FORCE AND MOTION

PART – IV

11TH VOL - II
UNIT – 6 GRAVITATION

INTRODUCTION

We are amazed looking at the glittering sky; we wonder how the Sun rises in the East and sets in the West, why there are comets or why stars twinkle. The sky has been an object of curiosity for human beings from time immemorial. We have always wondered about the motion of stars, the Moon, and the planets. From Aristotle to Stephen Hawking, great minds have tried to understand the movement of celestial objects in space and what causes their motion.

The 'Theory of Gravitation' was developed by Newton in the late 17th century to explain the motion of celestial objects and terrestrial objects and answer most of the queries raised. In spite of the study of gravitation and its effect on celestial objects, spanning last three centuries, "gravitation" is still one of the active areas of research in physics today. In 2017, the Nobel Prize in Physics was given for the detection of 'Gravitational waves' which was theoretically predicted by Albert Einstein in the year 1915. Understanding planetary motion, the formation of stars and galaxies, and recently massive objects like black holes and their life cycle have remained the focus of study for the past few centuries in physics.

Geocentric Model of Solar System

In the second century, Claudius Ptolemy, a famous Greco-Roman astronomer, developed a theory to explain the motion of celestial objects like the Sun, the Moon, Mars, Jupiter etc. This theory was called the geocentric model. According to the geocentric model, the Earth is at the center of the universe and all celestial objects including the Sun, the Moon, and other planets orbit the Earth. Ptolemy's model closely matched with the observations of the sky with our naked eye. But later, astronomers found that even though Ptolemy's model successfully explained the motion of the Sun and the Moon up to a certain level, the motion of Mars and Jupiter could not be explained effectively.

Heliocentric Model of Nicholas Copernicus

In the 15th century, a Polish astronomer, Nicholas Copernicus (1473-1543) proposed a new model called the 'Heliocentric model' in which the Sun was considered to be at the center of the solar system and all planets including the Earth orbited the Sun in circular orbits. This model successfully explained the motion of all celestial objects.

Around the same time, Galileo, a famous Italian physicist discovered that all objects close to Earth were accelerated towards the Earth at the same rate. Meanwhile, a noble man called Tycho Brahe (1546-1601) spent his entire lifetime in recording the observations of the stellar and planetary positions with his naked eye. The data that he compiled were analyzed later by his assistant Johannes Kepler (1571–1630) and eventually the analysis led to the deduction of the laws of the planetary motion. These laws are termed as 'Kepler's laws of planetary motion'.

Kepler's Laws of Planetary Motion

Law of orbits:

Each planet moves around the Sun in an elliptical orbit with the Sun at one of the foci.

The closest point of approach of the planet to the Sun 'P' is called perihelion and the farthest point 'A' is called aphelion (Figure 6.1). The semi-major axis is 'a' and semi-minor axis is 'b'. In fact, both Copernicus and Ptolemy considered planetary orbits to be circular, but Kepler discovered that the actual orbits of the planets are elliptical.

Law of area:

The radial vector (line joining the Sun to a planet) sweeps equal areas in equal intervals of time.

In Figure 6.2, the white shaded portion is the area DA swept in a small interval of time Dt, by a planet around the Sun. Since the Sun is not at the center of the ellipse, the planets travel faster when they are nearer to the Sun and slower when they are farther from it, to cover equal area in equal intervals of time. Kepler discovered the law of area by carefully noting the variation in the speed of planets.

Law of period:

The square of the time period of revolution of a planet around the Sun in its elliptical orbit is directly proportional to the cube of the semi-major axis of the ellipse. It can be written as:

$$T^2 \propto a^3$$

$$\frac{T^2}{a^3} = constant$$

where, T is the time period of revolution for a planet and a is the semi-major axis. Physically this law implies that as the distance of the planet from the Sun increases, the time period also increases but not at the same rate.

In Table 6.1, the time period of revolution of planets around the Sun along with their semi-major axes are given. From column four, we can realize that $\frac{T^2}{a^3}$ is nearly a constant endorsing Kepler's third law.

Planet	a ($10^{10} m$)	T (years)	$\frac{T^2}{a^3}$
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84	2.98
Neptune	450	165	2.99

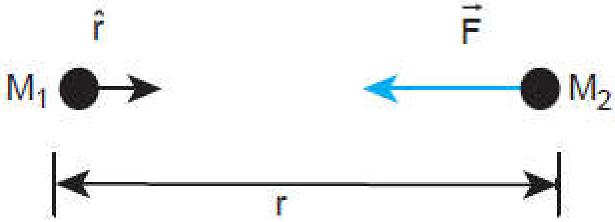
Universal Law of Gravitation

Even though Kepler's laws were able to explain the planetary motion, they failed to explain the forces responsible for it. It was Isaac Newton who analyzed Kepler's laws, Galileo's observations and deduced the law of gravitation.

Newton's law of gravitation states that a particle of mass M_1 attracts any other particle of mass M_2 in the universe with an attractive force. The strength of this force of attraction was found to be directly proportional to the product of their masses and is inversely proportional to the square of the distance between them. In mathematical form, it can be written as:

$$\vec{F} = -\frac{GM_1M_2}{r^2}\hat{r}$$

where $\hat{r}$ is the unit vector from M_1 towards M_2 as shown in Figure 6.3, and G is the Gravitational constant that has the value of 6.626×10^{-11}. Nm2kg^{-2}, and r is the distance between the two masses M_1 and M_2. In Figure 6.3, the vector $\vec{F}$ denotes the gravitational force experienced by M_2 due to M_1. Here the negative sign indicates that the gravitational force is always attractive in nature and the direction of the force is along the line joining the two masses.



In cartesian coordinates, the square of the distance is expressed as $r^2=(x^2+y^2+z^2)$ This is dealt in unit 2.

EXAMPLE

Consider two point masses m_1 and m_2 which are separated by a distance of 10 meter as shown in the following figure. Calculate the force of attraction between them and draw the directions of forces on each of them. Take m_1= 1 kg and m_2 = 2 kg

Solution

The force of attraction is given by

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

From the figure, r =10 m.

First, we can calculate the magnitude of the force

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{100}$$
$$=13.34\times10^{-13}N.$$

It is to be noted that this force is very small. This is the reason we do not feel the gravitational force of attraction between each other. The small value of G plays a very crucial role in deciding the strength of the force.

The force of attraction ($\vec{F}_{21}$) experienced by the mass m_2 due to m_1 is in the negative 'y' direction ie., $\hat{r}=-\hat{j}$.According to Newton's third law, the mass m_2 also exerts equal and opposite force on m_1. So the force of attraction ($\vec{F}_{12}$) experienced by m1 due to m2 is in the direction of positive 'y' axis ie., $\hat{r}=\hat{j}$.

$$\vec{F}_{21} = -13.34\times10^{-13}\hat{j}$$

$$\vec{F}_{12} = 13.34\times10^{-13}\hat{j}$$

The direction of the force is shown in the figure,
Gravitational force of attraction between m_1 and m_2
$\vec{F}_{12}$=-$\vec{F}_{21}$ which confirms Newton's third law.

Important features of gravitational force:

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As the distance between two masses increases, the strength of the force tends to decrease because of inverse dependence on r^2. Physically it implies that the planet Uranus experiences less gravitational force from the Sun than the Earth since Uranus is at larger distance from the Sun compared to the Earth.

The gravitational forces between two particles always constitute an action- reaction pair. It implies that the gravitational force exerted by the Sun on the Earth is always towards the Sun. The reaction-force is exerted by the Earth on the Sun. The direction of this reaction force is towards Earth.

The torque experienced by the Earth due to the gravitational force of the Sun is given by

$$\vec{\tau} = \vec{r}\times\vec{F} = \vec{r}\times\left(-\frac{GM_S M_E}{r^2}\hat{r}\right) = 0$$

$$\text{Since } \vec{r} = r\,\hat{r},\ \left(\hat{r}\times\hat{r}\right) = 0$$

- So $\bar{\tau} = \frac{d\bar{L}}{dt} = 0$. It implies that angular momentum $\dot{L}$ is a constant vector. The angular momentum of the Earth about the Sun is constant throughout the motion. It is true for all the planets. In fact, this constancy of angular momentum leads to the Kepler's second law.
- The expression $\bar{F} = -\frac{GM_1M_2}{r^2}\hat{r}$ has one inherent assumption that both M_1 and M_2 are treated as point masses. When it is said that Earth orbits around the Sun due to Sun's gravitational force, we assumed Earth and Sun to be point masses. This assumption is a good approximation because the distance between the two bodies is very much larger than their diameters. For some irregular and extended objects separated by a small distance, we cannot directly use the equation (6.3). Instead, we have to invoke separate mathematical treatment which will be brought forth in higher classes.
- However, this assumption about point masses holds even for small distance for one special case. To calculate force of attraction between a hollow sphere of mass M with uniform density and point mass m kept outside the hollow sphere, we can replace the hollow sphere of mass M as equivalent to a point mass M located at the center of the hollow sphere. The force of attraction between the hollow sphere of mass M and point mass m can be calculated by treating the hollow sphere also as another point mass. Essentially the entire mass of the hollow sphere appears to be concentrated at the center of the hollow sphere.
- There is also another interesting result. Consider a hollow sphere of mass M. If we place another object of mass 'm' inside this hollow sphere as in Figure 6.5(b), the force experienced by this mass 'm' will be zero. This calculation will be dealt with in higher classes.
- The triumph of the law of gravitation is that it concludes that the mango that is falling down and the Moon orbiting the Earth are due to the same gravitational force.

Newton's inverse square Law:

Newton considered the orbits of the planets as circular. For circular orbit of radius r, the centripetal acceleration towards the center is

$$a = -\frac{v^2}{r}$$

Here v is the velocity and r, the distance of the planet from the center of the orbit

The velocity in terms of known quantities r and T, is

$$v = \frac{2\pi r}{T}$$

Here T is the time period of revolution of the planet. Substituting this value of v in equation (6.4) we get,

$$a = -\frac{\left(\frac{2\pi r}{T}\right)^2}{r} = -\frac{4\pi^2 r}{T^2}$$

Substituting the value of 'a' from (6.6) in Newton's second law, F=ma, where 'm' is the mass of the planet.

$$F = -\frac{4\pi^2 mr}{T^2}$$

From Kepler's third law,

$$\frac{r^3}{T^2} = k(constant)$$

$$\frac{r}{T^2} = \frac{k}{r^2}$$

By substituting equation 6.9 in the force expression, we can arrive at the law of gravitation.

$$F = -\frac{4\pi^2 mk}{r^2}$$

Here negative sign implies that the force is attractive and it acts towards the center. In equation (6.10), mass of the planet 'm' comes explicitly. But Newton strongly felt that according to his third law, if Earth is attracted by the Sun, then the Sun must also be attracted by the Earth with the same magnitude of force. So he felt that the Sun's mass (M) should also occur explicitly in the expression for force (6.10). From this insight, he equated the constant $4\pi^2$k to GM which turned out to be the law of gravitation

$$F = -\frac{GMm}{r^2}$$

Again the negative sign in the above equation implies that the gravitational force is attractive.

In the above discussion we assumed that the orbit of the planet to be circular which is not true as the orbit of the planet around the Sun is elliptical. But this circular orbit assumption is justifiable because planet's orbit is very close to being circular and there is only a very small deviation from the circular shape.

EXAMPLE

Moon and an apple are accelerated by the same gravitational force due to Earth. Compare the acceleration of the two.

The gravitational force experienced by the apple due to Earth

$$F = -\frac{GM_E M_A}{R^2}$$

Here M_A– Mass of the apple, M_E– Mass of the Earth and R – Radius of the Earth.

Equating the above equation with Newton's second law,

$$M_A a_A = -\frac{GM_E M_A}{R^2}.$$

Simplifying the above equation we get,

$$a_A = -\frac{GM_E}{R^2}$$

Here a_A is the acceleration of apple that is equal to 'g'. Similarly the force experienced by Moon due to Earth is given by

$$F = -\frac{GM_E M_m}{R_m^2}.$$

Here R_m- distance of the Moon from the Earth, M_m – Mass of the Moon

The acceleration experienced by the Moon is given by

$$a_m = -\frac{GM_E}{R_m^2}.$$

The ratio between the apple's acceleration to Moon's acceleration is given by

$$\frac{a_A}{a_m} = \frac{R_m^2}{R^2}.$$

From the Hipparchrus measurement, the distance to the Moon is 60 times that of Earth radius. R_m = 60R.

$$a_A / a_m = \frac{(60R)^2}{R^2} = 3600.$$

The apple's acceleration is 3600 times the acceleration of the Moon.

The same result was obtained by Newton using his gravitational formula. The apple's acceleration is measured easily and it is 9.8 m s−2. Moon orbits the Earth once in 27.3 days and by using the centripetal acceleration formula, (Refer unit 3).

$$\frac{a_A}{a_m} = \frac{9.8}{0.00272} = 3600$$

which is exactly what he got through his law of gravitation.

Gravitational Constant

In the law of gravitation, the value of gravitational constant G plays a very important role. The value of G explains why the gravitational force between the Earth and the Sun is so great while the same force between two small objects (for example between two human beings) is negligible.

The force experienced by a mass 'm' which is on the surface of the Earth (Figure 6.7) is given by

$$F = -\frac{GM_E m}{R_E{}^2}$$

M_E-mass of the Earth, m - mass of the object, R_E- radius of the Earth.

Equating Newton's second law, F =mg, to equation (6.11) we get,

$$mg = -\frac{GM_E m}{R_E^2}$$

$$g = -\frac{GM_E}{R_E^2}$$

Now the force experienced by some other object of mass M at a distance r from the center of the Earth is given by,

$$F = -\frac{GM_E M}{r^2}$$

Using the value of g in equation (6.12), the force F will be,

$$F = -gM\frac{R_E^2}{r^2}$$

From this it is clear that the force can be calculated simply by knowing the value of g. It is to be noted that in the above calculation G is not required.

