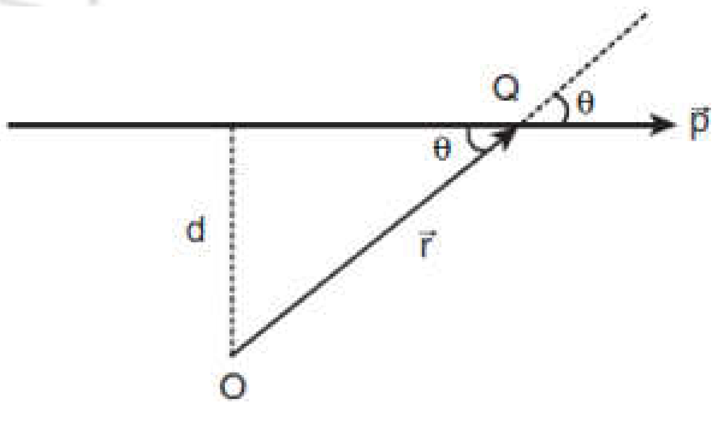
The angular momentum is zero (L = 0), if the linear momentum is zero (p = 0) or if the particle is at the origin (r = 0) or if r and $\dot{p}$ are parallel or antiparallel to each other (θ= 00 or 1800).

There is a misconception that the angular momentum is a quantity that is associated only with rotational motion. It is not true. The angular momentum is also associated with bodies in the linear motion. Let us understand the same with the following example.

EXAMPLE

A particle of mass (m) is moving with constant velocity (v). Show that its angular momentum about any point remains constant throughout the motion.

Solution



Let the particle of mass m move with constant velocity v. As it is moving with constant velocity, its path is a straight line. Its momentum ($\dot{p}$ =mv) is also directed along the same path. Let us fix an origin (O) at a perpendicular distance (d) from the path. At a particular instant, we can connect the particle which is at positon Q with a position vector $\left(\dot{r} = Q\dot{Q}\right)$).

Take, the angle between the r and $\dot{p}$ as θ. The magnitude of angular momentum of that particle at that instant is,

$$L = OQ\,p\sin\theta = OQ\,mv\sin\theta = mv\left(OQ\sin\theta\right)$$

The term (OQ sinθ) is the perpendicular distance (d) between the origin and line along which the mass is moving. Hence, the angular momentum of the particle about the origin is,

$$L = mvd$$

The above expression for angular momentum L, does not have the angle θ. As the momentum (p = mv) and the perpendicular distance (d) are constants, the angular momentum of the particle is also constant. Hence, the angular momentum is associated with bodies with linear motion also. If the straight path of the particle passes through the origin, then the angular momentum is zero, which is also a constant

Angular Momentum and Angular Velocity

Let us consider a rigid body rotating about a fixed axis. A point mass m in the body will execute a circular motion about the fixed axis as shown in Figure

The point mass m is at a distance r from the axis of rotation. Its linear momentum at any instant is tangential to the circular path. Then the angular momentum L is perpendicular to r and $\dot{p}$. Hence, it is directed along the axis of rotation. The angle θ between r and $\dot{p}$ in this case is 90o. The magnitude of the angular momentum L could be written as,

$$L = r\,mv\sin 90^{\circ} = r\,mv$$

where, v is the linear velocity. The relation between linear velocity v and angular velocity ω in a circular motion is, v = rω. Hence,

$$L = rmr\omega$$
$$L = (mr^2)\omega$$

The directions of L and ω are along the axis of rotation. The above expression can be written in the vector notation as,

$$\vec{L} = (mr^2)\vec{\omega}$$

As discussed earlier, the term mr2 in equations 5.22 and 5.23 is called moment of inertia (I) of the point mass. A rigid body is made up of many such point masses. Hence, the moment of inertia of a rigid body is the sum of moments of inertia of all such individual point masses that constitute the body $\left(I = \sum m_i r_i^2\right)$. Hence, the angular momentum of the rigid body can be written as,

$$\vec{L} = \left(\sum m_i r_i^2\right)\vec{\omega}$$
$$\vec{L} = I\vec{\omega}$$

The study about moment of inertia (I) is reserved for Section 5.4.

Torque Angular Momentum

We have the expression for magnitude of angular momentum of a rigid body as, L = Iω. The expression for magnitude of torque on a rigid body is, τ = Iα

We can further write the expression for torque as,

$$\tau = I\frac{d\omega}{dt} \quad \because \left(\alpha = \frac{d\omega}{dt}\right)$$

Where, ω is angular velocity and α is angular acceleration. We can also write equation 5.26 as,

$$\tau = \frac{d(I\omega)}{dt}$$

$$\tau = \frac{dL}{dt}$$

The above expression says that an external torque on a rigid body fixed to an axis produces rate of change of angular momentum in the body about that axis. This is the Newton's second law in rotational motion as it is in the form of $F = \frac{dp}{dt}$ which holds good for translational motion.

Conservation of angular momentum:

From the above expression we could conclude that in the absence of external torque, the angular momentum of the rigid body or system of particles is conserved.

$$\text{If } \tau = 0 \text{ then, } \frac{dL}{dt} = 0; \ L = \text{constant}$$

The above expression is known as law of conservation of angular momentum. We will learn about this law further in section 5.5.

EQUILIBRIUM OF RIGID BODIES

When a body is at rest without any motion on a table, we say that there is no force acting on the body. Actually it is wrong because, there is gravitational force acting on the body downward and also the normal force exerted by table on the body upward. These two forces cancel each other and thus there is no net force acting on the body. There is a lot of difference between the terms "no force" and "no net force" acting on a body. The same argument holds good for rotational conditions in terms of torque or moment of force.

A rigid body is said to be in mechanical equilibrium when both its linear momentum and angular momentum remain constant.

When the linear momentum remains constant, the net force acting on the body is zero.

$$\vec{F}_{net} = 0$$

In this condition, the body is said to be in translational equilibrium. This implies that the vector sum of different forces $\dot{F}_1, \dot{F}_2, \dot{F}_3, \ldots$ acting in different directions on the body is zero.

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n = 0$$

If the forces $\dot{F}_1, \dot{F}_2, \dot{F}_3, \ldots$ act in different directions on the body, we can resolve them into horizontal and vertical components and then take the resultant in the respective directions. In this case there will be horizontal as well as vertical equilibria possible.

Similarly, when the angular momentum remains constant, the net torque acting on the body is zero.

$$\vec{\tau}_{net} = 0$$

Under this condition, the body is said to be in rotational equilibrium. The vector sum of different torques $\dot{\tau}_1, \dot{\tau}_2, \dot{\tau}_3, \ldots$ producing different senses of rotation on the body is zero.

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots + \vec{\tau}_n = 0$$

Thus, we can also conclude that a rigid body is in mechanical equilibrium when the net force and net torque acts on the body is zero.

$$\vec{F}_{net} = 0 \text{ and } \vec{\tau}_{net} = 0$$

As the forces and torques are vector quantities, the directions are to be taken with proper sign conventions.

Types of Equilibrium

Based on the above discussions, we come to a conclusion that different types of equilibrium are possible based on the different conditions. They are consolidated in Table 5.2.