In the year 1798, Henry Cavendish experimentally determined the value of gravitational constant 'G' by using a torsion balance. He calculated the value of 'G' to be equal to 6.75 $\times 10^{-11}$Nm2kg^{-2} . Using modern techniques a more accurate value of G could be measured. Th e currently accepted value of G is 6.67259 $\times 10^{-11}$Nm2kg^{-2} .

GRAVITATIONAL FIELD AND GRAVITATIONAL POTENTIAL

Gravitational field

Force is basically due to the interaction between two particles. Depending upon the type of interaction we can have two kinds of forces:
Contact forces and
Non-contact forces (Figure 6.8).

Contact forces are the forces applied where one object is in physical contact with the other. The movement of the object is caused by the physical force exerted through the contact between the object and the agent which exerts force.

Consider the case of Earth orbiting around the Sun. Though the Sun and the Earth are not physically in contact with each other, there exists an interaction between them. This is because of the fact that the Earth experiences the gravitational force of the Sun. This gravitational force is a non-contact force.

It sounds mysterious that the Sun attracts the Earth despite being very far from it and without touching it. For contact forces like push or pull, we can calculate the strength of the force since we can feel or see. But how do we calculate the strength of non-contact force at different distances? To understand and calculate the strength of non-contact forces, the concept of 'field' is introduced.

The gravitational force on a particle of mass 'm_2' due to a particle of mass 'm_1' is

$$\vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

where $\hat{r}$ is a unit vector that points from m_1 to m_2 along the line joining the masses m_1 and m_2.

The gravitational field intensity $\vec{E}_1$ (here after called as gravitational field) at a point which is at a distance r from m_1 is defined as the gravitational force experienced by unit mass placed at that point. It given by the ratio $\frac{\overline{F}_{21}}{m_2}$ (where m_2 is the mass of the object on which $\vec{F}_{21}$ acts)

Using $\vec{E}_1 = \frac{\overline{F}_{21}}{m_2}$ in equation (6.14) we get,

$$\vec{E}_1 = -\frac{Gm_1}{r^2}\hat{r}$$

$\vec{E}_1$ is a vector quantity that points towards the mass m1 and is independent of mass m2, Here m2 is taken to be of unit magnitude . The unit is $\hat{r}$ along the line between m_1 and the point in question. The field $\vec{E}_1$ is due to the mass m_1. In general, the gravitational field intensity due to a mass M at a distance r is given by

$$\vec{E} = -\frac{GM}{r^2}\hat{r}$$

Now in the region of this gravitational field, a mass 'm' is placed at a point P (Figure 6.9). Mass 'm' interacts with the field $\vec{E}_1$ and experiences an attractive force due to M as shown in Figure 6.9. The gravitational force experienced by 'm' due to 'M' is given by

$$\vec{F}_m = m\vec{E}$$

Now we can equate this with Newton's second law $\vec{F} = m\vec{a}$

$$m\vec{a} = m\vec{E}$$

$$\vec{a} = \vec{E}$$

In other words, equation (6.18) implies that the gravitational field at a point is equivalent to the acceleration experienced by a particle at that point. However, it
is to be noted that $\vec{a}$ and $\vec{E}$ are separate physical quantities that have the same magnitude and direction. The gravitational field $\vec{E}$ is the property of the source and acceleration $\vec{a}$ is the effect experienced by the test mass (unit mass) which is placed in the gravitational field $\vec{E}$. The noncontact interaction between two masses can now be explained using the concept of "Gravitational field".

Points to be noted:

The strength of the gravitational field decreases as we move away from the mass M as depicted in the Figure 6.10. The magnitude of $\vec{E}$ decreases as the distance r increases.

Figure 6.10 shows that the strength of the gravitational field at points P, Q, and R is given by $|\vec{E}_P| < |\vec{E}_Q| < |\vec{E}_R|$. It can be understood by comparing the length of the vectors at points P, Q, and R.

The "field" concept was introduced as a mathematical tool to calculate gravitational interaction. Later it was found that field is a real physical quantity and it carries energy and momentum in space. The concept of field is inevitable in understanding the behavior of charges.

The unit of gravitational field is Newton per kilogram (N/kg) or m s^{-2}.

Superposition principle for Gravitational field

Consider 'n' particles of masses $m_1, m_2, m_3, ...$ distributed in space at positions $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...$etc, with respect to point P. The total gravitational field at a point P due to all the masses is given by the vector sum of the gravitational field due to the individual masses (Figure 6.11). This principle is known as superposition of gravitational fields.

$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 + \ldots \vec{E}_n$$

$$= -\frac{Gm_1}{r_1^2}\hat{r}_1 - \frac{Gm_2}{r_2^2}\hat{r}_2 - \ldots - \frac{Gm_n}{r_n^2}\hat{r}_n$$

$$= -\sum_{i=1}^{n} \frac{Gm_i}{r_i^2}\hat{r}_i \cdot$$

Instead of discrete masses, if we have continuous distribution of a total mass M, then the gravitational field at a point P is calculated using the method of integration.

EXAMPLE

Two particles of masses m_1 and m_2 are placed along the x and y axes respectively at a distance 'a' from the origin. Calculate the gravitational field at a point P shown in figure below.

Solution

Gravitational field due to m_1 at a point P is given by

$$\vec{E}_1 = -\frac{Gm_1}{a^2}\hat{j}$$

Gravitational field due to m_2 at the point p is given by,

$$\vec{E}_2 = -\frac{Gm_2}{a^2}\hat{i}$$

$$\vec{E}_{total} = -\frac{Gm_1}{a^2}\hat{j} - \frac{Gm_2}{a^2}\hat{i}$$

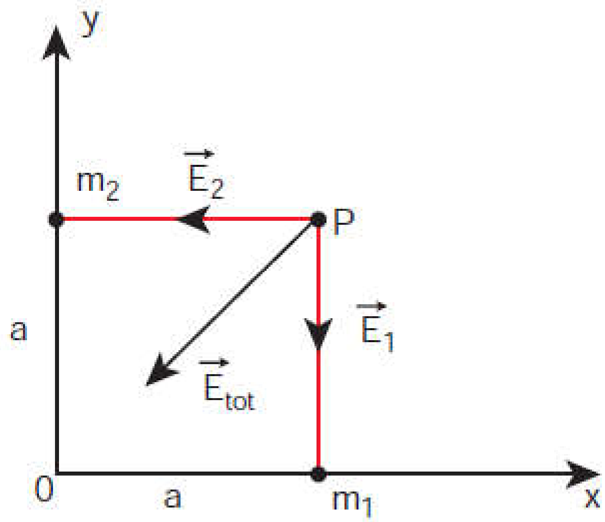
$$= -\frac{G}{a^2}\left(m_1\hat{j} + m_2\hat{i}\right)$$

The direction of the total gravitational field is determined by the relative value of m_1 and m_2

When $m_1 = m_2 = m$

$$\vec{E}_{total} = -\frac{Gm}{a^2}\left(\hat{i} + \hat{j}\right)$$

($\hat{i} + \hat{j} = \hat{j} + \hat{i}$ as vectors obeys commutation law).



$\vec{E}_{total}$ points towards the origin of the co-ordinate system and the magnitude of $\vec{E}_{total}$ is $\frac{Gm}{a^2}$.

EXAMPLE

Qualitatively indicate the gravitational field of Sun on Mercury, Earth, and Jupiter shown in figure.

Since the gravitational field decreases as distance increases, Jupiter experiences a weak gravitational field due to the Sun. Since Mercury is the nearest to the Sun, it experiences the strongest gravitational field.

Gravitational Potential Energy

The concept of potential energy and its physical meaning were dealt in unit 4. The gravitational force is a conservative force and hence we can define a gravitational potential energy associated with this conservative force field.

Two masses m_1 and m_2 are initially separated by a distance r′. Assuming m_1 to be fixed in its position, work must be done on m_2 to move the distance from r′ to r as shown in Figure 6.12(a).

To move the mass m_2 through an infinitesimal displacement $d\vec{r}$ from $\vec{r}$ to $\vec{r} + d\vec{r}$ (shown in the Figure 6.12(b)), work has to be done externally. This infinitesimal work is given by

$$dW = \vec{F}_{ext}.d\vec{r}$$

The work is done against the gravitational force, therefore,

$$\left|\vec{F}_{ext}\right| = \left|\vec{F}_G\right| = \frac{Gm_1m_2}{r^2}$$

Substituting Equation (6.22) in 6.21, we get

$$dW = \frac{Gm_1m_2}{r^2}\hat{r}.d\vec{r}$$

Also we know,

$$d\vec{r} = dr\,\hat{r} \tag{6.24}$$

$$\Rightarrow dW = \frac{Gm_1m_2}{r^2}\hat{r}.\left(dr\,\hat{r}\right) \tag{6.25}$$

$$\hat{r}.\hat{r} = 1\left(since\ both\ are\ unit\ vectors\right)$$

$$\therefore\ dW = \frac{Gm_1m_2}{r^2}dr \tag{6.26}$$

Thus the total work done for displacing the particle from r′ to r is

$$W = \int_{r'}^{r} dW = \int_{r'}^{r} \frac{Gm_1m_2}{r^2}dr$$

$$W = -\left(\frac{Gm_1m_2}{r}\right)_{r'}^{r}$$

$$W = -\frac{Gm_1m_2}{r} + \frac{Gm_1m_2}{r'}$$

$$W = U(r) - U(r')$$

$$\text{where } U(r) = \frac{-Gm_1m_2}{r}$$

This work done W gives the gravitational potential energy difference of the system of masses m_1 and m_2 when the separation between them are r and r′ respectively.

Case 1: If r<r′

Since gravitational force is attractive, m_2 is attracted by m_1.Then m_2 can move from r to r′ without any external work (Figure 6.13). Here work is done by the system spending its internal energy and hence the work done is said to be negative.

Case 2: If r>r′

Work has to be done against gravity to move the object from r to r′. Therefore work is done on the body by external force and hence work done is positive.

It is to be noted that only potential energy difference has physical significance. Now gravitational potential energy can be discussed by choosing one point as the reference point

Let us choose $r' = \infty$. Then the second term in the equation (6.28) becomes zero.

$$W = -\frac{Gm_1m_2}{r} + 0$$

Now we can define gravitational potential energy of a system of two masses m_1 and m_2 separated by a distance r as the amount of work done to bring the mass m_2 from infinity to a distance r assuming m_1 to be fixed in its position and is written as $U(r) = -\frac{Gm_1m_2}{r}$. It is to be noted that the gravitational potential energy of the system consisting of two masses m_1 and m_2 separated by a distance r, is the gravitational potential energy difference of the system when the masses are separated by an infinite distance and by distance r. $U(r) = U(r) - U(\infty)$. Here we choose $U(\infty)$=0 as the reference point. The gravitational potential energy $U(r)$ is always negative because when two masses come together slowly from infinity, work is done by the system.

The unit of gravitational potential energy $U(r)$ is Joule and it is a scalar quantity. The gravitational potential energy depends upon the two masses and the distance between them.

Gravitational potential energy near the surface of the Earth

It is already discussed in chapter 4 that when an object of mass m is raised to a height h, the potential energy stored in the object is mgh (Figure 6.14). This can be derived using the general expression for gravitational potential energy

Consider the Earth and mass system, with r, the distance between the mass m and the Earth's centre. Then the gravitational potential energy,

$$U = -\frac{GM_em}{r}$$

Here r = R_e+h, where R_e is the radius of the Earth. h is the height above the Earth's surface

$$U = -G\frac{M_em}{(R_e + h)}$$

If h << Re, equation (6.31) can be modified as

$$U = -G\frac{M_e m}{R_e\left(1+h/R_e\right)}$$

$$U = -G\frac{M_e m}{R_e}\left(1+h/R_e\right)^{-1}$$

By using Binomial expansion and neglecting the higher order terms, we get

$$U = -G\frac{M_e m}{R_e}\left(1-\frac{h}{R_e}\right).$$

We know that, for a mass m on the Earth's surface,

$$G\,\frac{M_e m}{R_e} = mgR_e$$

Substituting equation (6.34) in (6.33) we get,

$$U = -mgR_e + mgh$$

It is clear that the first term in the above expression is independent of the height h. For example, if the object is taken from height h_1 to h_2,then the potential energy at h_1 is

$$U\left(h_1\right) = -mgR_e + mgh_1$$

and the potential energy at h_2 is

$$U\left(h_2\right) = -mgR_e + mgh_2$$

The potential energy difference between h1 and h_2 is

$$U(h_2) - U(h_1) = mg(h_1 - h_2).$$

The term mgR_e in equations (6.36) and (6.37) plays no role in the result. Hence in the equation (6.35) the first term can be omitted or taken to zero. Thus it can be stated that The gravitational potential energy stored in the particle of mass m at a height h from the surface of the Earth is U =mgh. On the surface of the Earth, U = 0, since h is zero.

It is to be noted that mgh is the work done on the particle when we take the mass m from the surface of the Earth to a height h. This work done is stored as a gravitational potential energy in the mass m. Even though mgh is gravitational potential energy of the system (Earth and mass m), we can take mgh as the gravitational potential energy of the mass m since Earth is stationary when the mass moves to height h.

Gravitational potential V(r)

It is explained in the previous sections that the gravitational field $\dot{E}$ depends only on the source mass which creates the field. It is a vector quantity. We can also define a scalar quantity called "gravitational potential" which depends only on the source mass.

The gravitational potential at a distance r due to a mass is defined as the amount of work required to bring unit mass from infinity to the distance r and it is denoted as V(r). In other words, the gravitational potential at distance r is equivalent to gravitational potential energy per unit mass at the same distance r. It is a scalar quantity and its unit is J kg^{-1}

We can determine gravitational potential from gravitational potential energy. Consider two masses m_1 and m_2 separated by a distance r which has gravitational potential energy U (r) - (Figure 6.15). The gravitational potential due to mass m_1 at a point P which is at a distance r from m_1 is obtained by making m_2 equal to unity (m_2 = 1kg). Thus the gravitational potential V r - due to mass m_1 at a distance r is

$$V(r) = -\frac{Gm_1}{r}$$

Gravitational field and gravitational force are vector quantities whereas the gravitational potential and gravitational potential energy are scalar quantities. The motion of particles can be easily analyzed using scalar quantities than vector quantities. Consider the example of a falling apple:

Figure 6.16 shows an apple which falls on Earth due to Earth's gravitational force. This can be explained using the concept of gravitational potential V (r) - as follows.

The gravitational potential V (r) - at a point of height h from the surface of the Earth is given by,

$$V(r = R + h) = -\frac{GM_e}{(R + h)}$$

The gravitational potential V (r) - on the surface of Earth is given by,

$$V(r = R) = -\frac{GM_e}{R}$$

Thus we see that

$$V(r = R) < V(r = R + h).$$

It is already discussed in the previous section that the gravitational potential energy near the surface of the Earth at height h is mgh. The gravitational potential at this point is simply V(h)= U(h)/m = gh. In fact, the gravitational potential on the surface of the Earth is zero since h is zero. So the apple falls from a region of a higher gravitational potential to a region of lower gravitational potential. In general, the mass will move from a region of higher gravitational potential to a region of lower gravitational potential.

EXAMPLE

Water falls from the top of a hill to the ground. Why?

This is because the top of the hill is a point of higher gravitational potential than the surface of the Earth i.e. V_{hill} V_{ground}

The motion of particles can be analysed more easily using scalars like U(r) or V(r) than vector quantities like $\vec{F}$ or $\vec{E}$. In modern theories of physics, the concept of potential plays a vital role.

EXAMPLE

Consider four masses m_1, m_2, m_3, and m_4 arranged on the circumference of a circle as shown in figure below

Calculate

(a) The gravitational potential energy of the system of 4 masses shown in figure.

(b) The gravitational potential at the point O due to all the 4 masses

Solution

The gravitational potential energy U(r) can be calculated by finding the sum of gravitational potential energy of each pair of particles.

$$U = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_1m_4}{r_{14}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_2m_4}{r_{24}} - \frac{Gm_3m_4}{r_{34}}$$

Here r_{12}, r_{13} ... are distance between pair of particles

$$r_{14}{}^2 = R^2 + R^2 = 2R^2$$

$$r_{14} = \sqrt{2}R = r_{12} = r_{23} = r_{34}$$

$$r_{13} = r_{24} = 2R$$

$$U = -\frac{Gm_1m_2}{\sqrt{2}R} - \frac{Gm_1m_3}{2R} - \frac{Gm_1m_4}{\sqrt{2}R} - \frac{Gm_2m_3}{\sqrt{2}R} - \frac{Gm_2m_4}{2R} - \frac{Gm_3m_4}{\sqrt{2}R}$$

$$U = -\frac{G}{R}\left[\frac{m_1m_2}{\sqrt{2}} + \frac{m_1m_3}{2} + \frac{m_1m_4}{\sqrt{2}} + \frac{m_2m_3}{\sqrt{2}} + \frac{m_2m_4}{2} + \frac{m_3m_4}{\sqrt{2}}\right]$$

If all the masses are equal, then $m_1 = m_2 = m_3 = m_4 = M$

$$U = -\frac{GM^2}{R}\left[\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}}\right]$$

$$U = -\frac{GM^2}{R}\left[1 + \frac{4}{\sqrt{2}}\right]$$

$$U = -\frac{GM^2}{R}\left[1 + 2\sqrt{2}\right]$$

The gravitational potential V(r) at a point O is equal to the sum of the gravitational potentials due to individual mass. Since potential is a scalar, the net potential at point O is the algebraic sum of potentials due to each mass.

$$V_O(r) = -\frac{Gm_1}{R} - \frac{Gm_2}{R} - \frac{Gm_3}{R} - \frac{Gm_4}{R}$$

$$\text{If } m_1 = m_2 = m_3 = m_4 = M$$

$$V_O(r) = -\frac{4GM}{R}$$

ACCELERATION DUE TO GRAVITY OF THE EARTH

When objects fall on the Earth, the acceleration of the object is towards the Earth. From Newton's second law, an object is accelerated only under the action of a force. In the case of Earth, this force is the gravitational pull of Earth. This force produces a constant acceleration near the Earth's surface in all bodies, irrespective of their masses. The gravitational force exerted by Earth on the mass m near the surface of the Earth is given by

$$\vec{F} = -\frac{GmM_e}{R_e^2}\hat{r}$$

Now equating Gravitational force to Newton's second law,

$$m\vec{a} = -\frac{GmM_e}{R_e^2}\hat{r}$$

hence, acceleration is,

$$\vec{a} = -\frac{GM_e}{R_e^2}\hat{r}$$

The acceleration experienced by the object near the surface of the Earth due to its gravity is called acceleration due to gravity. It is denoted by the symbol g. The magnitude of acceleration due to gravity is

$$|g| = \frac{GM_e}{R_e^{\,2}}.$$

It is to be noted that the acceleration experienced by any object is independent of its mass. The value of g depends only on the mass and radius of the Earth. Infact, Galileo arrived at the same conclusion 400 years ago that all objects fall towards the Earth with the same acceleration through various quantitative experiments. The acceleration due to gravity g is found to be 9.8 m s^{-2} on the surface of the Earth near the equator.

Variation of g with altitude, depth and latitude

Consider an object of mass m at a height h from the surface of the Earth. Acceleration experienced by the object due to Earth is

$$g' = \frac{GM}{(R_e + h)^2}$$

$$g' = \frac{GM}{R_e^{\,2}\left(1 + \frac{h}{R_e}\right)^2}$$

$$g' = \frac{GM}{R_e^{\,2}}\left(1 + \frac{h}{R_e}\right)^{-2}$$

If h <<R_e

We can use Binomial expansion. Taking the terms upto first order

$$g' = \frac{GM}{R_e^{\ 2}}\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

We find that g′<g . This means that as altitude h increases the acceleration due to gravity g decreases

EXAMPLE

Calculate the value of g in the following two cases:
(a) If a mango of mass ½ kg falls from a tree from a height of 15 meters, what is the acceleration due to gravity when it begins to fall?
Solution

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = 9.8\left(1 - \frac{2 \times 15}{6400 \times 10^3}\right)$$

$$g' = 9.8\left(1 - 0.469 \times 10^{-5}\right)$$

$$\text{But} \quad 1 - 0.00000469 \cong 1$$

Therefore g′=g

(b) Consider a satellite orbiting the Earth in a circular orbit of radius 1600 km above the surface of the Earth. What is the acceleration experienced by the satellite due to Earth's gravitational force?

Solution

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = g\left(1 - \frac{2 \times 1600 \times 10^3}{6400 \times 10^3}\right)$$

$$g' = g\left(1 - \frac{2}{4}\right)$$

$$g' = g\left(1 - \frac{1}{2}\right) = g/2$$

The above two examples show that the acceleration due to gravity is a constant near the surface of the Earth.

Variation of g with depth:

Consider a particle of mass m which is in a deep mine on the Earth. (Example: coal mines in Neyveli). Assume the depth of the mine as d. To calculate g′ at a depth d, consider the following points.

The part of the Earth which is above the radius ($R_e - d$) do not contribute to the acceleration. The result is proved earlier and is given as

$$g' = \frac{GM'}{(R_e - d)^2}$$

Here M′ is the mass of the Earth of radius ($R_e - d$)
Assuming the density of Earth ρ to be constant

$$\rho = \frac{M}{V}$$

where M is the mass of the Earth and V its volume, Thus,

$$\rho = \frac{M'}{V'}$$

$$\frac{M'}{V'} = \frac{M}{V} \text{ and } M' = \frac{M}{V}V'$$

$$M' = \left(\frac{M}{\frac{4}{3}\pi R_e^{\,3}}\right)\left(\frac{4}{3}\pi (R_e - d)^3\right)$$

$$M' = \frac{M}{R_e^{\,3}}(R_e - d)^3$$

$$g' = G\frac{M}{R_e^{\,3}}(R_e - d)^3 . \frac{1}{\left(R_e - d\right)^2}$$

$$g' = GM\frac{R_e\left(1 - \frac{d}{R_e}\right)}{R_e^{\,3}}$$

$$g' = GM\frac{\left(1 - \frac{d}{R_e}\right)}{R_e^{\,2}}$$

Thus

$$g' = g\left(1 - \frac{d}{R_e}\right)$$

Here also g ′ < g . As depth increases, g′ decreases. It is very interesting to know that acceleration due to gravity is maximum on the surface of the Earth but decreases when we go either upward or downward.

Variation of g with latitude:

Whenever we analyze the motion of objects in rotating frames [explained in chapter 3] we must take into account the centrifugal force. Even though we treat the Earth as an inertial frame, it is not exactly correct because the Earth spins about its own axis. So when an object is on the surface of the Earth, it experiences a centrifugal force that depends on the latitude of the object on Earth. If the Earth were not spinning, the force on the object would have been mg. However, the object experiences an additional centrifugal force due to spinning of the Earth.

This centrifugal force is given by $m\omega^2R'$.

$$R' = R\cos\lambda$$

where λ is the latitude. The component of centrifugal acceleration experienced by the object in the direction opposite to g is

$$a_{PQ} = \omega^2 R' \cos\lambda = \omega^2 R\cos^2\lambda$$
$$since\, R' = R\cos\lambda$$

Therefore,

$$g' = g - \omega^2 R\cos^2\lambda$$

From the expression (6.52), we can infer that at equator, λ=0; g′=g-ω^2R. The acceleration due to gravity is minimum. At poles λ= 90; g′=g, it is maximum. At the equator, g′is minimum.

EXAMPLE

Find out the value of g′ in your school laboratory?

Solution

Calculate the latitude of the city or village where the school is located. The information is available in Google search. For example, the latitude of Chennai is approximately 13 degree.

$$g' = g - \omega^2 R\cos^2\lambda$$

Here $\omega^2R = (2x3.14/86400)^2 \times (6400x10^3) = 3.4x10^{-2}$ m s^{-2}.

It is to be noted that the value of λ should be in radian and not in degree. 13 degree is equivalent to 0.2268 rad.

$$g' = 9.8 - \left(3.4\times10^{-2}\right)\times\left(\cos 0.2268\right)^2$$

$$g' = 9.7677 \text{ m s}^{-2}$$

ESCAPE SPEED AND ORBITAL SPEED

Hydrogen and helium are the most abundant elements in the universe but Earth's atmosphere consists mainly of nitrogen and oxygen. The following discussion brings forth the reason why hydrogen and helium are not found in abundance on the Earth's atmosphere. When an object is thrown up with some initial speed it will reach a certain height after which it will fall back to Earth. If the same object is thrown again with a higher speed, it reaches a greater height than the previous one and falls back to Earth. This leads to the question of what should be the speed of an object thrown vertically up such that it escapes the Earth's gravity and would never come back.

Consider an object of mass M on the surface of the Earth. When it is thrown up with an initial speed vi, the initial total energy of the object is

$$E_i = \frac{1}{2}Mv_i^2 - \frac{GMM_E}{R_E}$$

where, M_E is the mass of the Earth and R_E- the radius of the Earth. The term $-\frac{GMM_E}{R_E}$ is the potential energy of the mass M.

When the object reaches a height far away from Earth and hence treated as approaching infinity, the gravitational potential energy becomes zero $[U(\infty) = 0]$ and the kinetic energy becomes zero as well. Therefore the final total energy of the object becomes zero. This is for minimum energy and for minimum speed to escape. Otherwise Kinetic energy can be nonzero.

$$E_f = 0$$

According to the law of energy conservation,

$$E_i = E_f$$

Substituting (6.53) in (6.54) we get,

$$\frac{1}{2}Mv_i^2 - \frac{GMM_E}{R_E} = 0$$

$$\frac{1}{2}Mv_i^2 = \frac{GMM_E}{R_E}$$

Consider the escape speed, the minimum speed required by an object to escape Earth's gravitational field, hence replace v_i with v_e . i.e,

$$\frac{1}{2}Mv_e^{\,2} = \frac{GMM_E}{R_E}$$

$$v_e^{\,2} = \frac{GMM_E}{R_E} \cdot \frac{2}{M}$$

$$v_e^{\,2} = \frac{2GM_E}{R_E}$$

Using $g = \frac{GM_E}{R_e^2}$

$$v_e^{\,2} = 2gR_E$$

$$v_e = \sqrt{2gR_E}$$

From equation (6.56) the escape speed depends on two factors: acceleration due to gravity and radius of the Earth. It is completely independent of the mass of the object. By substituting the values of g (9.8 m s^{-2}) and R_e = 6400 km , the escape speed of the Earth is v_e = 11.2 kms^{-1} . The escape speed is independent of the direction in which the object is thrown. Irrespective of whether the object is thrown vertically up, radially outwards or tangentially it requires the same initial speed to escape Earth's gravity. It is shown in Figure 6.19

Lighter molecules such as hydrogen and helium have enough speed to escape from the Earth, unlike the heavier ones such as nitrogen and oxygen. (The average speed of hydrogen and helium atoms compaired with the escape speed of the Earth,is presented in the kinetic theory of gases, unit 9).

Satellites, orbital speed and time period

We are living in a modern world with sophisticated technological gadgets and are able to communicate to any place on Earth. This advancement was made possible because of our understanding of solar system. Communication mainly depends on the satellites that orbit the

Earth (Figure 6.20). Satellites revolve around the Earth just like the planets revolve around the Sun. Kepler's laws are applicable to manmade satellites also.

For a satellite of mass M to move in a circular orbit, centripetal force must be acting on the satellite. This centripetal force is provided by the Earth's gravitational force.

$$\frac{Mv^2}{(R_E + h)} = \frac{GMM_E}{(R_e + h)^2}$$

$$v^2 = \frac{GM_E}{(R_E + h)}$$

$$v = \sqrt{\frac{GM_E}{(R_E + h)}}$$

As h increases, the speed of the satellite decreases.