EXAMPLE

Arun and Babu carry a wooden log of mass 28 kg and length 10 m which has almost uniform thickness. They hold it at 1 m and 2 m from the ends respectively. Who will bear more weight of the log? [g = 10 ms^{-2}]

Solution

Let us consider the log is in mechanical equilibrium. Hence, the net force and net torque on the log must be zero. The gravitational force acts at the center of mass of the log downwards. It is cancelled by the normal reaction forces R_A and R_B applied upwards by Arun and Babu at points A and B respectively. These reaction forces are the weights borne by them.

The total weight, W = mg = 28 × 10 = 280 N, has to be borne by them together. The reaction forces are the weights borne by each of them separately. Let us show all the forces acting on the log by drawing a free body diagram of the log.

For translational equilibrium:

The net force acting on the log must be zero.

$$R_A + (-mg) + R_B = 0$$

Here, the forces R_A an R_B are taken positive as they act upward. The gravitational force acting downward is taken negative.

$$R_A + R_B = mg$$

For rotational equilibrium:

The net torque acting on the log must be zero. For ease of calculation, we can take the torque caused by all the forces about the point A on the log. The\ forces are perpendicular to the distances. Hence,

$$(0R_A) + (-4mg) + (7R_B) = 0.$$

Here, the reaction force R_A cannot produce any torque as the reaction forces pass through the point of reference A. The torque of force mg produces a clockwise turn about the point A which is taken negative and torque of force R_B causes anticlockwise turn about A which is taken positive.

$$7R_B = 4mg$$

$$R_B = \frac{4}{7}mg$$

$$R_B = \frac{4}{7} \times 28 \times 10 = 160\ N$$

By substituting for R_B we get,

$$R_A = mg - R_B$$

$$R_A = 28 \times 10 - 160 = 280 - 160 = 120\ N$$

As R_B is greater than R_A, it is concluded that Babu bears more weight than Arun. The one closer to center of mass of the log bears more weight.

Couple

Consider a thin uniform rod AB. Its center of mass is at its midpoint C. Let two forces which are equal in magnitude and opposite in direction be applied at the two ends A and B of the rod perpendicular to it. The two forces are separated by a distance of 2r as shown in Figure 5.13.

As the two equal forces are opposite in direction, they cancel each other and the net force acting on the rod is zero. Now the rod is in translational equilibrium. But, the rod is not in rotational equilibrium. Let us see how it is not in rotational equilibrium. The moment of the force applied at the end A taken with respect to the center point C, produces an anticlockwise rotation. Similarly, the moment of the force applied at the end B also produces an anticlockwise rotation. The moments of both the forces cause the same sense of rotation in the rod. Thus, the rod undergoes a rotational motion or turning even though the rod is in translational equilibrium.

A pair of forces which are equal in magnitude but opposite in direction and separated by a perpendicular distance so that their lines of action do not coincide that causes a turning effect is called a couple. We come across couple in many of our daily activities as shown in Figure 5.14.

Principle of Moments

Consider a light rod of negligible mass which is pivoted at a point along its length. Let two parallel forces F1 and F2 act at the two ends at distances d1 and d2 from the point of pivot and the normal reaction force N at the point of pivot as shown in Figure 5.15. If the rod has to remain stationary in horizontal position, it should be in translational and rotational equilibrium. Then, both the net force and net torque must be zero.

For net force to be zero, $-F_1 + N - F_2 = 0$

$$N = F_1 + F_2$$

For net torque to be zero, $d_1F_1 - d_2 F_2 = 0$

$$d_1F_1 = d_2F_2$$

The above equation represents the principle of moments. This forms the principle for beam balance used for weighing goods with the condition $d_1 = d_2$; $F_1 = F_2$. We can rewrite the equation 5.33 as,

$$\frac{F_1}{F_2} = \frac{d_2}{d_1}$$

If F_1 is the load and F_2 is our effort, we get advantage when, $d_1 < d_2$. This implies that $F_1 > F_2$. Hence, we could lift a large load with small effort. The ratio $\left(\frac{d_2}{d_1}\right)$ is called mechanical advantage of the simple lever. The pivoted point is called fulcrum.

Mechanical Advantage MA= $\frac{d_2}{d_1}$

There are many simple machines that work on the above mentioned principle.

Center of Gravity

Each rigid body is made up of several point masses. Such point masses experience gravitational force towards the center of Earth. As the size of Earth is very large compared to any practical rigid body we come across in daily life, these forces appear to be acting parallelly downwards as shown in Figure 5.16

The resultant of these parallel forces always acts through a point. This point is called center of gravity of the body (with respect to Earth). The center of gravity of a body is the point at which the entire weight of the body acts irrespective of the position and orientation of the body. The center of gravity and center of mass of a rigid body coincide when the gravitational field is uniform across the body. The concept of gravitational field is dealt in Unit 6.

We can also determine the center of gravity of a uniform lamina of even an irregular shape by pivoting it at various points by trial and error. The lamina remains horizontal when pivoted at the point where the net gravitational force acts, which is the center of gravity as shown in Figure 5.17. When a body is supported at the center of gravity, the sum of the torques acting on all the point masses of the rigid body becomes zero. Moreover the weight is compensated by the normal reaction force exerted by the pivot. The body is in static equilibrium and hence it remains horizontal.

There is also another way to determine the center of gravity of an irregular lamina. If we suspend the lamina from different points like P, Q, R as shown in Figure 5.18, the vertical lines PP', QQ', RR' all pass through the center of gravity. Here, reaction force acting at the point of suspension and the gravitational force acting at the center of gravity cancel each other and the torques caused by them also cancel each other.

Bending of Cyclist in Curves

Let us consider a cyclist negotiating a circular level road (not banked) of radius r with a speed v. The cycle and the cyclist are considered as one system with mass m. The center gravity of the system is C and it goes in a circle of radius r with center at O. Let us choose the line OC as X-axis and the vertical line through O as Z-axis as shown in Figure 5.19.

The system as a frame is rotating about Z-axis. The system is at rest in this rotating frame. To solve problems in rotating frame of reference, we have to apply a centrifugal force (pseudo force) on the system which will be $\frac{mv^2}{r}$ This force will act through the center of gravity. The forces acting on the system are, (i) gravitational force (mg), (ii) normal force (N), (iii) frictional force (f) and (iv) centrifugal force $\left(\frac{mv^2}{r}\right)$. As the system is in equilibrium in the rotational frame of reference, the net external force and net external torque must be zero. Let us consider all torques about the point A in Figure 5.20.

The torque due to the gravitational force about point A is (mgAB) which causes a clockwise turn that is taken as negative. The torque due to the centripetal force is $\left(\frac{mv^2}{r}BC\right)$ which causes an anticlockwise turn that is taken as positive.

$$-mg\,AB + \frac{mv^2}{r}BC = 0$$

$$mg\,AB = \frac{mv^2}{r}BC$$

From Δ ABC,
AB = ACsinθ and BC = ACcosθ

$$mg\,AC\,\sin\theta = \frac{mv^2}{r}AC\cos\theta$$

$$\tan\theta = \frac{v^2}{rg}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

While negotiating a circular level road of radius r at velocity v, a cyclist has to bend by an angle θ from vertical given by the above expression to stay in equilibrium (i.e. to avoid a fall).