Time period of the satellite:

The distance covered by the satellite during one rotation in its orbit is equal to $2\pi(R_E + h)$ and time taken for it is the time period, T. Then

$$Speed\ v = \frac{Distance\ travelled}{Time\ taken} = \frac{2\pi(R_E + h)}{T}$$

From equation (6.58)

$$\sqrt{\frac{GM_E}{(R_E + h)}} = \frac{2\pi(R_E + h)}{T}$$

$$T = \frac{2\pi}{\sqrt{GM_E}}(R_E + h)^{3/2}$$

Squaring both sides of the equation (6.60), we get

$$T^2 = \frac{4\pi^2}{GM_E}\left(R_E + h\right)^3$$

$$\frac{4\pi^2}{GM_E} = constant\ say\ c$$

$$T^2 = c\left(R_E + h\right)^3$$

Equation (6.61) implies that a satellite orbiting the Earth has the same relation between time and distance as that of Kepler's law of planetary motion. For a satellite orbiting near the surface of the Earth, h is negligible compared to the radius of the Earth R_E. Then,

$$T^2 = \frac{4\pi^2}{GM_E} R_E{}^3$$

$$T^2 = \frac{4\pi^2}{GM_E / R_E{}^2} R_E$$

$$T^2 = \frac{4\pi^2}{g} R_E$$

$$since\ GM_E / R_E{}^2 = g$$

$$T = 2\pi\sqrt{\frac{R_E}{g}}$$

By substituting the values of $R_E = 6.4 \times 10^6$ m and $g = 9.8$ m s^{-2}, the orbital time period is obtained as $T \cong 85$ minutes.

EXAMPLE

Moon is the natural satellite of Earth and it takes 27 days to go once around its orbit. Calculate the distance of the Moon from the surface of the Earth assuming the orbit of the Moon as circular.

Solution

We can use Kepler's third law,

$$T^2 = c\left(R_E + h\right)^3$$

$$T^{2/3} = c^{1/3}\left(R_E + h\right)$$

$$\left(\frac{T^2}{c}\right)^{1/3} = \left(R_E + h\right)$$

$$\left(\frac{T^2 G M_E}{4\pi^2}\right)^{\frac{1}{3}} = \left(R_E + h\right);$$

$$c = \frac{4\pi^2}{G M_E}$$

$$h = \left(\frac{T^2 G M_E}{4\pi^2}\right)^{1/3} - R_E$$

Here h is the distance of the Moon from the surface of the Earth. Here,

$$R_E - \text{radius of the Earth} = 6.4 \times 10^6 m$$

$$M_E - \text{mass of the Earth} = 6.02 \times 10^{24} kg$$

$$G - \text{Universal gravitational constant} = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$$

By substituting these values, the distance to the Moon from the surface of the Earth is calculated to be 3.77×10^5 km.

Energy of an Orbiting Satellite

The total energy of a satellite orbiting the Earth at a distance h from the surface of Earth is calculated as follows; The total energy of the satellite is the sum of its kinetic energy and the gravitational potential energy. The potential energy of the satellite is,

$$U = -\frac{GM_s M_E}{(R_E + h)}$$

Here M_s - mass of the satellite, M_E - mass of the Earth, R_E - radius of the Earth.

The Kinetic energy of the satellite is

$$K.E = \frac{1}{2} M_s v^2$$

Here v is the orbital speed of the satellite and is equal to

$$v = \sqrt{\frac{GM_E}{(R_E + h)}}$$

Substituting the value of v in (6.64), the kinetic energy of the satellite becomes,

$$K.E = \frac{1}{2}\frac{GM_E M_s}{\left(R_E + h\right)}$$

Therefore the total energy of the satellite is

$$E = \frac{1}{2}\frac{GM_E M_s}{\left(R_E + h\right)} - \frac{GM_s M_E}{\left(R_E + h\right)}$$

$$E = -\frac{GM_s M_E}{2\left(R_E + h\right)}$$

The negative sign in the total energy implies that the satellite is bound to the Earth and it cannot escape from the Earth.

As h approaches ∞ , the total energy tends to zero. Its physical meaning is that the satellite is completely free from the influence of Earth's gravity and is not bound to Earth at large distances.

EXAMPLE

Calculate the energy of the (i) Moon orbiting the Earth and (ii) Earth orbiting the Sun.

Solution

Assuming the orbit of the Moon to be circular, the energy of Moon is given by,

$$E_m = -\frac{GM_E M_m}{2R_m}$$

where M_E is the mass of Earth 6.02×10^{24} kg; M_m is the mass of Moon 7.35×10^{22} kg; and R_m is the distance between the Moon and the center of the Earth 3.84×10^5km

$$G = 6.67 \times 10^{-11} \text{N m}^2 \text{kg}^{-2}.$$

$$E_m = -\frac{6.67 \times 10^{-11} \times 6.02 \times 10^{24} \times 7.35 \times 10^{22}}{2 \times 3.84 \times 10^5 \times 10^3}$$

$$E_m = -38.42 \times 10^{-19} \times 10^{46}$$

$$E_m = -38.42 \times 10^{46}\ Joule$$

The negative energy implies that the Moon is bound to the Earth.

Same method can be used to prove that the energy of the Earth is also negative.

Geo-stationary and polar satellite

The satellites orbiting the Earth have different time periods corresponding to different orbital radii. Can we calculate the orbital radius of a satellite if its time period is 24 hours?

Kepler's third law is used to find the radius of the orbit.

$$T^2 = \frac{4\pi^2}{GM_E}\left(R_E + h\right)^3$$

$$\left(R_E + h\right)^3 = \frac{GM_E T^2}{4\pi^2}$$

$$R_E + h = \left(\frac{GM_E T^2}{4\pi^2}\right)^{1/3}$$

Substituting for the time period (24 hrs = 86400 seconds), mass, and radius of the Earth, h turns out to be 36,000 km. Such satellites are called "geo-stationary satellites", since they appear to be stationary when seen from Earth.

India uses the INSAT group of satellites that are basically geo-stationary satellites for the purpose of telecommunication. Another type of satellite which is placed at a distance of 500 to 800 km from the surface of the Earth orbits the Earth from north to south direction. This type of satellite that orbits Earth from North Pole to South Pole is called a polar satellite. The time period of a polar satellite is nearly 100 minutes and the satellite completes many revolutions in a day. A Polar satellite covers a small strip of area from pole to pole during one revolution. In the next revolution it covers a different strip of area since the Earth would have moved by a small angle. In this way polar satellites cover the entire surface area of the Earth.

Weightlessness Weight of an object

Objects on Earth experience the gravitational force of Earth. The gravitational force acting on an object of mass m is mg. This force always acts downwards towards the center of the Earth. When we stand on the floor, there are two forces acting on us. One is the gravitational force, acting downwards and the other is the normal force exerted by the floor upwards on us to keep us at rest. The weight of an object $\dot{W}$ is defined as the downward force whose magnitude W is equal to that of upward force that must be applied to the object to hold it at rest or at constant velocity relative to the earth. The direction of weight is in the direction of gravitational force. So the magnitude of weight of an object is denoted as, W=N=mg. Note that even though magnitude of weight is equal to mg, it is not same as gravitational force acting on the object.

Apparent weight in elevators

Everyone who used an elevator would have felt a jerk when the elevator takes off or stops. Why does it happen? Understanding the concept of weight is crucial for explaining this effect. Let us consider a man inside an elevator in the following scenarios. When a man is standing in the elevator, there are two forces acting on him.

1. Gravitational force which acts downward. If we take the vertical direction as positive y direction, the gravitational force acting on the man is $\vec{F}_G = -mg\hat{j}$
2. The normal force exerted by floor on the man which acts vertically upward, $\vec{N} = N\hat{j}$

Case (i) When the elevator is at rest

The acceleration of the man is zero. Therefore the net force acting on the man is zero. With respect to inertial frame (ground), applying Newton's second law on the man,

$$\vec{F}_G + \vec{N} = 0$$
$$-mg\hat{j} + N\hat{j} = 0$$

By comparing the components, we can write

$$N - mg = 0 \text{ (or) } N = mg$$

Since weight, W =N, the apparent weight of the man is equal to his actual weight.

Case (ii) When the elevator is moving uniformly in the upward or downward direction

In uniform motion (constant velocity), the net force acting on the man is still zero. Hence, in this case also the apparent weight of the man is equal to his actual weight. It is shown in Figure 6.23(a)

Case (iii) When the elevator is accelerating upwards

If an elevator is moving with upward acceleration $(\vec{a} = a\hat{j})$ with respect to inertial frame (ground), applying Newton's second law on the man,

$$\vec{F}_G + \vec{N} = m\vec{a}$$

Writing the above equation in terms of unit vector in the vertical direction,

$$-mg\hat{j} + N\hat{j} = ma\hat{j}$$

By comparing the components,

$$N = m\,(g + a\,)$$

Therefore, apparent weight of the man is greater than his actual weight. It is shown in Figure 6.23(b)

Case (iv) When the elevator is accelerating downwards

If the elevator is moving with downward acceleration $(\dot{a} = -a\,\dot{j})$ by applying Newton's second law on the man, we can write

$$\vec{F}_G + \vec{N} = m\vec{a}$$

Writing the above equation in terms of unit vector in the vertical direction

$$-mg\hat{j} + N\hat{j} = -ma\hat{j}$$

By comparing the components,

$$N = m\,(g - a\,)$$

Therefore, apparent weight W = N = m(g-a) of the man is lesser than his actual weight. It is shown in Figure 6.23(c)

Weightlessness of freely falling bodies

Freely falling objects experience only gravitational force. As they fall freely, they are not in contact with any surface (by neglecting air friction). The normal force acting on the object is zero. The downward acceleration is equal to the acceleration due to the gravity of the Earth. i.e (a = g). From equation (6.69) we get.

$$a = g \quad \therefore N = m(g - g) = 0.$$

This is called the state of weightlessness. When the lift falls (when the lift wire cuts) with downward acceleration a = g, the person inside the elevator is in the state of weightlessness or free fall. It is shown in Figure 6.23(d)

When the apple was falling from the tree it was weight less. As soon as it hit Newton's head, it gained weight! and Newton gained physics!

Weightlessness in satellites:

There is a wrong notion that the astronauts in satellites experience no gravitational force because they are far away from the Earth. Actually the Earth satellites that orbit very close to Earth experience only gravitational force. The astronauts inside the satellite also experience the same gravitational force. Because of this, they cannot exert any force on the floor of the satellite. Thus, the floor of the satellite also cannot exert any normal force on the astronaut. Therefore, the astronauts inside a satellite are in the state of weightlessness. Not only the astronauts, but all the objects in the satellite will be in the state of weightlessness which is similar to that of a free fall. It is shown in the Figure 6.24.

ELEMENTARY IDEAS OF ASTRONOMY

Astronomy is one of the oldest sciences in the history of mankind. In the olden days, astronomy was an inseparable part of physical science. It contributed a lot to the development of physics in the 16th century. In fact Kepler's laws and Newton's theory of gravitation were formulated and verified using astronomical observations and data accumulated over the centuries by famous astronomers like Hippachrus, Aristachrus, Ptolemy, Copernicus and Tycho Brahe. Without Tycho

Brahe's astronomical observations, Kepler's laws would not have emerged. Without Kepler's laws, Newton's theory of gravitation would not have been formulated.

It was mentioned in the beginning of this chapter that Ptolemy's geocentric model was replaced by Copernicus' heliocentric model. It is important to analyze and explain the shortcoming of the geocentric model over heliocentric model.

Heliocentric system over geocentric system

When the motion of the planets are observed in the night sky by naked eyes over a period of a few months, it can be seen that the planets move eastwards and reverse their motion for a while and return to eastward motion again. This is called "retrograde motion" of planets.

Figure 6.25 shows the retrograde motion of the planet Mars. Careful observation for a period of a year clearly shows that Mars initially moves eastwards (February to June), then reverses its path and moves backwards (July, August, September). It changes its direction of motion once again and continues its forward motion (October onwards). In olden days, astronomers recorded the retrograde motion of all visible planets and tried to explain the motion. According to Aristotle, the other planets and the Sun move around the Earth in the circular orbits. If it was really a circular orbit it was not known how the planet could reverse its motion for a brief interval. To explain this retrograde motion, Ptolemy introduced the concept of "epicycle" in his geocentric model. According to this theory, while the planet orbited the Earth, it also underwent another circular motion termed as "epicycle". A combination of epicycle and circular motion around the Earth gave rise to retrograde motion of the planets with respect to Earth (Figure 6.26). Essentially Ptolemy retained the Earth centric idea of Aristotle and added the epicycle motion to it.

But Ptolemy's model became more and more complex as every planet was found to undergo retrograde motion. In the 15th century, the Polish astronomer Copernicus proposed the heliocentric model to explain this problem in a simpler manner. According to this model, the Sun is at the center of the solar system and all planets orbited the Sun. The retrograde motion of planets with respect to Earth is because of the relative motion of the planet with respect to Earth. The retrograde motion from the heliocentric point of view is shown in Figure 6.27.

Figure 6.27 shows that the Earth orbits around the Sun faster than Mars. Because of the relative motion between Mars and Earth, Mars appears to move backwards from July to October. In the same way the retrograde motion of all other planets was explained successfully by the Copernicus model. It was because of its simplicity, the heliocentric model slowly replaced the geocentric model. Historically, if any natural phenomenon has one or more explanations, the simplest one is usually accepted. Though this was not the only reason to disqualify the geocentric model, a detailed discussion on correctness of the Copernicus model over to Ptolemy's model can be found in astronomy books

Kepler's Third Law and The Astronomical Distance

When Kepler derived his three laws, he strongly relied on Tycho Brahe's astronomical observation. In his third law, he formulated the relation between the distance of a planet from the Sun to the time period of revolution of the planet. Astronomers cleverly used geometry and trigonometry to calculate the distance of a planet from the Sun in terms of the distance between Earth and Sun. Here we can see how the distance of Mercury and Venus from the Sun were measured. The Venus and Mercury, being inner planets with respect to Earth, the maximum angular distance they can subtend at a point on Earth with respect to the Sun is 46 degree for Venus and 22.5 degree for Mercury. It is shown in the Figure 6.28

Figure 6.29 shows that when Venus is at maximum elongation (i.e., 46 degree) with respect to Earth, Venus makes 90 degree to Sun. This allows us to find the distance between Venus and Sun. The distance between Earth and Sun is taken as one Astronomical unit (1 AU).

The trigonometric relation satisfied by this right angled triangle is shown in Figure 6.29.

$$\sin\theta = \frac{r}{R}$$

where R = 1 AU.

$$r = R\sin\theta = (1AU)(\sin 46^\circ)$$

Here sin46 - 0.77. Hence Venus is at a distance of 0.77 AU from Sun. Similarly, the distance between Mercury (θ is 22.5 degree) and Sun is calculated as 0.38 AU. To find the distance of exterior planets like Mars and Jupiter, a slightly different method is used. The distances of planets from the Sun is given in the table below.

Planet	semi major axis of the orbit(a)	Period T (years)	a^3/T^2
Mercury	0.389 AU	87.77	7.64
Venus	0.724 AU	224.70	7.52
Earth	1.000 AU	365.25	7.50
Mars	1.524 AU	686.98	7.50
Jupiter	5.200 AU	4332.62	7.49
Saturn	9.510 AU	10,759.20	7.40

It is to be noted that to verify the Kepler's law we need only high school level geometry and trigonometry.

Measurement of radius of the Earth

Around 225 B.C a Greek librarian "Eratosthenes" who lived at Alexandria measured the radius of the Earth with a small error when compared with results using modern measurements. The technique he used involves lower school geometry and brilliant insight. He observed that during noon time of summer solstice the Sun's rays cast no shadow in the city Syne which was located 500 miles away from Alexandria. At the same day and same time he found that in Alexandria the Sun's rays made 7.2 degree with local vertical as shown in the Figure 6.30. He realized that this difference of 7.2 degree was due to the curvature of the Earth.

$$\frac{1}{8} radian. \text{ So } \theta = \frac{1}{8} rad;$$

If S is the length of the arc between the cities of Syne and Alexandria, and if R is radius of Earth, then

$$S = R\theta = 500\, miles,$$

so radius of the Earth

$$R = \frac{500}{\theta} \text{ miles}$$

$$R = 500 \frac{miles}{\frac{1}{8}}$$

$$\text{R} = 4000 \text{ miles}$$

1 mile is equal to 1.609 km. So, he measured the radius of the Earth to be equal to R = 6436 km, which is amazingly close to the correct value of 6378 km.

The distance of the Moon from Earth was measured by a famous Greek astronomer Hipparchus in the 3rd century BC.

Interesting Astronomical Facts
Lunar eclipse and measurement of shadow of Earth

On January 31, 2018 there was a total lunar eclipse which was observed from various places including Tamil Nadu. It is possible to measure the radius of shadow of the Earth at the point where the Moon crosses. Figure 6.31 illustrates this.

When the Moon is inside the umbra shadow, it appears red in color. As soon as the Moon exits from the umbra 1737shadow, it appears in crescent shape. Figure 6.32 is the photograph taken by digital camera during Moon's exit from the umbra shadow.

By finding the apparent radii of the Earth's umbra shadow and the Moon, the ratio of the these radii can be calculated. This is shown in Figures 6.33 and 6.34.

The apparent radius of Earth's umbra shadow = R_s = 13.2 cm

The apparent radius of the Moon = R_m= 5.15 cm

$$\text{The ratio } \frac{R_s}{R_m} \approx 2.56$$

The radius of the Earth's umbra shadow is $R_s=2.56\times R_m$

The radius of Moon R_m=1737 km

The radius of the Earth's umbra shadow

is $R_s = 2.56\times 1737 km \cong 4446 km$.

The correct radius is 4610 km.

The percentage of error in the calculation

$$= \frac{4610-4446}{4610}\times 100 = 3.5\%.$$

The error will reduce if the pictures taken using a high quality telescope are used. It is to be noted that this calculation is done using very simple mathematics.

Early astronomers proved that Earth is spherical in shape by looking at the shape of the shadow cast by Earth on the Moon during lunar eclipse

Why there are no lunar eclipse and solar eclipse every month?

If the orbits of the Moon and Earth lie on the same plane, during full Moon of every month, we can observe lunar eclipse. If this is so during new Moon we can observe solar eclipse. But Moon's orbit is tilted 5° with respect to Earth's orbit. Due to this 5° tilt, only during certain periods of the year, the Sun, Earth and Moon align in straight line leading to either lunar eclipse or solar eclipse depending on the alignment. This is shown in Figure 6.35

Why do we have seasons on Earth?

The common misconception is that 'Earth revolves around the Sun, so when the Earth is very far away, it is winter and when the Earth is nearer, it is summer'. Actually, the seasons in the Earth arise due to the rotation of Earth around the Sun with 23.5° tilt. This is shown in Figure 6.36

Due to this 23.5° tilt, when the northern part of Earth is farther to the Sun, the southern part is nearer to the Sun. So when it is summer in the northern hemisphere, the southern hemisphere experience winter.

Star's apparent motion and spinning of the Earth

The Earth's spinning motion can be proved by observing star's position over a night. Due to Earth's spinning motion, the stars in sky appear to move in circular motion about the pole star as shown in Figure 6.37

Recent developments of astronomy and gravitation

Till the 19th century astronomy mainly depended upon either observation with the naked eye or telescopic observation. After the discovery of the electromagnetic spectrum at the end of the 19th century, our understanding of the universe increased enormously. Because of this development in the late 19th century it was found that Newton's law of gravitation could not explain certain phenomena and showed some discrepancies. Albert Einstein formulated his 'General theory of relativity' which was one of the most successful theories of 20^{th} century in the field of gravitation.

In the twentieth century both astronomy and gravitation merged together and have grown in manifold. The birth and death of stars were more clearly understood. Many Indian physicists made very important contributions to the field of astrophysics and gravitation.

Subramanian Chandrasekar formulated the theory of black holes and explained the life of stars. These studies brought him the Nobel prize in the year 1983. Another very notable Indian astrophysicist Meghnad Saha discovered the ionization formula which was useful in classifying stars. This formula is now known as "Saha ionization formula". In the field of gravitation Amal Kumar Raychaudhuri solved an equation now known as "Raychaudhuri equation" which was a very important contribution. Another notable Indian Astrophysicist Jayant V Narlikar made pioneering contribution in the field of astrophysics and has written interesting books on astronomy and astrophysics. IUCAA (Inter University Center for Astronomy and Astrophysics) is one of the important Indian research institutes where active research in astrophysics and gravitation are conducted. The institute was founded by Prof. J.V. Narlikar. Students are encouraged to read more about the recent developments in these fields.

UNIT - 10 OSCILLATIONS

INTRODUCTION

Have you seen the Thanjavur Dancing Doll (In Tamil, it is called 'Thanjavur thalayattibommai')?. It is a world famous Indian cultural doll (Figure 10.1). What does this doll do when disturbed? It will dance such that the head and body move continuously in a to and fro motion, until the movement gradually stops. Similarly, when we walk on the road, our hands and legs will move front and back. Again similarly, when a mother swings a cradle to make her child sleep, the cradle is made to move in to and fro motion. All these motions are diff erent from the motion that we have discussed so far. Th ese motions are shown in Figure 10.2. Generally, they are known as oscillatory motion or vibratory motion. A similar motion occurs even at atomic levels. When the temperature is raised, the atoms in a solid vibrate about their rest position (mean position or equilibrium position). Th e study of vibrational motion is very important in engineering applications, such as, designing the structure of building, mechanical equipments, etc.

Periodic and nonperiodic motion

Motion in physics can be classified as repetitive (periodic motion) and non- repetitive (non-periodic motion).

Periodic motion

Any motion which repeats itself in a fixed time interval is known as periodic motion.

Examples : Hands in pendulum clock, swing of a cradle, the revolution of the Earth around the Sun, waxing and waning of Moon, etc.

Non-Periodic motion

Any motion which does not repeat itself after a regular interval of time is known as non-periodic motion.

Example : Occurance of Earth quake, eruption of volcano, etc.

E X A M P L E

Classify the following motions as periodic and non-periodic motions?.

a. Motion of Halley's comet.
b. Motion of clouds.
c. Moon revolving around the Earth

Solution

a. Periodic motion
b. Non-periodic motion
c. Periodic motion

E X A M P L E

Which of the following functions of time represent periodic and non-periodic motion?.

a. $\sin \omega t + \cos \omega t$
b. $\ln \omega t$

Solution

a. Periodic
b. Non-periodic

Oscillatory motion

When an object or a particle moves back and forth repeatedly for some duration of time its motion is said to be oscillatory (or vibratory). Examples; our heart beat, swinging motion of the wings of an insect, grandfather's clock (pendulum clock), etc. Note that all oscillatory motion are periodic whereas all periodic motions need not be oscillation in nature. see Figure 10.3

SIMPLE HARMONIC MOTION (SHM)

Simple harmonic motion is a special type of oscillatory motion in which the acceleration or force on the particle is directly proportional to its displacement from a fixed point and is always directed towards that fixed point. In one dimensional case, let x be the displacement of the particle and a_x be the acceleration of the particle, then

$$a_x \propto x$$

$$a_x = -\,b\,x$$

where b is a constant which measures acceleration per unit displacement and dimensionally it is equal to T^{-2}. By multiplying by mass of the particle on both sides of equation (10.2) and from Newton's second law, the force is

$$F_x = -\,k\,x$$

where k is a force constant which is defined as force per unit length. The negative sign indicates that displacement and force (or acceleration) are in opposite directions. This means that when the displacement of the particle is taken towards right of equilibrium position (x takes positive value), the force (or acceleration) will point towards equilibrium (towards left) and similarly, when the displacement of the particle is taken towards left of equilibrium position (x takes negative value), the force (or acceleration) will point towards equilibrium (towards right). This type of force is known as restoring force because it always directs the particle executing simple harmonic motion to restore to its original (equilibrium or mean) position. This force (restoring force) is central and attractive whose center of attraction is the equilibrium position.

In order to represent in two or three dimensions, we can write using vector notation

$$\vec{F} = -k\vec{r}$$

where $\vec{r}$ the displacement of the particle from the chosen origin. Note that the force and displacement have a linear relationship. This means that the exponent of force
$\vec{F}$ and the exponent of displacement $\vec{r}$ are unity. The sketch between cause (magnitude of force $|\vec{F}|$) and effect (magnitude of displacement $|\vec{r}|$) is a straight line passing through second and fourth quadrant as shown in. By measuring slope $\frac{1}{k}$ one can find the numerical value of force constant k.

The projection of uniform circular motion on a diameter of SHM

Consider a particle of mass m moving with uniform speed v along the circumference of a circle whose radius is r in anti-clockwise direction (as shown in Figure 10.6). Let us assume that the origin of the coordinate system coincides with the center O of the circle. If ω is the angular velocity of the particle and θ the angular displacement of the particle at any instant of time t, then $\theta = \omega t$. By projecting the uniform circular motion on its diameter gives a simple harmonic motion. This means that we can associate a map (or a relationship) between uniform circular (or revolution) motion to vibratory motion. Conversely, any vibratory motion or revolution can be mapped to uniform circular motion. In other words, these two motions are similar in nature.

Let us first project the position of a particle moving on a circle, on to its vertical diameter or on to a line parallel to vertical diameter as shown in Figure 10.7. Similarly, we can do it for horizontal axis or a line parallel to horizontal axis.

As a specific example, consider a spring mass system (or oscillation of pendulum) as shown in Figure 10.8. When the spring moves up and down (or pendulum moves to and fro), the motion of the mass or bob is mapped to points on the circular motion.

Thus, if a particle undergoes uniform circular motion then the projection of the particle on the diameter of the circle (or on a line parallel to the diameter) traces straight line motion which is simple harmonic in nature. The circle is known as reference circle of the simple harmonic motion. The simple harmonic motion can also be defined as the motion of the projection of a particle on any diameter of a circle of reference.

Displacement, velocity, acceleration and its graphical representation – SHM

The distance travelled by the vibrating particle at any instant of time t from its mean position is known as displacement. Let P be the position of the particle on a circle of radius A at some instant of time t as shown in Figure 10.9. Then its displacement y at that instant of time t can be derived as follows In ΔOPN

$$\sin\theta = \frac{ON}{OP} \Rightarrow ON = OP\sin\theta$$

But $\theta = \omega t$, ON = y and OP = A

$$y = A\sin\omega t$$

The displacement y takes maximum value (which is equal to A) when sin ωt=1.This maximum displacement from the mean positionis known as amplitude (A) of the vibrating particle. For simple harmonic motion, the amplitude is constant. But, in general, for any motion other than simple harmonic, the amplitude need not be constant, it may vary with time.

Velocity

The rate of change of displacement is velocity. Taking derivative of equation (10.6) with respect to time, we get

$$v = \frac{dy}{dt} = \frac{d}{dt}(\text{A}\sin\omega t)$$

For circular motion (of constant radius), amplitude A is a constant and further, for uniform circular motion, angular velocity ω is a constant. Therefore,

$$v = \frac{dy}{dt} = A\ \omega \cos \omega t$$

Using trigonometry identity,

$$\sin^2 \omega t + \cos^2 \omega t = 1 \Rightarrow \cos \omega t = \sqrt{1 - \sin^2 \omega t}$$

we get

$$v = A\omega\sqrt{1 - \sin^2 \omega t}$$

From equation (10.6),

$$\sin \omega t = \frac{y}{A}$$

$$v = A\omega\sqrt{1 - \left(\frac{y}{A}\right)^2}$$

$$v = \omega\sqrt{A^2 - y^2}$$

From equation (10.8), when the displacement y = 0, the velocity v = ωA (maximum) and for the maximum displacement y = A, the velocity v = 0 (minimum).