EXAMPLE

A cyclist while negotiating a circular path with speed 20 m s^{-1} is found to bend an angle by 30o with vertical. What is the radius of the circular path? (given, g = 10 m s^{-2})

Solution

Speed of the cyclist, v = 20 m s^{-1}
Angle of bending with vertical, θ = 30^o

Equation for angle of bending, $\tan\theta = \frac{V^2}{rg}$

Rewriting the above equation for radius

$$r = \frac{v^2}{\tan\theta\, g}$$

Substituting,

$$r = \frac{(20)^2}{(\tan 30^\circ)\times 10} = \frac{20\times 20}{(\tan 30^\circ)\times 10}$$

$$= \frac{400}{\left(\frac{1}{\sqrt{3}}\right)\times 10}$$

$$r = \left(\sqrt{3}\right)\times 40 = 1.732\times 40$$

$$r = 69.28\,\text{m}$$

MOMENT OF INERTIA

In the expressions for torque and angular momentum for rigid bodies (which are considered as bulk objects), we have come across a term $\sum m_i r_i^2$ This quantity is called moment of inertia (I) of the bulk object. For point mass mi at a distance r_i from the fixed axis, the moment of inertia is given as $m_i r_i^2$

Moment of inertia for point mass,

$$I = m_i r_i^2$$

Moment of inertia for bulk object

$$I = \sum m_i r_i^2$$

In translational motion, mass is a measure of inertia; in the same way, for rotational motion, moment of inertia is a measure of rotational inertia. The unit of moment of inertia is, kg m^2. Its dimension is M L^2. In general, mass is an invariable quantity of matter (except for motion comparable to that of light). But, the moment of inertia of a body is not an invariable quantity. It depends not only on the mass of the body, but also on the way the mass is distributed around the axis of rotation.

To find the moment of inertia of a uniformly distributed mass; we have to consider an infinitesimally small mass (dm) as a point mass and take its position (r) with respect to an axis. The moment of inertia of this point mass can now be written as,

$$dI = (dm)r^2$$

We get the moment of inertia of the entire bulk object by integrating the above expression.

$$I = \int dI = \int (dm)r^2$$

$$I = \int r^2 dm$$

We can use the above expression for determining the moment of inertia of some of the common bulk objects of interest like rod, ring, disc, sphere etc.

Moment of Inertia of a Uniform Rod

Let us consider a uniform rod of mass (M) and length (ℓ) as shown in Figure 5.21. Let us find an expression for moment of inertia of this rod about an axis that passes through the center of mass and perpendicular to the rod. First an origin is to be fixed for the coordinate system so that it coincides with the center of mass, which is also the geometric center of the rod. The rod is now along the x axis. We take an infinitesimally small mass (dm) at a distance (x) from the origin. The moment of inertia (dI) of this mass (dm) about the axis is,

$$dI = (dm)\, x^2$$

As the mass is uniformly distributed, the mass per unit length (λ) of the rod is $\lambda = \frac{M}{\ell}$

The (dm) mass of the infinitesimally small length as, dm $= \lambda\ dx = \frac{M}{\ell} dx$

The moment of inertia (I) of the entire rod can be found by integrating dI,

$$I = \int dI = \int (dm) x^2 = \int \left(\frac{M}{\ell} dx \right) x^2$$

$$I = \frac{M}{\ell} \int x^2 dx$$

As the mass is distributed on either side of the origin, the limits for integration are taken from $-\ell/2$ to $\ell/2$.

$$I = \frac{M}{\ell}\int_{-\ell/2}^{\ell/2} x^2 dx = \frac{M}{\ell}\left[\frac{x^3}{3}\right]_{-\ell/2}^{\ell/2}$$

$$I = \frac{M}{\ell}\left[\frac{\ell^3}{24} - \left(-\frac{\ell^3}{24}\right)\right] = \frac{M}{\ell}\left[\frac{\ell^3}{24} + \frac{\ell^3}{24}\right]$$

$$I = \frac{M}{\ell}\left[2\left(\frac{\ell^3}{24}\right)\right]$$

$$I = \frac{1}{12}M\ell^2 \qquad (5.41)$$

EXAMPLE

Find the moment of inertia of a uniform rod about an axis which is perpendicular to the rod and touches any one end of the rod.

Solution

The concepts to form the integrand to find the moment of inertia could be borrowed from the earlier derivation. Now, the origin is fixed to the left end of the rod and the limits are to be taken from 0 to ℓ.

$$I = \frac{M}{\ell}\int_0^{\ell} x^2 dx = \frac{M}{\ell}\left[\frac{x^3}{3}\right]_0^{\ell} = \frac{M}{\ell}\left[\frac{\ell^3}{3}\right]$$

$$I = \frac{1}{3}M\ell^2$$

Moment of Inertia of a Uniform Ring

Let us consider a uniform ring of mass M and radius R. To find the moment of inertia of the ring about an axis passing through its center and perpendicular to the plane, let us take an infinitesimally small mass (dm) of length (dx) of the ring. This (dm) is located at a distance R, which is the radius of the ring from the axis.

The moment of inertia (dI) of this small mass (dm) is,

$$dI = (dm)R^2$$

The length of the ring is its circumference ($2\pi R$). As the mass is uniformly distributed, the mass per unit length (λ) is,

$$\lambda = \frac{\text{mass}}{\text{length}} = \frac{M}{2\pi R}$$

The mass (dm) of the infinitesimally small length is, $dm = \lambda\, dx = \frac{M}{2\pi R} dx$

Now, the moment of inertia (I) of the entire ring is,

$$I = \int dI = \int (dm)R^2 = \int \left(\frac{M}{2\pi R} dx\right) R^2$$

$$I = \frac{MR}{2\pi} \int dx$$

To cover the entire length of the ring, the limits of integration are taken from 0 to $2\pi R$.

$$I = \frac{MR}{2\pi} \int_0^{2\pi R} dx$$

$$I = \frac{MR}{2\pi} [x]_0^{2\pi R} = \frac{MR}{2\pi} [2\pi R - 0]$$

$$I = MR^2 \qquad (5.42)$$

Moment of Inertia of a Uniform Disc

Consider a disc of mass M and radius R. This disc is made up of many infinitesimally small rings as shown in Figure 5.23. Consider one such ring of mass (dm) and thickness (dr) and radius (r). The moment of inertia (dI) of this small ring is,

$$dI = (dm)r^2$$

As the mass is uniformly distributed, the mass per unit area (σ) is,

$$\sigma = \frac{\text{mass}}{\text{area}} = \frac{M}{\pi R^2}$$

The mass of the infinitesimally small ring is,

$$dm = \sigma 2\pi r dr = \frac{M}{\pi R^2} 2\pi r dr$$

where, the term ($2\pi r$ dr) is the area of this elemental ring ($2\pi r$ is the length and dr is the thickness). $dm = \frac{2M}{R^2} rdr$

$$dI = \frac{2M}{R^2} r^3 dr$$

The moment of inertia (I) of the entire disc is,

$$I = \int dI$$

$$I = \int_0^R \frac{2M}{R^2} r^3 dr = \frac{2M}{R^2} \int_0^R r^3 dr$$

$$I = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2}\left[\frac{R^4}{4} - 0\right]$$

$$I = \frac{1}{2} MR^2$$

Radius of Gyration

For bulk objects of regular shape with uniform mass distribution, the expression for moment of inertia about an axis involves their total mass and geometrical features like radius, length, breadth, which take care of the shape and the size of the objects. But, we need an expression for the moment of inertia which could take care of not only the mass, shape and size of objects, but also its orientation to the axis of rotation. Such an expression should be general so that it is applicable even for objects of irregular shape and non-uniform distribution of mass. The general expression for moment of inertia is given as,

$$I = MK^2$$

where, M is the total mass of the object and K is called the radius of gyration.

The radius of gyration of an object is the perpendicular distance from the axis of rotation to an equivalent point mass, which would have the same mass as well as the same moment of inertia of the object.

As the radius of gyration is distance, its unit is m. Its dimension is L.

A rotating rigid body with respect to any axis, is considered to be made up of point masses m_1, m_2, m_3, . . .m_n at perpendicular distances (or positions) r_1, r_2, r_3 . . . r_n respectively as shown in Figure 5.24.

The moment of inertia of that object can be written as,

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2$$

If we take all the n number of individual masses to be equal,

$$m = m_1 = m_2 = m_3 = \ldots = m_n$$

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \cdots + mr_n^2$$

$$= m\left(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2\right)$$

$$= nm\left(\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}\right)$$

$$I = MK^2$$

where, nm is the total mass M of the body and K is the radius of gyration.

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}}$$

The expression for radius of gyration indicates that it is the root mean square (rms) distance of the particles of the body from the axis of rotation.

In fact, the moment of inertia of any object could be expressed in the form, $I = MK^2$

For example, let us take the moment of inertia of a uniform rod of mass M and length ℓ. Its moment of inertia with respect to a perpendicular axis passing through the center of mass is, $I = \frac{1}{12} M \ell^2$

In terms of radius of gyration, I =MK^2

$$\text{Hence, } MK^2 = \frac{1}{12}M\ell^2$$

$$K^2 = \frac{1}{12}\ell^2$$

$$K = \frac{1}{\sqrt{12}}\ell \text{ or } K = \frac{1}{2\sqrt{3}}\ell \text{ or } K = (0.289)\ell$$

EXAMPLE

Find the radius of gyration of a disc of mass M and radius R rotating about an axis passing through the center of mass and perpendicular to the plane of the disc.

Solution

The moment of inertia of a disc about an axis passing through the center of mass and perpendicular to the disc is, $I = \frac{1}{2}MR^2$

In terms of radius of gyration, I=MK^2

$$\text{Hence, } MK^2 = \frac{1}{2}MR^2; \quad K^2 = \frac{1}{2}R^2$$

$$K = \frac{1}{\sqrt{2}}R \text{ or } K = \frac{1}{1.414}R \text{ or } K = (0.707)R$$

From the case of a rod and also a disc, we can conclude that the radius of gyration of the rigid body is always a geometrical feature like length, breadth, radius or their combinations with a positive numerical value multiplied to it.

Obesity and associated ailments like back pain, joint pain etc. are due to the shift in center of mass of the body. Due to this shift in center of mass, unbalanced torque acting on the body leads to ailments. As the mass is spread away from center of the body the moment of inertia is more and turning will also be diffi cult.

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Parallel axis theorem:

Parallel axis theorem states that the moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its center of mass and the product of the mass of the body and the square of the perpendicular distance between the two axes.

If I_C is the moment of inertia of the body of mass M about an axis passing through the center of mass, then the moment of inertia I about a parallel axis at a distance d from it is given by the relation,

$$I = I_C + Md^2$$

Let us consider a rigid body as shown in Figure 5.25. Its moment of inertia about an axis AB passing through the center of mass is I_C. DE is another axis parallel to AB at a perpendicular distance d from AB. The moment of inertia of the body about DE is I. We attempt to get an expression for I in terms of I_C. For this, let us consider a point mass m on the body at position x from its center of mass.

The moment of inertia of the point mass about the axis DE is, $m(x + d)^2$.

The moment of inertia I of the whole body about DE is the summation of the above expression.

$$I = \sum m(x+d)^2$$

This equation could further be written as,

$$I = \sum m\left(x^2 + d^2 + 2xd\right)$$

$$I = \sum\left(mx^2 + md^2 + 2dmx\right)$$

$$I = \sum mx^2 + \sum md^2 + 2d\sum mx$$

Here, Σmx^2 is the moment of inertia of the body about the center of mass. Hence, $I_C = \sum mx^2$

The term, $\Sigma mx = 0$ because, x can take positive and negative values with respect to the axis AB. The summation (Σmx) will be zero.

$$\text{Thus, } I = I_C + \sum md^2 = I_C + \left(\sum m\right)d^2$$

Here, Σm is the entire mass M of the object ($\Sigma m = M$)

$$I = I_C + Md^2$$

Hence, the parallel axis theorem is proved.

Perpendicular axis theorem:

This perpendicular axis theorem holds good only for plane laminar objects. The theorem states that the moment of inertia of a plane laminar body about an axis perpendicular to its plane is equal to the sum of moments of inertia about two perpendicular axes lying in the plane of the body such that all the three axes are mutually perpendicular and have a common point.

Let the X and Y-axes lie in the plane and Z-axis perpendicular to the plane of the laminar object. If the moments of inertia of the body about X and Y-axes are I_X and I_Y respectively and I_Z is the moment of inertia about Z-axis, then the perpendicular axis theorem could be expressed as,

$$I_Z = I_X + I_Y$$

To prove this theorem, let us consider a plane laminar object of negligible thickness on which lies the origin (O). The X and Y-axes lie on the plane and Z-axis is perpendicular to it as shown in Figure 5.26. The lamina is considered to be made up of a large number of particles of mass m. Let us choose one such particle at a point P which has coordinates (x, y) at a distance r from O.

The moment of inertia of the particle about Z-axis is, mr^2

The summation of the above expression gives the moment of inertia of the entire lamina about Z-axis as, $I_Z = \sum mr^2$

$$\text{Here, } r^2 = x^2 + y^2$$

$$\text{Then, } I_Z = \sum m\left(x^2 + y^2\right)$$

$$I_Z = \sum mx^2 + \sum my^2$$

In the above expression, the term $\sum mx^2$ is the moment of inertia of the body about the Y-axis and similarly the term $\sum my^2$ is the moment of inertia about X-axis. Thus,

$$I_X = \sum my^2 \quad \text{and} \quad I_Y = \sum mx^2$$

Substituting in the equation for I_z gives,

$$I_Z = I_X + I_Y$$

Thus, the perpendicular axis theorem is proved.

EXAMPLE

Find the moment of inertia of a disc of mass 3 kg and radius 50 cm about the following axes.

1. axis passing through the center and perpendicular to the plane of the disc,
2. axis touching the edge and perpendicular to the plane of the disc and
3. axis passing through the center and lying on the plane of the disc.