As displacement increases from zero to maximum, the velocity decreases from maximum to zero. This is repeated.

Since velocity is a vector quantity, equation (10.7) can also be deduced by resolving in to components.

Acceleration

The rate of change of velocity is acceleration.

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(A\omega \cos \omega t\right)$$

$$a = -\omega^2 A \sin \omega t = -\omega^2 y$$

$$\therefore \quad a = \frac{d^2 y}{dt^2} = -\omega^2 y$$

From the Table 10.1 and figure 10.10, we observe that at the mean position

Table 10.1 Displacement, velocity and acceleration at different instant of time.

Time	0	$\frac{T}{4}$	$\frac{2T}{4}$	$\frac{3T}{4}$	$\frac{4T}{4} = T$
ωt	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Displacement $y = A \sin \omega t$	0	A	0	$-A$	0
Velocity $v = A \omega \cos \omega t$	$A\omega$	0	$-A\omega$	0	$A\omega$
Acceleration $a = -A\omega^2 \sin \omega t$	0	$-A\omega^2$	0	$A\omega^2$	0

(y = 0), velocity of the particle is maximum but the acceleration of the particle is zero. At the extreme position (y = ±A), the velocity of the particle is zero but the acceleration is maximum $\pm A\omega^2$ acting in the opposite direction.

E X A M P L E

Which of the following represent simple harmonic motion?

a. $x = A \sin \omega t + B \cos \omega t$
b. $x = A \sin \omega t + B \cos 2\omega t$
c. $x = A e^{i\omega t}$
d. $x = A \ln \omega t$

Solution

a. $x = A \sin \omega t + B \cos \omega t$

$$\frac{dx}{dt} = A\,\omega \cos \omega t - B\,\omega \sin \omega t$$

$$\frac{d^2x}{dt^2} = -\,\omega^2\,(A \sin \omega t + B \cos \omega t)$$

$$\frac{d^2x}{dt^2} = -\,\omega^2 x$$

This differential equation is similar to the differential equation of SHM (equation 10.10). Therefore, $x = A \sin \omega t + B \cos \omega t$ represents SHM.

b. $x = A \sin \omega t + B \cos 2\omega t$

$$\frac{dx}{dt} = A\,\omega \cos \omega t - B\,(2\omega) \sin 2\omega t$$

$$\frac{d^2x}{dt^2} = -\,\omega^2\,(A \sin \omega t + 4B \cos 2\omega t)$$

$$\frac{d^2x}{dt^2} \neq -\,\omega^2 x$$

This differential equation is not like the differential equation of a SHM (equation 10.10). Therefore, $x = A \sin \omega t + B \cos 2\omega t$ does not represent SHM.

c. $x = A e^{i\omega t}$

$$\frac{dx}{dt} = A\,\omega e^{i\omega t}$$

$$\frac{d^2x}{dt^2} = -A\,\omega^2\, e^{i\omega t} = -\omega^2 x$$

This differential equation is like the differential equation of SHM (equation 10.10). Therefore, $x = A e^{i\omega t}$ represents SHM.

d. x = A ln ωt

$$\frac{dx}{dt} = \left(\frac{A}{\omega t}\right)\omega = \frac{A}{t}$$

$$\frac{d^2x}{dt^2} = -\frac{A}{t^2} \Rightarrow \frac{d^2x}{dt^2} \neq -\omega^2 x$$

This differential equation is not like the differential equation of a SHM (equation 10.10). Therefore, x = A ln ωt does not represent SHM.

E X A M P L E

Consider a particle undergoing simple harmonic motion. The velocity of the particle at position x_1 is v_1 and velocity of the particle at position x_2 is v_2. Show that the ratio of time period and amplitude is

$$\frac{T}{A} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 x_2^2 - v_2^2 x_1^2}}$$

Solution

$$v = \omega\sqrt{A^2 - x^2} \Rightarrow v^2 = \omega^2 (A^2 - x^2)$$

Therefore, at position x_1,

$$v_1^2 = \omega^2 (A^2 - x_1^2)$$

Similarly, at position x_2,

$$v_2^2 = \omega^2 (A^2 - x_2^2)$$

$$v_1^2 - v_2^2 = \omega^2 (A^2 - x_1^2) - \omega^2 (A^2 - x_2^2)$$

$$= \omega^2 (x_2^2 - x_1^2)$$

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} \Rightarrow T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

$$\frac{v_1^2}{v_2^2} = \frac{\omega^2\left(A^2 - x_1^2\right)}{\omega^2\left(A^2 - x_2^2\right)} \Rightarrow A = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$$

$$\frac{T}{A} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 x_2^2 - v_2^2 x_1^2}}$$

Time period, frequency, phase, phase difference and epoch in SHM.
Time period

The time period is defined as the time taken by a particle to complete one oscillation. It is usually denoted by T. For one complete revolution, the time taken is t = T, therefore

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$$

Then, the displacement of a particle executing simple harmonic motion can be written either as sine function or cosine function.

$$y(t) = A \sin \frac{2\pi}{T} t \quad \text{or} \quad y(t) = A \cos \frac{2\pi}{T} t$$

where T represents the time period. Suppose the time t is replaced by t + T, then the function

$$\begin{aligned} y(t+T) &= A \sin \frac{2\pi}{T}(t+T) \\ &= A \sin\left(\frac{2\pi}{T} t + 2\pi\right) \\ &= A \sin \frac{2\pi}{T} t = y(t) \end{aligned}$$
$$y(t+T) = y(t)$$

Thus, the function repeats after one time period. This y(t) is an example of periodic function.

Frequency and angular frequency

The number of oscillations produced by the particle per second is called frequency. It is denoted by f. SI unit for frequency is s^{-1} or hertz (In symbol, Hz). Mathematically, frequency is related to time period by

$$f = \frac{1}{T}$$

The number of cycles (or revolutions) per second is called angular frequency. It is usually denoted by the Greek small letter 'omega', ω. Comparing equation (10.11) and equation (10.12), angular frequency and frequency are related by

$$\omega = 2\pi f$$

SI unit for angular frequency is rad s^{-1}. (read it as radian per second)

Phase

The phase of a vibrating particle at any instant completely specifies the state of the particle. It expresses the position and direction of motion of the particle at that instant with respect to its mean position (Figure 10.11).

$$y = A \sin(\omega t + \varphi_0)$$

where $\omega t + \varphi 0 = \varphi$ is called the phase of the vibrating particle. At time t = 0 s (initial time), the phase $\varphi = \varphi 0$ is called epoch (initial phase) where $\varphi 0$ is called the angle of epoch. Phase difference: Consider two particles executing simple harmonic motions. Their equations are $y1 = A \sin(\omega t + \varphi 1)$ and $y_2 = A \sin(\omega t + \varphi 2)$, then the phase difference $\Delta\varphi = (\omega t + \varphi 2) - (\omega t + \varphi 1) = \varphi 2 - \varphi 1$.

E X A M P L E

A nurse measured the average heart beats of a patient and reported to the doctor in terms of time period as 0.8 s. Express the heart beat of the patient in terms of number of beats measured per minute.

Solution

Let the number of heart beats measured be f. Since the time period is inversely proportional to the heart beat, then

$$f = \frac{1}{T} = \frac{1}{0.8} = 1.25 s^{-1}$$

One minute is 60 second,

$$(1 \text{ second} = \frac{1}{60} \text{ minute} \Rightarrow 1 \text{ s}^{-1} = 60 \text{ min}^{-1})$$

$$f = 1.25 \text{ s}^{-1} \Rightarrow f = 1.25 \times 60 \text{ min}^{-1} = 75 \text{ beats}$$
per minute

E X A M P L E

Calculate the amplitude, angular frequency, frequency, time period and initial phase for the simple harmonic oscillation given below

a. $y = 0.3 \sin(40\pi t + 1.1)$
b. $y = 2 \cos(\pi t)$
c. $y = 3 \sin(2\pi t - 1.5)$

Solution

Simple harmonic oscillation equation is $y = A \sin(\omega t + \varphi_0)$ or $y = A \cos(\omega t + \varphi_0)$

a. For the wave, $y = 0.3 \sin(40\pi t + 1.1)$
Amplitude is $A = 0.3$ unit
Angular frequency $\omega = 40\pi$ rad s^{-1}
Frequency $f = \frac{\omega}{2\pi} = \frac{40\pi}{2\pi} = 20\,Hz$
Time period $T = \frac{1}{f} = \frac{1}{20} = 0.05s$
Initial phase is $\varphi_0 = 1.1$ rad

b. For the wave, $y = 2 \cos(\pi t)$

Amplitude is A = 2 unit
Angular frequency $\omega = \pi$ rad s^{-1}
Frequency $f = \frac{\omega}{2\pi} = \frac{\pi}{2\pi} = 0.5\,Hz$
Time period $T = \frac{1}{f} = \frac{1}{0.5} = 2s$
Initial phase is $\varphi_0 = 0$ rad

c. For the wave, $y = 3\sin(2\pi t + 1.5)$
Amplitude is A = 3 unit
Angular frequency $\omega = 2\pi$ rad s^{-1}
Frequency $f = \frac{\omega}{2\pi} = \frac{2\pi}{2\pi} = 1Hz$
Time period $T = \frac{1}{f} = \frac{1}{1} = 1s$
Initial phase is $\varphi_0 = 1.5$ rad

E X A M P L E

Show that for a simple harmonic motion, the phase difference between

a. displacement and velocity is $\frac{\pi}{2}$ radian or 90°.
b. velocity and acceleration is $\frac{\pi}{2}$ radian or 90
c. displacement and acceleration is π radian or 180°.

Solution

a. The displacement of the particle executing simple harmonic motion y=A sinωt
Velocity of the particle is
$$v = A\omega\cos\omega t = A\omega\sin\left(\omega t + \frac{\pi}{2}\right)$$
The phase difference between displacement and velocity is $\frac{\pi}{2}$

b. The velocity of the particle is v = A ω cos ωt
Acceleration of the particle is

$$a = -A\omega^2 \sin \omega t = A\omega^2 \cos\left(\omega t + \frac{\pi}{2}\right)$$

The phase difference between velocity and acceleration is $\frac{\pi}{2}$

c. The displacement of the particle is y = A sinωt
Acceleration of the particle is

$$a = -A\omega^2 \sin \omega t = A\omega^2 \sin(\omega t + \pi)$$

The phase difference between displacement and acceleration is π.

ANGULAR SIMPLE HARMONIC MOTION

Time period and frequency of angular SHM

When a body is allowed to rotate freely about a given axis then the oscillation is known as the angular oscillation. The point at which the resultant torque acting on the body is taken to be zero is called mean position. If the body is displaced from the mean position, then the resultant torque acts such that it is proportional to the angular displacement and this torque has a tendency to bring the body towards the mean position. (Note: Torque is explained in unit 5)

Let $\dot{\theta}$ be the angular displacement of the body and the resultant torque $\dot{\tau}$ acting on the body is

$$\vec{\tau} \propto \vec{\theta}$$

$$\vec{\tau} = -\kappa\vec{\theta}$$

K is the restoring torsion constant, which is torque per unit angular displacement. If I is the moment of inertia of the body and $\dot{\alpha}$ is the angular acceleration then

$$\vec{\tau} = I\vec{\alpha} = -\kappa\vec{\theta}$$

But $\vec{\alpha} = \frac{d^2\bar{\theta}}{dt^2}$ and therefore,

$$\frac{d^2\vec{\theta}}{dt^2} = -\frac{\kappa}{I}\vec{\theta}$$

This differential equation resembles simple harmonic differential equation.

So, comparing equation (10.17) with simple harmonic motion given in equation (10.10), we have

$$\omega = \sqrt{\frac{\kappa}{I}}\, rad\, s^{-1}$$

The frequency of the angular harmonic motion (from equation 10.13) is

$$f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}\, Hz$$

The time period (from equation 10.12) is

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \text{ second}$$

Comparison of Simple Harmonic Motion and Angular Simple Harmonic Motion

In linear simple harmonic motion, the displacement of the particle is measured in terms of linear displacement $\vec{r}$ The restoring force is $\vec{F} = -k\vec{r}$, where k is a spring constant or force constant which is force per unit displacement. In this case, the inertia factor is mass of the body executing simple harmonic motion.

In angular simple harmonic motion, the displacement of the particle is measured in terms of angular displacement $\vec{\theta}$. Here, the spring factor stands for torque constant i.e., the moment of the couple to produce unit angular displacement or the restoring torque per unit angular displacement. In this case, the inertia factor stands for moment of inertia of the body executing angular simple harmonic oscillation.