Solution

The mass, M = 3 kg, radius R = 50 cm = 50×10^{-2} m = 0.5 m

The moment of inertia (I) about an axis passing through the center and perpendicular to the plane of the disc is,

$$I = \frac{1}{2} MR^2$$

$$I = \frac{1}{2} \times 3 \times (0.5)^2 = 0.5 \times 3 \times 0.5 \times 0.5$$

$$I = 0.375 \, kg\,m^2$$

The moment of inertia (I) about an axis touching the edge and perpendicular to the plane of the disc by parallel axis theorem is,

$$I = I_C + Md^2$$

$$\text{where, } I_C = \frac{1}{2} MR^2 \text{ and } d = R$$

$$I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

$$I = \frac{3}{2} \times 3 \times (0.5)^2 = 1.5 \times 3 \times 0.5 \times 0.5$$

$$I = 1.125\,kg\,m^2$$

The moment of inertia (I) about an axis passing through the center and lying on the plane of the disc is,

$$I_Z = I_X + I_Y$$

$$\text{where, } I_X = I_Y = I \text{ and } I_Z = \frac{1}{2}MR^2$$

$$I_Z = 2I;\ I = \frac{1}{2}I_Z$$

$$I = \frac{1}{2} \times \frac{1}{2}MR^2 = \frac{1}{4}MR^2$$

$$I = \frac{1}{4} \times 3 \times (0.5)^2 = 0.25 \times 3 \times 0.5 \times 0.5$$

$$I = 0.1875\,kg\,m^2$$

EXAMPLE

Find the moment of inertia about the geometric center of the given structure made up of one thin rod connecting two similar solid spheres as shown in Figure.

Solution

The structure is made up of three objects; one thin rod and two solid spheres.

The mass of the rod, M = 3 kg and the total length of the rod, ℓ = 80 cm = 0.8 m

The moment of inertia of the rod about its center of mass is,

$$I_{rod} = \frac{1}{12}M\ell^2 \quad I_{rod} = \frac{1}{12}\times 3\times(0.8)^2 = \frac{1}{4}\times 0.64$$

$$I_{rod} = 0.16\,kg\,m^2$$

The mass of the sphere, M = 5 kg and the radius of the sphere, R = 10 cm = 0.1m $I_C = \frac{2}{5}MR^2$

The moment of inertia of the sphere about geometric center of the structure is,

$$I_{sph} = I_C + Md^2$$

Where, d = 40 cm + 10 cm = 50 cm = 0.5 m

$$I_{sph} = \frac{2}{5}MR^2 + Md^2$$

$$I_{sph} = \frac{2}{5}\times 5\times(0.1)^2 + 5\times(0.5)^2$$

$$I_{sph} = (2\times 0.01) + (5\times 0.25) = 0.02 + 1.25$$

$$I_{sph} = 1.27\,kg\,m^2$$

As there are one rod and two similar solid spheres we can write the total moment of inertia (I) of the given geometric structure as,

$$I = I_{rod} + (2\times I_{sph})$$

$$I = (0.16) + (2\times 1.27) = 0.16 + 2.54$$

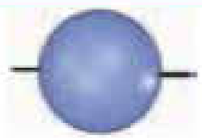
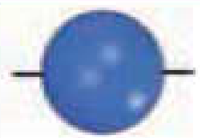
$$I = 2.7\,kg\,m^2$$

Moment of Inertia of Different Rigid Bodies

The moment of inertia of different objects about different axes is given in the Table 5.3.

Table 5.3 Moment of Inertia of Different Rigid Bodies

No.	Object	About an axis	Diagram	Moment of Inertia (I) kg m²	Radius of Gyration (K)	Ratio $\left(\frac{K^2}{R^2}\right)$
1.	**Thin Uniform Rod** Mass = M Length = ℓ	Passing through the center and perpendicular to the length		$\frac{1}{12}M\ell^2$	$\frac{1}{\sqrt{12}}$	--
		Touching one end and perpendicular to the length		$\frac{1}{3}M\ell^2$	$\frac{1}{\sqrt{3}}$	
2.	**Thin Uniform Rectangular Sheet** Mass = M; Length = ℓ; Breadth = b	Passing through the center and perpendicular to the plane of the sheet		$\frac{1}{12}M(\ell^2+b^2)$	$\sqrt{\frac{(\ell^2+b^2)}{12}}$	--
3.	Thin Uniform Ring Mass = M Radius = R	Passing through the center and perpendicular to the plane		MR^2	R	1
		Touching the edge perpendicular to the plane (perpendicular tangent)		$2MR^2$	$(\sqrt{2})R$	2
		Passing through the center lying on the plane (along diameter)		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Touching the edge parallel to the plane (parallel tangent)		$\frac{3}{2}MR^2$	$\left(\sqrt{\frac{3}{2}}\right)R$	$\frac{3}{2}$
4.	**Thin Uniform Disc** Mass = M Radius = R	Passing through the center and perpendicular to the plane		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Touching the edge perpendicular to the plane (perpendicular tangent to the plane)		$\frac{3}{2}MR^2$	$\left(\sqrt{\frac{3}{2}}\right)R$	$\frac{3}{2}$
		Passing through the center lying on the plane (along diameter)		$\frac{1}{4}MR^2$	$\left(\frac{1}{2}\right)R$	$\frac{1}{4}$
		Touching the edge parallel to the plane (parallel tangent to the plane)		$\frac{5}{4}MR^2$	$\left(\sqrt{\frac{5}{4}}\right)R$	$\frac{5}{4}$

5.	**Thin Uniform Hollow Cylinder** Mass = M Length = ℓ; Radius = R	Passing through the center and along the axis of the cylinder		MR^2	R	1
		Passing perpendicular to the length and passing through the center		$M\left(\frac{R^2}{2}+\frac{\ell^2}{12}\right)$	$\sqrt{\frac{R^2}{2}+\frac{\ell^2}{12}}$	- -
6.	Uniform Solid Cylinder Mass = M Length = ℓ; Radius = R	Passing through the center and along the axis of the cylinder		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Passing perpendicular to the length and passing through the center		$M\left(\frac{R^2}{4}+\frac{\ell^2}{12}\right)$	$\sqrt{\frac{R^2}{4}+\frac{\ell^2}{12}}$	- -
7.	Thin Hollow Sphere (Thin Spherical Shell) Mass = M Radius = R	Passing through the center (along diameter)		$\frac{2}{3}MR^2$	$\left(\sqrt{\frac{2}{3}}\right)R^2$	$\frac{2}{3}$
		Touching the edge (tangent)		$\frac{5}{3}MR^2$	$\left(\sqrt{\frac{5}{3}}\right)R$	$\frac{5}{3}$
8.	Uniform Solid Sphere Mass = M Radius = R	Passing through the center (along diameter)		$\frac{2}{5}MR^2$	$\left(\sqrt{\frac{2}{5}}\right)R^2$	$\frac{2}{5}$
		Touching the edge (tangent)		$\frac{7}{5}MR^2$	$\left(\sqrt{\frac{7}{5}}\right)R^2$	$\frac{7}{5}$

ROTATIONAL DYNAMICS

The relations among torque, angular acceleration, angular momentum, angular velocity and moment of inertia were seen in Section 5.2. In continuation to that, in this section, we will learn the relations among the other dynamical quantities like work, kinetic energy in rotational motion of rigid bodies. Finally a comparison between the translational and rotational quantities is made with a tabulation.

Effect of Torque on Rigid Bodies

A rigid body which has non zero external torque (τ) about the axis of rotation would have an angular acceleration (α) about that axis. The scalar relation between the torque and angular acceleration is,

$$\tau = I\alpha$$

where, I is the moment of inertia of the rigid body. The torque in rotational motion is equivalent to the force in linear motion.

EXAMPLE

A disc of mass 500 g and radius 10 cm can freely rotate about a fixed axis as shown in figure. light and inextensible string is wound several turns around it and 100 g body is suspended at its free end. Find the acceleration of this mass. [Given: The string makes the disc to rotate and does not slip over it. g = 10 m s^{-2}.]

Solution

Let the mass of the disc be m1 and its radius R. The mass of the suspended body is m_2.

$$m_1 = 500\text{ g} = 500\times10^{-3}\text{ kg} = 0.5\text{ kg}$$
$$m_2 = 100\text{ g} = 100\times10^{-3}\text{ kg} = 0.1\text{ kg}$$
$$R = 10\text{ cm} = 10\times10^{-2}\text{ m} = 0.1\text{ m}$$

As the light inextensible string is wound around the disc several times it makes the disc rotate without slipping over it. The translational acceleration of m_2 and tangential acceleration of m_1 will be the same. Let us draw the free body diagram (FBD) of m_1 and m_2 separately.

Its gravitational force (m_1g) acts downward and normal force N exerted by the fixed support at the center acts upward. The tension T acts downward at the edge. The gravitational force (m_1g) and the normal force (N) cancel each other. $m_1g = N$

The tension T produces a torque (R T), which produces a rotational motion in the disc with angular acceleration, $\left(\alpha = \frac{a}{R^2}\right)$. Here, a is the linear acceleration of a point at the edge of the disc. If the moment of inertia of the disc is I and its radius of gyration is K, then

$$RT = I\alpha; \quad RT = \left(m_1K^2\right)\frac{a}{R}$$

$$T = \left(m_1K^2\right)\frac{a}{R^2}$$

FBD of the body:

Its gravitational force (m_2g) acts downward and the tension T acts upward. As ($T < m_2g$), there is a resultant force (m2a) acting on it downward.

Substituting for T from the equation for disc,

$$m_2g-\left(m_1K^2\right)\frac{a}{R^2}=m_2a$$

$$m_2g=\left(m_1K^2\right)\frac{a}{R^2}+m_2a$$

$$m_2g=\left[\left(m_1\frac{K^2}{R^2}\right)+m_2\right]a$$

$$a=\frac{m_2}{\left[\left(m_1\frac{K^2}{R^2}\right)+m_2\right]}g$$

The expression $\left(\frac{K^2}{R^2}\right)$ for a disc rotating about an axis passing through the center and perpendicular to the plane is $\frac{K^2}{R^2}=\frac{1}{2}$. Now the expression for acceleration further simplifies as,

$$a=\frac{m_2}{\left[\left(\frac{m_1}{2}\right)+m_2\right]}g\ ;\quad a=\frac{2m_2}{\left[m_1+2m_2\right]}g$$

substituting the values,

$$a=\frac{2\times0.1}{\left[0.5+0.2\right]}\times10=\frac{0.2}{0.7}\times10$$

$$a=2.857\,\mathrm{m\,s^{-2}}$$

Conservation of Angular Momentum

When no external torque acts on the body, the net angular momentum of a rotating rigid body remains constant. This is known as law of conservation of angular momentum.

$$\tau = \frac{dL}{dt}$$

$$\text{If } \tau = 0 \text{ then, } L = \text{constant}$$

As the angular momentum is L = Iω, the conservation of angular momentum could further be written for initial and final situations as,

$$I_i\omega_i = I_f\omega_f \text{ (or) } I\omega = \text{constant}$$

The above equations say that if I increases ω will decrease and vice-versa to keep the angular momentum constant.

There are several situations where the principle of conservation of angular momentum is applicable. One striking example is an ice dancer as shown in Figure 5.27. The dancer spins slowly when the hands are stretched out and spins faster when the hands are brought close to the body. Stretching of hands away from body increases moment of inertia, thus the angular velocity decreases resulting in slower spin. When the hands are brought close to the body, the moment of inertia decreases, and thus the angular velocity increases resulting in faster spin.

A diver while in air as in Figure 5.28 curls the body close to decrease the moment of inertia, which in turn helps to increase the number of somersaults in air.

EXAMPLE

A jester in a circus is standing with his arms extended on a turn table rotating with angular velocity ω. He brings his arms closer to his body so that his moment of inertia is reduced to one third of the original value. Find his new angular velocity. [Given: There is no external torque on the turn table in the given situation.]

Solution

Let the moment of inertia of the jester with his arms extended be I. As there is no external torque acting on the jester and the turn table, his total angular momentum is conserved. We can write the equation,

$$I_i\omega_i = I_f\omega_f$$

$$I\omega = \frac{1}{3}I\omega_f \quad \because\left(I_f = \frac{1}{3}I\right)$$

$$\omega_f = 3\omega$$

The above result tells that the final angular velocity is three times that of initial angular velocity.

Work done by Torque

Let us consider a rigid body rotating about a fixed axis. Figure 5.29 shows a point P on the body rotating about an axis perpendicular to the plane of the page. A tangential force F is applied on the body.

It produces a small displacement ds on the body. The work done (dw) by the force is,

$$dw = Fds$$

As the distance ds, the angle of rotation dθ and radius r are related by the expression,

$$ds = r\,d\theta$$

The expression for work done now becomes,

$$dw = F\,ds;\ dw = F\,r\,d\theta$$

The term (Fr) is the torque τ produced by the force on the body.

$$dw = \tau d\theta$$

This expression gives the work done by the external torque τ, which acts on the body rotating about a fixed axis through an angle dθ.

The corresponding expression for work done in translational motion is,

$$dw = F ds$$

Kinetic Energy in Rotation

Let us consider a rigid body rotating with angular velocity ω about an axis as shown in Figure 5.30. Every particle of the body will have the same angular velocity ω and different tangential velocities v based on its positions from the axis of rotation.

Let us choose a particle of mass m_i situated at distance r_i from the axis of rotation. It has a tangential velocity v_i given by the relation, $v_i = r_i \omega$. The kinetic energy KE_i of the particle is,

$$KE_i = \frac{1}{2} m_i v_i^2$$

Writing the expression with the angular velocity,

$$KE_i = \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} (m_i r_i^2) \omega^2$$

For the kinetic energy of the whole body, which is made up of large number of such particles, the equation is written with summation as,

$$KE = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2$$

where, the term $\sum m_i r_i^2$ is the moment of inertia I of the whole body. $I = \sum m_i r_i^2$

Hence, the expression for KE of the rigid body in rotational motion is,

$$KE = \frac{1}{2} I\omega^2$$

This is analogous to the expression for kinetic energy in translational motion.

$$KE = \frac{1}{2} Mv^2$$

Relation between rotational kinetic energy and angular momentum

Let a rigid body of moment of inertia I rotate with angular velocity ω.