Table 10.2 Comparision of simple harmonic motion and angular harmonic motion

S.No	Simple Harmonic Motion	Angular Harmonic Motion
1.	The displacement of the particle is measured in terms of linear displacement $\vec{r}$.	The displacement of the particle is measured in terms of angular displacement $\vec{\theta}$ (also known as angle of twist).
2.	Acceleration of the particle is $\vec{a}=-\omega^2\vec{r}$	Angular acceleration of the particle is $\vec{\alpha}=-\omega^2\vec{\theta}$.
3.	Force, $\vec{F}=m\vec{a}$, where m is called mass of the particle.	Torque, $\vec{\tau}=I\vec{\alpha}$, where I is called moment of inertia of a body.
4.	The restoring force $\vec{F}=-k\vec{r}$, where k is restoring force constant.	The restoring torque $\vec{\tau}=-\kappa\vec{\theta}$, where the symbol κ (Greek alphabet is pronounced as '*kappa*') is called restoring torsion constant. It depends on the property of a particular torsion fiber.
5.	Angular frequency, $\omega=\sqrt{\frac{k}{m}}$ rad s^{-1}	Angular frequency, $\omega=\sqrt{\frac{\kappa}{I}}$ rad s^{-1}

LINEAR SIMPLE HARMONIC OSCILLATOR (LHO)

Horizontal oscillations of a spring-mass system

Consider a system containing a block of mass m attached to a massless spring with stiffness constant or force constant or spring constant k placed on a smooth horizontal surface (frictionless surface) as shown in Figure 10.13. Let x_0 be the equilibrium position or mean position of mass m when it is left undisturbed. Suppose the mass is displaced through a small displacement x towards right from its equilibrium position and then released, it will oscillate back and forth about its mean position x_0. Let F be the restoring force (due to stretching of the spring) which is proportional to the amount of displacement of block. For one dimensional motion, mathematically, we have

$$F \propto x$$
$$F = -k\,x$$

where negative sign implies that the restoring force will always act opposite to the direction of the displacement. This equation is called Hooke's law (refer to unit 7). Notice that, the restoring force is linear with the displacement (i.e., the exponent of force and displacement are unity). This is not always true; in case if we apply a very large stretching force, then the amplitude of oscillations becomes very large (which means, force is proportional to displacement containing higher powers of x) and therefore, the oscillation of the system is not linear and hence, it is called non-linear oscillation. We restrict ourselves only to linear oscillations throughout our discussions, which means Hooke's law is valid (force and displacement have a linear relationship).

From Newton's second law, we can write the equation for the particle executing simple harmonic motion

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Comparing the equation (10.21) with simple harmonic motion equation (10.10), we get

$$\omega^2 = \frac{k}{m}$$

which means the angular frequency or natural frequency of the oscillator is

$$\omega = \sqrt{\frac{k}{m}} \text{ rad } s^{-1}$$

The frequency of the oscillation is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hertz}$$

and the time period of the oscillation is

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \text{ seconds}$$

Notice that in simple harmonic motion, the time period of oscillation is independent of amplitude. This is valid only if the amplitude of oscillation is small. The solution of the differential equation of a SHM may be written as

$$x(t) = A\sin(\omega t + \varphi)$$

or

$$x(t) = A\cos(\omega t + \varphi)$$

where A, ω and ϕ are constants. General solution for differential equation 10.21 is x(t) = A sin(ωt +φ)+ B cos(ωt +φ) where A and B are contants.

Vertical oscillations of a spring

Let us consider a massless spring with stiff ness constant or force constant k attached to a ceiling as shown in Figure 10.15. Let the length of the spring before loading mass m be L. If the block of mass m is attached to the other end of spring, then the spring elongates by a length l. Let F1 be the restoring force due to stretching of spring. Due to mass m, the gravitational force acts vertically downward. We can draw free-body diagram for this system as shown in Figure 10.15. When the system is under equilibrium,

$$F_1 + mg = 0$$

But the spring elongates by small displacement l, therefore

$$F_1 \propto l \Rightarrow F_1 = -k\,l$$

Substituting equation (10.28) in equation (10.27), we get

$$-k\,l + mg = 0$$
$$\mathrm{mg} = \mathrm{kl}$$
or
$$\frac{m}{k} = \frac{l}{g}$$

Suppose we apply a very small external force on the mass such that the mass further displaces downward by a displacement y, then it will oscillate up and down. Now, the restoring force due to this stretching of spring (total extension of spring is y + l) is

$$\mathrm{F}_2 \propto (y + l)$$
$$\mathrm{F}_2 = -k\,(y + l) = -ky - kl$$

Since, the mass moves up and down with acceleration $\frac{d^2 y}{dt^2}$ by drawing the free body diagram for this case, we get

$$-ky - kl + mg = m\frac{d^2 y}{dt^2}$$

The net force acting on the mass due to this stretching is

$$F = \mathrm{F}_2 + mg$$
$$F = -ky - kl + mg$$

The gravitational force opposes the restoring force. Substituting equation (10.29) in equation (10.32), we get

$$F = -ky - kl + kl = -ky$$

Applying Newton's law, we get

$$m\frac{d^2y}{dt^2} = -k\,y$$

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y$$

The above equation is in the form of simple harmonic differential equation. Therefore, we get the time period as

$$T = 2\pi\sqrt{\frac{m}{k}}\ \text{second}$$

The time period can be rewritten using equation (10.29)

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}}\ \text{second}$$

The acceleration due to gravity g can be computed from the formula

$$g = 4\pi^2\left(\frac{l}{T^2}\right)\ \text{m s}^{-2}$$

E X A M P L E

A spring balance has a scale which ranges from 0 to 25 kg and the length of the scale is 0.25m. It is taken to an unknown planet X where the acceleration due to gravity is 11.5 m s^{-1}. Suppose a body of mass M kg is suspended in this spring and made to oscillate with a period of 0.50 s. Compute the gravitational force acting on the body.

Solution

Let us first calculate the stiff ness constant of the spring balance by using equation (10.29),

$$k = \frac{mg}{l} = \frac{25\times 11.5}{0.25} = 1150\ N\,m^{-1}$$

The time period of oscillations is given by $T = 2\pi\sqrt{\frac{M}{k}}$ where M is the mass of the body. Since, M is unknown, rearranging, we get

$$M = \frac{kT^2}{4\pi^2} = \frac{(1150)(0.5)^2}{4\pi^2} = 7.3kg$$

The gravitational force acting on the body is W = Mg = 7.3 × 11.5 = 83.95 N ≈ 84 N

Combinations of springs

Spring constant or force constant, also called as stiffness constant, is a measure of the stiffness of the spring. Larger the value of the spring constant, stiffer is the spring. This implies that we need to apply more force to compress or elongate the spring. Similarly, smaller the value of spring constant, the spring can be stretched (elongated) or compressed with lesser force. Springs can be connected in two ways. Either the springs can be connected end to end, also known as series connection, or alternatively, connected in parallel. In the following subsection, we compute the effective spring constant when

a. Springs are connected in series
b. Springs are connected in parallel

Springs connected in series

When two or more springs are connected in series, we can replace (by removing) all the springs in series with an equivalent spring (effective spring) whose net effect is the same as if all the springs are in series connection. Given the value of individual spring constants k_1, k_2, k_3,... (known quantity), we can establish a mathematical relationship to find out an effective (or equivalent) spring constant k_s (unknown quantity). For simplicity, let us consider only two springs whose spring constant are k_1 and k_2 and which can be attached to a mass m as shown in Figure 10.17. The results thus obtained can be generalized for any number of springs in series.

Let F be the applied force towards right as shown in Figure 10.18. Since the spring constants for different spring are different and the connection points between them is not rigidly fixed, the strings can stretch in different lengths. Let x1 and x2 be the elongation of springs from their equilibrium position (un-stretched position) due to the applied force F. Then, the net displacement of the mass point is

$$x = x_1 + x_2$$

$$F = -k_s (x_1 + x_2) \Rightarrow x_1 + x_2 = -\frac{F}{k_s} \quad (10.38)$$

For springs in series connection

$$-k_1 x_1 = -k_2 x_2 = F$$

$$\Rightarrow x_1 = -\frac{F}{k_1} \text{ and } x_2 = -\frac{F}{k_2}$$

Therefore, substituting equation (10.39) in equation (10.38), the effective spring constant can be calculated as

$$-\frac{F}{k_1} - \frac{F}{k_2} = -\frac{F}{k_s}$$

$$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2}$$

Or

$$k_s = \frac{k_1 k_2}{k_1 + k_2} \text{ Nm}^{-1}$$

Suppose we have n springs connected in series, the effective spring constant in series is

$$\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \dots + \frac{1}{k_n} = \sum_{i=1}^{n} \frac{1}{k_i}$$

$$\frac{1}{k_s} = \frac{n}{k} \Rightarrow k_s = \frac{k}{n}$$

This means that the effective spring constant reduces by the factor n. Hence, for springs in series connection, the effective spring constant is lesser than the individual spring constants.

From equation (10.39), we have,

$$k_1x_1 = k_2x_2$$

Then the ratio of compressed distance or elongated distance x_1 and x_2 is

$$\frac{x_2}{x_1} = \frac{k_1}{k_2}$$

The elastic potential energy stored in first and second springs are $V_1 = \frac{1}{2}k_1x_1^2$ and $V_2 = \frac{1}{2}k_2x_2^2$ respectively. Then, their ratio is

$$\frac{V_1}{V_2} = \frac{\frac{1}{2}k_1x_1^2}{\frac{1}{2}k_2x_2^2} = \frac{k_1}{k_2}\left(\frac{x_1}{x_2}\right)^2 = \frac{k_2}{k_1}$$

E X A M P L E

Consider two springs whose force constants are 1 N m^{-1} and 2 N m^{-1} which are connected in series. Calculate the effective spring constant (k_s) and comment on k_s.

Solution

$$k_1 = 1\ \text{N m}^{-1},\ k_2 = 2\ \text{N m}^{-1}$$

$$k_s = \frac{k_1 k_2}{k_1 + k_2}\ \text{N m}^{-1}$$

$$k_s = \frac{1\times 2}{1+2} = \frac{2}{3}\ \text{N m}^{-1}$$

$$k_s < k_1 \text{ and } k_s < k_2$$

Therefore, the effective spring constant is lesser than both k_1 and k_2.

Springs connected in parallel

When two or more springs are connected in parallel, we can replace (by removing) all these springs with an equivalent spring (effective spring) whose net effect is same as if all the springs are in parallel connection. Given the values of individual spring constants to be k_1,k_2,k_3, ... (known quantities), we can establish a mathematical relationship to find out an effective (or equivalent) spring constant k_p (unknown quantity). For simplicity, let us consider only two springs of spring constants $k_1$and k_2 attached to a mass m as shown in Figure 10.19. The results can be generalized to any number of springs in parallel

Let the force F be applied towards right as shown in Figure 10.20. In this case, both the springs elongate or compress by the same amount of displacement. Therefore, net force for the displacement of mass m is

$$F = -k_p x$$

where k_p is called effective spring constant. Let the first spring be elongated by a displacement x due to force F_1 and second spring be elongated by the same displacement x due to force F_2, then the net force

$$F = -k_1 x - k_2 x$$

Equating equations (10.46) and (10.45), we get

$$k_p = k_1 + k_2$$

Generalizing, for n springs connected in parallel,

$$k_p = \sum_{i=1}^{n} k_i$$

If all spring constants are identical i.e., $k_1 = k_2 = \ldots = k_n = k$ then

$$kp = n\,k$$

This implies that the effective spring constant increases by a factor n. Hence, for the springs in parallel connection, the effective spring constant is greater than individual spring constant.

E X A M P L E

Consider two springs with force constants 1 N m^{-1} and 2 N m^{-1} connected in parallel. Calculate the effective spring constant (k_p) and comment on k_p.

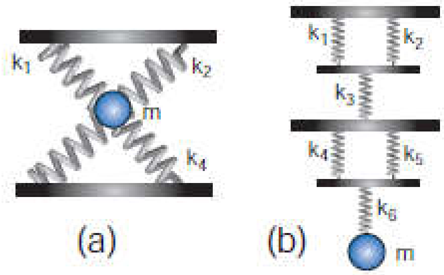
Solution

$$k_1 = 1\,\mathrm{N\,m^{-1}},\ k_2 = 2\,\mathrm{N\,m^{-1}}$$
$$k_p = k_1 + k_2\,\mathrm{N\,m^{-1}}$$
$$k_p = 1 + 2 = 3\,\mathrm{N\,m^{-1}}$$
$$k_p > k_1 \text{ and } k_p > k_2$$

Therefore, the effective spring constant is greater than both k_1 and k_2.

E X A M P L E

Calculate the equivalent spring constant for the following systems and also compute if all the spring constants are equal:



Solution

a. Since k_1 and k_2 are parallel, $k_u = k_1 + k_2$ Similarly, k_3 and k_4 are parallel, therefore, $k_d = k_3 + k_4$ But k_u and k_d are in series,

therefore, $k_{eq} = \frac{k_u k_d}{k_u + k_d}$ If all the spring constants are equal then, $k_1 = k_2 = k_3 = k_4 = k$
Which means, $k_u = 2k$ and $k_d = 2k$

Hence, $k_{eq} = \frac{4k^2}{4k} = k$

b. Since k_1 and k_2 are parallel, $k_A = k_1 + k_2$ Similarly, k_4 and k_5 are parallel, therefore, $k_B = k_4 + k_5$ But k_A, k_3, k_B, and k_6 are in series, therefore,

$$\frac{1}{k_{eq}} = \frac{1}{k_A} + \frac{1}{k_3} + \frac{1}{k_B} + \frac{1}{k_6}$$

If all the spring constants are equal then, $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k$ which means, $k_A = 2_k$ and $k_B = 2_k$

$$\frac{1}{k_{eq}} = \frac{1}{2k} + \frac{1}{k} + \frac{1}{2k} + \frac{1}{k} = \frac{3}{k}$$

$$k_{eq} = \frac{k}{3}$$

E X A M P L E

A mass m moves with a speed v on a horizontal smooth surface and collides with a nearly massless spring whose spring constant is k. If the mass stops after collision, compute the maximum compression of the spring.

Solution

When the mass collides with the spring, from the law of conservation of energy "the loss in kinetic energy of mass is gain in elastic potential energy by spring".

Let x be the distance of compression of spring, then the law of conservation of energy

$$\frac{1}{2}m\,v^2 = \frac{1}{2}\,k\,x^2 \Rightarrow x = v\sqrt{\frac{m}{k}}$$

Oscillations of a simple pendulum in SHM and laws of simple pendulum

Simple pendulum

A pendulum is a mechanical system which exhibits periodic motion. It has a bob with mass m suspended by a long string (assumed to be massless and inextensible string) and the other end is fixed on a stand as shown in Figure 10.21 (a). At equilibrium, the pendulum does not oscillate and hangs vertically downward. Such a position is known as mean position or equilibrium position. When a pendulum is displaced through a small displacement from its equilibrium position and released, the bob of the pendulum executes to and fro motion. Let l be the length of the pendulum which is taken as the distance between the point of suspension and the centre of gravity of the bob. Two forces act on the bob of the pendulum at any displaced position, as shown in the Figure 10.21 (d),

1. The gravitational force acting on the body $\left(\vec{F} = m\vec{g}\right)$ which acts vertically

downwards.

2. The tension in the string $\vec{T}$ which acts along the string to the point of suspension

Resolving the gravitational force into its components:

Normal component:

The component along the string but in opposition to the direction of tension, F_{as} = mg cosθ.

Tangential component:

The component perpendicular to the string i.e., along tangential direction of arc of swing, F_{ps} = mg sinθ.
Therefore, The normal component of the force is, along the string,

$$T - W_{as} = m\frac{v^2}{l}$$

Here v is speed of bob

$$T - mg\cos\theta = m\frac{v^2}{l}$$

From the Figure 10.21, we can observe that the tangential component W_{ps} of the gravitational force always points towards the equilibrium position i.e., the direction in which it always points opposite to the direction of displacement of the bob from the mean position. Hence, in this case, the tangential force is nothing but the restoring force. Applying Newton's second law along tangential direction, we have

$$m\frac{d^2s}{dt^2} + F_{ps} = 0 \Rightarrow m\frac{d^2s}{dt^2} = -F_{ps}$$

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad (10.51)$$

where, s is the position of bob which is measured along the arc. Expressing arc length in terms of angular displacement i.e.,

$$s = l\,\theta$$

then its acceleration,

$$\frac{d^2s}{dt^2} = l\,\frac{d^2\theta}{dt^2}$$

Substituting equation (10.53) in equation (10.51), we get

$$l\frac{d^2\theta}{dt^2} = -\,g\,\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\,\frac{g}{l}\,\sin\theta$$

Because of the presence of sin θ in the above differential equation, it is a non-linear differential equation (Here, homogeneous second order). Assume "the small oscillation approximation", $\sin\theta \approx \theta$, the above differential equation becomes linear differential equation.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\,\theta$$

This is the well known oscillatory diff erential equation. Therefore, the angular frequency of this oscillator (natural frequency of this system) is

$$\omega^2 = \frac{g}{l}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{g}{l}}\ \text{ in rad } s^{-1}$$

The frequency of oscillations is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}\ \text{ in Hz}$$

and time period of oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}} \text{ in second}$$

Laws of simple pendulum

The time period of a simple pendulum

a. **Depends on the following laws**
Law of length

For a given value of acceleration due to gravity, the time period of a simple pendulum is directly proportional to the square root of length of the pendulum.

$$T \propto \sqrt{l}$$

Law of acceleration

For a fi xed length, the time period of a simple pendulum is inversely proportional to square root of acceleration due to gravity.

$$T \propto \frac{1}{\sqrt{g}}$$

b. **Independent of the following factors**
Mass of the bob

Th e time period of oscillation is independent of mass of the simple pendulum. This is similar to free fall. Therefore, in a pendulum of fixed length, it does not matter whether an elephant swings or an ant swings. Both of them will swing with the same time period.

Amplitude of the oscillations

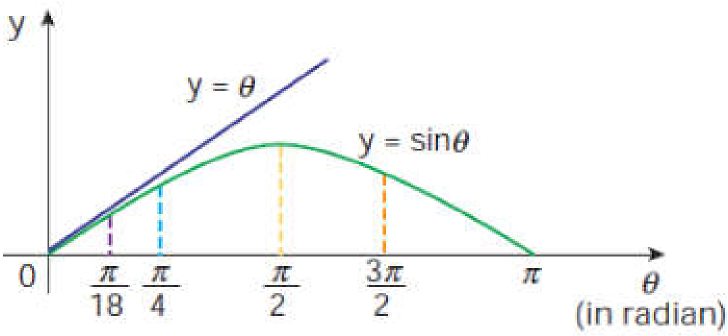
For a pendulum with small angle approximation (angular displacement is very small), the time period is independent of amplitude of the oscillation.

E X A M P L E

In simple pendulum experiment, we have used small angle approximation . Discuss the small angle approximation.

θ (in degrees)	θ(in radian)	sin θ
0	0	0
5	0.087	0.087
10	0.174	0.174
15	0.262	0.256
20	0.349	0.342
25	0.436	0.422
30	0.524	0.500
35	0.611	0.574
40	0.698	0.643
45	0.785	0.707

For θ in radian, $\sin\theta \approx \theta$ for very small angles



This means that "for θ as large as 10 degrees, sin θ is nearly the same as θ when θ is expressed in radians". As θ increases in value sinθ gradually becomes different from θ

Pendulum length due to effect of temperature

Suppose the suspended wire is affected due to change in temperature. The rise in temperature affects length by

$$l = l_o(1 + \alpha\,\Delta t)$$

where lo is the original length of the wire and l is final length of the wire when the temperature is raised. Let Δt is the change in temperature and α is the co-efficient of linear expansion.

$$\text{Then, } T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{l_0(1+\alpha\Delta t)}{g}}$$

$$= 2\pi\sqrt{\frac{l_0}{g}}\sqrt{(1+\alpha\Delta t)}$$

$$T = T_0(1+\alpha\Delta t)^{\frac{1}{2}} \approx T_0\left(1+\frac{1}{2}\alpha\Delta t\right)$$

$$\Rightarrow \quad \frac{T}{T_0} - 1 = \frac{T - T_0}{T_0} = \frac{\Delta T}{T_0} = \frac{1}{2}\alpha\Delta t$$

where ΔT is the change in time period due to the effect of temperature and T_0 is the time period of the simple pendulum with original length l_0.

E X A M P L E

If the length of the simple pendulum is increased by 44% from its original length, calculate the percentage increase in time period of the pendulum.

Solution

Since

$$T \propto \sqrt{l}$$

Therefore,

$$T = \text{constant } \sqrt{l}$$

$$\frac{T_f}{T_i} = \sqrt{\frac{l + \frac{44}{100} l}{l}} = \sqrt{1.44} = 1.2$$

Therefore, $T_f = 1.2\ T_i = T_i + 20\%\ T_i$

Oscillation of liquid in a U-tube:

Consider a U-shaped glass tube which consists of two open arms with uniform crosssectional area A. Let us pour a non-viscous uniform incompressible liquid of density ρ in the U-shaped tube to a height h as shown in the Figure 10.22. If the liquid and tube are not disturbed then the liquid surface will be in equilibrium position O. It means the pressure as measured at any point on the liquid is the same and also at the surface on the arm (edge of the tube on either side), which balances with the atmospheric pressure. Due to this the level of liquid in each arm will be the same. By blowing air one can provide sufficient force in one arm, and the liquid gets disturbed from equilibrium position O, which means, the pressure at blown arm is higher than the other arm. This creates difference in pressure which will cause the liquid to oscillate for a very short duration of time about the mean or equilibrium position and finally comes to rest.

Time period of the oscillation is

$$T = 2\pi\sqrt{\frac{l}{2g}}\ \text{second}$$

ENERGY IN SIMPLE HARMONIC MOTION

a. Expression for Potential Energy

For the simple harmonic motion, the force and the displacement are related by Hooke's law

$$\vec{F} = -k\vec{r}$$

Since force is a vector quantity, in three dimensions it has three components. Further, the force in the above equation is a conservative force field; such a force can be derived from a scalar function which has only one component. In one dimensional case

$$F = -\,k\,x$$

As we have discussed in unit 4 of volume I, the work done by the conservative force field is independent of path. The potential energy U can be calculated from the following expression.

$$F = -\frac{dU}{dx}$$

Comparing (10.63) and (10.64), we get

$$-\frac{dU}{dx} = -\,kx$$

$$dU = k\,x\,dx$$

This work done by the force F during a small displacement dx stores as potential energy

$$U(x) = \int_0^x k\,x'\,dx' = \frac{1}{2}k\left(x'\right)^2\Big|_0^x = \frac{1}{2}kx^2$$

From equation (10.22), we can substitute the value of force constant k = m ω2 in equation (10.65),

$$U(x)=\frac{1}{2}m\omega^2x^2$$

where ω is the natural frequency of the oscillating system. For the particle executing simple harmonic motion from equation (10.6), we get

$$x = A \sin \omega t$$

$$U(\mathrm{t})=\frac{1}{2}m\omega^2A^2\sin^2\omega t$$

b. Expression for Kinetic Energy

Kinetic energy

$$KE=\frac{1}{2}mv_x^2=\frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

Since the particle is executing simple harmonic motion, from equation (10.6)

$$x = A \sin \omega t$$

Therefore, velocity is

$$v_x=\frac{dx}{dt}=A\omega\cos\omega t$$

$$=A\omega\sqrt{1-\left(\frac{x}{A}\right)^2}$$

$$v_x=\omega\sqrt{A^2-x^2}$$

Hence,

$$KE=\frac{1}{2}mv_x^2=\frac{1}{2}m\omega^2\left(A^2-x^2\right)$$

$$KE=\frac{1}{2}m\omega^2A^2\cos^2\omega t$$

c. Expression for Total Energy

Total energy is the sum of kinetic energy and potential energy

$$E = KE + U$$

$$E=\frac{1}{2}m\omega^2(A^2-x^2)+\frac{1}{2}m\omega^2x^2$$

Hence, cancelling x^2 term,

$$E=\frac{1}{2}m\omega^2A^2 = \text{constant}$$

Alternatively, from equation (10.67) and equation (10.72), we get the total energy as

$$E=\frac{1}{2}m\omega^2A^2\sin^2\omega t+\frac{1}{2}m\omega^2A^2\cos^2\omega t$$
$$=\frac{1}{2}m\omega^2A^2(\sin^2\omega t+\cos^2\omega t)$$

From trigonometry identity, $(\sin^2 \omega t + \cos^2 \omega t) = 1$

$$E=\frac{1}{2}m\omega^2A^2 = \text{constant}$$

Thus the amplitude of simple harmonic oscillator, can be expressed in terms of total energy.

$$A=\sqrt{\frac{2E}{m\omega^2}}=\sqrt{\frac{2E}{k}}$$

EXAMPLE

Write down the kinetic energy and total energy expressions in terms of linear momentum, For one-dimensional case.

Solution

Kinetic energy is $KE = \frac{1}{2}mv_x^2$

Multiply numerator and denominator by m

$$KE = \frac{1}{2m} m^2 v_x^2 = \frac{1}{2m} (m v_x)^2 = \frac{1}{2m} p_x^2$$

where, px is the linear momentum of the particle executing simple harmonic motion.

Total energy can be written as sum of kinetic energy and potential energy, therefore, from equation (10.73) and also from equation (10.75), we get

$$E = KE + U(x) = \frac{1}{2m} p_x^2 + \frac{1}{2} m\omega^2 x^2 = \text{constant}$$

E X A M P L E

Compute the position of an oscillating particle when its kinetic energy and potential energy are equal.

Solution

Since the kinetic energy and potential energy of the oscillating particle are equal,

$$\frac{1}{2} m\omega^2 (A^2 - x^2) = \frac{1}{2} m\omega^2 x^2$$

$$A^2 - x^2 = x^2$$

$$2x^2 = A^2$$

$$\Rightarrow x = \pm \frac{A}{\sqrt{2}}$$

TYPES OF OSCILLATIONS:
Free oscillations

When the oscillator is allowed to oscillate by displacing its position from equilibrium position, it oscillates with a frequency which is equal to the natural frequency of the oscillator. Such an oscillation or vibration is known as free oscillation or free vibration. In this case, the amplitude, frequency and the energy of the vibrating object remains constant.

Examples

a. Vibration of a tuning fork.
b. Vibration in a stretched string.
c. Oscillation of a simple pendulum.
d. Oscillationsof a spring-mass system.

Damped oscillations

During the oscillation of a simple pendulum (in previous case), we have assumed that the amplitude of the oscillation is constant and also the total energy of the oscillator is constant. But in reality, in a medium, due to the presence of friction and air drag, the amplitude of oscillation decreases as time progresses. It implies that the oscillation is not sustained and the energy of the SHM decreases gradually indicating the loss of energy. Th e energy lost is absorbed by the surrounding medium. Th is type of oscillatory motion is known as damped oscillation. In other words, if an oscillator moves in a resistive medium, its amplitude goes on decreasing and the energy of the oscillator is used to do work against the resistive medium. Th e motion of the oscillator is said to be damped and in this case, the resistive force (or damping force) is proportional to the velocity of the oscillator.

Examples

a. Th e oscillations of a pendulum (including air friction) or pendulum oscillating inside an oil fi lled container.
b. Electromagnetic oscillations in a tank circuit.
c. Oscillations in a dead beat and ballistic galvanometers.

Maintained oscillations

While playing in swing, the oscillations will stop aft er a few cycles, this is due to damping. To avoid damping we have to supply a push to sustain oscillations. By supplying energy from an external source, the amplitude of the oscillation can be made constant. Such vibrations are known as maintained vibrations.

Example:

The vibration of a tuning fork getting energy from a battery or from external power supply.

Forced oscillations

Any oscillator driven by an external periodic agency to overcome the damping is known as forced oscillator or driven oscillator. In this type of vibration, the body executing vibration initially vibrates with its natural frequency and due to the presence of external periodic force, the body later vibrates with the frequency of the applied periodic force. Such vibrations are known as forced vibrations.

Example:

Sound boards of stringed instruments.

Resonance

It is a special case of forced vibrations where the frequency of external periodic force (or driving force) matches with the natural frequency of the vibrating body (driven). As a result the oscillating body begins to vibrate such that its amplitude increases at each step and ultimately it has a large amplitude. Such a phenomenon is known as resonance and the corresponding vibrations are known as resonance vibrations.

Example

The breaking of glass due to sound

Soliders are not allowed to march on a bridge. This is to avoid resonant vibration of the bridge. While crossing a bridge, if the period of stepping on the ground by marching soldiers equals the natural frequency of the bridge, it may result in resonance vibrations. This may be so large that the bridge may collapse.