The angular momentum of a rigid body is, $L = I\omega$

The rotational kinetic energy of the rigid body is

$$KE = \frac{1}{2} I\omega^2$$

By multiplying the numerator and denominator of the above equation with I, we get a relation between L and KE as,

$$KE = \frac{1}{2} \frac{I^2\omega^2}{I} = \frac{1}{2} \frac{(I\omega)^2}{I}$$

$$KE = \frac{L^2}{2I}$$

EXAMPLE

Find the rotational kinetic energy of a ring of mass 9 kg and radius 3 m rotating with 240 rpm about an axis passing through its center and perpendicular to its plane. (rpm is a unit of speed of rotation which means revolutions per minute)

Solution

The rotational kinetic energy is, $KE = \frac{1}{2} I\omega^2$ The moment of inertia of the ring is, $I=MR^2$

$$I = 9 \times 3^2 = 9 \times 9 = 81\,\text{kg m}^2$$

The angular speed of the ring is,

$$\omega = 240\,\text{rpm} = \frac{240 \times 2\pi}{60}\,\text{rad s}^{-1}$$

$$KE = \frac{1}{2} \times 81 \times \left(\frac{240 \times 2\pi}{60}\right)^2 = \frac{1}{2} \times 81 \times (8\pi)^2$$

$$KE = \frac{1}{2} \times 81 \times 64 \times (\pi)^2 = 2592 \times (\pi)^2$$

$$KE \approx 25920\,\text{J} \quad \because (\pi)^2 \approx 10$$

$$KE = 25.920\,\text{kJ}$$

Power Delivered by Torque

Power delivered is the work done per unit time. If we differentiate the expression for work done with respect to time, we get the instantaneous power (P).

$$P = \frac{dw}{dt} = \tau \frac{d\theta}{dt} \quad \because (dw = \tau d\theta)$$

$$P = \tau\omega \tag{5.54}$$

The analogous expression for instantaneous power delivered in translational motion is,

$$P = \vec{F} \cdot \vec{v}$$

Comparison of Translational and Rotational Quantities

Many quantities in rotational motion have expressions similar to that of translational motion. The rotational terms are compared with the translational equivalents in Table 5.4.

S.No	Translational Motion	Rotational motion about a fixed axis
1	Displacement, x	Angular displacement, θ
2	Time, t	Time, t
3	Velocity, $v = \frac{dx}{dt}$	Angular velocity, $\omega = \frac{d\theta}{dt}$
4	Acceleration, $a = \frac{dv}{dt}$	Angular acceleration, $\alpha = \frac{d\omega}{dt}$
5	Mass, m	Moment of inertia, I
6	Force, $F = ma$	Torque, $\tau = I\,\alpha$
7	Linear momentum, $p = mv$	Angular momentum, $L = I\omega$
8	Impulse, $F\,\Delta t = \Delta p$	Impulse, $\tau\Delta t = \Delta L$
9	Work done, $w = F\,s$	Work done, $w = \tau\,\theta$
10	Kinetic energy, $KE = \frac{1}{2}mv^2$	Kinetic energy, $KE = \frac{1}{2}I\omega^2$
11	Power, $P = F\,v$	Power, $P = \tau\omega$

ROLLING MOTION

The rolling motion is the most commonly observed motion in daily life. The motion of wheel is an example of rolling motion. Round objects like ring, disc, sphere etc. are most suitable for rolling .

Let us study the rolling of a disc on a horizontal surface. Consider a point P on the edge of the disc. While rolling, the point undergoes translational motion along with its center of mass and rotational motion with respect to its center of mass.

Combination of Translation and Rotation

We will now see how these translational and rotational motions are related in rolling. If the radius of the rolling object is R, in one full rotation, the center of mass is displaced by $2\pi R$ (its circumference). One would agree that not only the center of mass, but all the points on the disc are displaced by the same $2\pi R$ after one full rotation. The only difference is that the center of mass takes a straight path; but, all the other points undergo a path which has a combination of the translational and rotational motion. Especially the point on the edge undergoes a path of a cycloid as shown in the Figure 5.31.

As the center of mass takes only a straight line path, its velocity v_{CM} is only translational velocity v_{TRANS} ($v_{CM} = v_{TRANS}$). All the other points have two velocities. One is the translational velocity v_{TRANS}, (which is also the velocity of center of mass) and the other is the rotational velocity v_{ROT} ($v_{ROT} = r\omega$). Here, r is the distance of the point from the center of mass and ω is the angular velocity. The rotational velocity v_{ROT} is perpendicular to the instantaneous position vector from the center of mass as shown in Figure 5.32(a). The resultant of these two velocities is v. This resultant velocity v is perpendicular to the position vector from the point of contact of the rolling object with the surface on which it is rolling as shown in Figure 5.32(b).

We shall now give importance to the point of contact. In pure rolling, the point of the rolling object which comes in contact with the surface is at momentary rest. This is the case with every point that is on the edge of the rolling object. As the rolling proceeds, all the points on the edge, one by one come in contact with the surface; remain at momentary rest at the time of contact and then take the path of the cycloid as already mentioned.

Hence, we can consider the pure rolling in two different ways.

1. The combination of translational motion and rotational motion about the center of mass.

(or)

2. The momentary rotational motion about the point of contact

As the point of contact is at momentary rest in pure rolling, its resultant velocity v is zero ($v = 0$). For example, in Figure 5.33, at the point of contact, v_{TRANS} is forward (to right) and v_{ROT} is backwards (to the left).

That implies that, v_{TRANS} and v_{ROT} are equal in magnitude and opposite in direction ($v = v_{TRANS} - v_{ROT} = 0$). Hence, we conclude that in pure rolling, for all the points on the edge, the magnitudes of v_{TRANS} and v_{ROT} are equal ($v_{TRANS} = v_{ROT}$). As $v_{TRANS} = v_{CM}$ and $v_{ROT} = R\omega$, in pure rolling we have,

$$v_{CM} = R\,\omega$$

We should remember the special feature of the equation 5.55. In rotational motion, as per the relation $v = r\omega$, the center point will not have any velocity as r is zero. But in rolling motion, it suggests that the center point has a velocity v_{CM} given by equation 5.55.

For the topmost point, the two velocities v_{TRANS} and v_{ROT} are equal in magnitude and in the same direction (to the right). Thus, the resultant velocity v is the sum of these two velocities, $v = v_{TRANS} + v_{ROT}$. In other form, $v = 2\,v_{CM}$ as shown in Figure 5.34.

Slipping and Sliding

When the round object moves, it always tends to roll on any surface which has a coefficient of friction any value greater than zero ($\mu > 0$). The friction that enabling the rolling motion is called rolling friction. In pure rolling, there is no relative motion of the point of contact with the surface. When the rolling object speeds up or slows down, it must accelerate or decelerate respectively. If this suddenly happens it makes the rolling object to slip or slide.

Sliding

Sliding is the case when $v_{CM} > R\omega$ (or $v_{TRANS} > v_{ROT}$). The translation is more

than the rotation. This kind of motion happens when sudden break is applied in a moving vehicles, or when the vehicle enters into a slippery road. In this case, the point of contact has more of v_{TRANS} than v_{ROT}. Hence, it has a resultant velocity v in the forward direction as shown in Figure 5.35. The kinetic frictional force (f_k) opposes the relative motion. Hence, it acts in the opposite direction of the relative velocity. This frictional force reduces the translational velocity and increases the rotational velocity till they become equal and the object sets on pure rolling. Sliding is also referred as forward slipping.

Slipping

Slipping is the case when $v_{CM} < R\omega$ (or $v_{TRANS} < v_{ROT}$). The rotation is more
than the translation. This kind of motion happens when we suddenly start the vehicle from rest or the vehicle is stuck in mud. In this case, the point of contact has more of v_{ROT} than v_{TRANS}. It has a resultant velocity v in the backward direction as shown in Figure 5.36. The kinetic frictional force (f_k) opposes the relative motion. Hence it acts in the opposite direction of the relative velocity. This frictional force reduces the rotational velocity and increases the translational velocity till they become equal and the object sets pure rolling. Slipping is sometimes empahasised as backward slipping.

EXAMPLE

A rolling wheel has velocity of its center of mass as 5 m s-1. If its radius is 1.5 m and angular velocity is 3 rad s^{-1}, then check whether it is in pure rolling or not.

Solution

Translational velocity (v_{TRANS}) or velocity of center of mass, $v_{CM} = 5$ m s^{-1}

The radius is, R = 1.5 m and the angular velocity is, $\omega = 3$ rad s^{-1}

Rotational velocity, $v_{ROT} = R\omega$

$$v_{ROT} = 1.5 \times 3$$

$$v_{ROT} = 4.5 \text{ m s}^{-1}$$

$v_{CM} > R\omega$ (or) $v_{TRANS} > R\omega$, It is not in pure rolling, but sliding.

Kinetic Energy in Pure Rolling

As pure is the combination of translational and rotational motion, we can write the total kinetic energy (KE) as the sum of kinetic energy due to translational motion (KE_{TRANS}) and kinetic energy due to rotational motion (KE_{ROT}).

$$KE = KE_{TRANS} + KE_{ROT}$$

If the mass of the rolling object is M, the velocity of center of mass is v_{CM}, its moment of inertia about center of mass is I_{CM} and angular velocity is ω, then

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

With center of mass as reference: The moment of inertia (I_{CM}) of a rolling object about the center of mass is,

$I_{CM}=MK^2$ and $v_{CM} = R\omega$. Here, K is radius of gyration.

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\left(MK^2\right)\frac{v_{CM}^2}{R^2}$$

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

$$KE = \frac{1}{2}Mv_{CM}^2\left(1 + \frac{K^2}{R^2}\right)$$

With point of contact as reference:

We can also arrive at the same expression by taking the momentary rotation happening with respect to the point of contact (another approach to rolling). If we take the point of contact as O, then,

$$KE = \frac{1}{2} I_o \omega^2$$

Here, Io is the moment of inertia of the object about the point of contact. By parallel axis theorem, $I_o = I_{CM} + MR^2$. Further we can write, $I_o = MK^2 + MR^2$. With $v_{CM} = R\omega$ or

$$\omega = \frac{v_{CM}}{R}$$

$$KE = \frac{1}{2}\left(MK^2 + MR^2\right)\frac{v_{CM}^2}{R^2}$$

$$KE = \frac{1}{2} M v_{CM}^2 \left(1 + \frac{K^2}{R^2}\right)$$

As the two equations 5.59 and 5.60 are the same, it is once again confirmed that the pure rolling problems could be solved by considering the motion as any one of the following two cases.

1. The combination of translational motion and rotational motion about the center of mass.
2. The momentary rotational motion about the point of contact.

EXAMPLE

A solid sphere is undergoing pure rolling. What is the ratio of its translational kinetic energy to rotational kinetic energy?

Solution

The expression for total kinetic energy in pure rolling is,

$$KE = KE_{TRANS} + KE_{ROT}$$

For any object the total kinetic energy as per equation 5.58 and 5.59 is,

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

$$KE = \frac{1}{2}Mv_{CM}^2\left(1+\frac{K^2}{R^2}\right)$$

Then,

$$\frac{1}{2}Mv_{CM}^2\left(1+\frac{K^2}{R^2}\right) = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

The above equation suggests that in pure rolling the ratio of total kinetic energy, translational kinetic energy and rotational kinetic energy is given as,

$$KE : KE_{TRANS} : KE_{ROT} :: \left(1+\frac{K^2}{R^2}\right) : 1 : \left(\frac{K^2}{R^2}\right)$$

$$\text{Now, } KE_{TRANS} : KE_{ROT} :: 1 : \left(\frac{K^2}{R^2}\right)$$

$$\text{For a solid sphere, } \frac{K^2}{R^2} = \frac{2}{5}$$

$$\text{Then, } KE_{TRANS} : KE_{ROT} :: 1 : \frac{2}{5} \quad \text{or}$$

$$KE_{TRANS} : KE_{ROT} :: 5 : 2$$

Rolling on Inclined Plane

Let us assume a round object of mass m and radius R is rolling down an inclined plane without slipping as shown in Figure 5.37. There are two forces acting on the object along the inclined plane. One is the component of gravitational force (mg sinθ) and the other is the static frictional force (f). The other component of gravitation force (mg cosθ) is cancelled by the normal force (N) exerted by the plane. As the motion is happening along the incline, we shall write the equation for motion from the free body diagram (FBD) of the object.

For translational motion, mg sinθ is the supporting force and f is the opposing force

$$mg \sin\theta - f = ma$$

For rotational motion, let us take the torque with respect to the center of the object. Then mg sinθ cannot cause torque as it passes through it but the frictional force f can set torque of Rf.

$$Rf = I\alpha$$

By using the relation, $a = r\,\alpha$, and moment of inertia $I=mK^2$, we get,

$$Rf = mK^2\frac{a}{R}; \quad f = ma\left(\frac{K^2}{R^2}\right)$$

Now equation (5.59) becomes,

$$mg \sin\theta - ma\left(\frac{K^2}{R^2}\right) = ma$$

$$mg\sin\theta = ma + ma\left(\frac{K^2}{R^2}\right)$$

$$a\left(1+\frac{K^2}{R^2}\right) = g\sin\theta$$

After rewriting it for acceleration, we get,

$$a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

We can also find the expression for final velocity of the rolling object by using third equation of motion for the inclined plane. $v^2 = u^2 + 2as$. If the body starts rolling from rest, u=0. When h is the vertical height of the incline, the length of the incline s is, $s = \frac{h}{\sin\theta}$

$$v^2 = 2\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}\left(\frac{h}{\sin\theta}\right) = \frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}$$

By taking square root,

$$v = \sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}$$

The time taken for rolling down the incline could also be written from first equation of motion as, v = u + at. For the object which starts rolling from rest, u=0. Then,

$$t = \frac{v}{a}$$

$$t = \left(\sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}\right)\left(\frac{\left(1+\frac{K^2}{R^2}\right)}{g\sin\theta}\right)$$

$$t = \sqrt{\frac{2h\left(1+\frac{K^2}{R^2}\right)}{g\sin^2\theta}} \qquad (5.64)$$

The equation suggests that for a given incline, the object with the least value of radius of gyration K will reach the bottom of the incline first.

EXAMPLE

Four round objects namely a ring, a disc, a hollow sphere and a solid sphere with same radius R start to roll down an incline at the same time. Find out which object will reach the bottom first.

Solution

For all the four objects namely the ring, disc, hollow sphere and solid sphere, the radii of gyration K are R, $\sqrt{\frac{1}{2}}R, \sqrt{\frac{2}{3}}R, \sqrt{\frac{2}{5}}R,$ ref Table (5.3)). With numerical values the radius of gyration K are 1R, 0.707R, 0.816R, 0.632R respectively. The expression for time taken for rolling has the radius of gyration K in the numerator as per equation 5.63

$$t = \sqrt{\frac{2h\left(1 + \frac{K^2}{R^2}\right)}{g \sin^2\theta}}$$

The one with least value of radius of gyration K will take the shortest time to reach the bottom of the inclined plane. The order of objects reaching the bottom is first, solid sphere; second, disc; third, hollow sphere and last, ring.