PHYSICS
TEST – 6

11th physics	**Unit 2**	**Kinematics**
	Unit 3	**Laws Of Motion**
	Unit 4	**Work, Energy And Power**

11TH VOL - I
UNIT 2 KINEMATICS

INTRODUCTION

Physics is basically an experimental science and rests on two pillars–Experiments and mathematics. Two thousand three hundred years ago the Greek librarian Eratosthenes measured the radius of the Earth. The size of the atom was measured much later, only in the beginning of the 20th century. The central aspect in physics is motion. Motion is found at all levels–from microscopic level (within the atom) to macroscopic and galactic level (planetary system and beyond). In short the entire Universe is governed by various types of motion. The study of various types of motion is expressed using the language of mathematics.

How do objects move? How fast or slow do they move? For example, when ten athletes run in a race, all of them do not run in the same manner. Their performance cannot be qualitatively recorded by usage of words like 'fastest', 'faster', 'average', 'slower' or 'slowest'. It has to be quantified. Quantifying means assigning numbers to each athlete's motion. Comparing these numbers one can analyse how fast or slow each athlete runs when compared to others. In this unit, the basic mathematics needed for analyzing motion in terms of its direction and magnitude is covered.

Kinematics is the branch of mechanics which deals with the motion of objects without taking force into account. The Greek word "kinema" means "motion".

CONCEPT OF REST AND MOTION

The concept of rest and motion can be well understood by the following elucidation. A person sitting in a moving bus is at rest with respect to a fellow passenger but is in motion with respect to a person outside the bus. The concepts of rest and motion have meaning only with respect to some reference frame. To understand rest or motion we need a convenient fixed reference frame.

Frame of Reference:

If we imagine a coordinate system and the position of an object is described relative to it, then such a coordinate system is called frame of reference. At any given instant of time, the frame of reference with respect to which the position of the object is described in terms of position coordinates (x, y, z) (i.e., distances of the given position of an object along the x, y, and z–axes.) is called "Cartesian coordinate system".

It is to be noted that if the x, y and z axesare drawn in anticlockwise direction thenthe coordinate system is called as "right–handed Cartesian coordinate system".Though other coordinate systems do exist,in physics we conventionally follow theright–handed coordinate system.

Illustrates the difference between left and right handed coordinate systems.

Point mass

To explain the motion of an object which has finite mass, the concept of "point mass" is required and is very useful. Let the mass of any object be assumed to be concentrated at a point. Then this idealized mass is called "point mass". It has no internal structure like shape and size. Mathematically a point mass has finite mass with zero dimension. Even though in reality a point mass does not exist, it often simplifies our calculations. It is to be noted that the term "point mass" is a relative term. It has meaning only with respect to a reference frame and with respect to the kind of motion that we analyse.

Examples

- To analyse the motion of Earth with respect to Sun, Earth can be treated as a point mass. This is because the distance between the Sun and Earth is very large compared to the size of the Earth.
- If we throw an irregular object like a small stone in the air, to analyse its motion it is simpler to consider the stone as a point mass as it moves in space. The size of the stone is very much smaller than the distance through which it travels.

Types of motion

In our day–to–day life the following kinds of motion are observed:

Linear motion

An object is said to be in linear motion if it moves in a straight line.

Examples

1. An athlete running on a straight track
2. A particle falling vertically downwards to the Earth.

Circular motion

Circular motion is defined as a motion described by an object traversing a circular path.

Examples

1. The whirling motion of a stone attached to a string
2. The motion of a satellite around the Earth

Rotational motion

If any object moves in a rotational motion about an axis, the motion is called 'rotation'. During rotation every point in the object transverses a circular path about an axis, (except the points located on the axis).

Examples

1. Rotation of a disc about an axis through its center
2. Spinning of the Earth about its own axis.

Vibratory motion

If an object or particle executes a to–and– fro motion about a fixed point, it is said to be in vibratory motion. This is sometimes also called oscillatory motion.

Examples

1. Vibration of a string on a guitar
2. Movement of a swing

Other types of motion like elliptical motion and helical motion are also possible.

Motion in One, Two and Three Dimensions

Let the position of a particle in space be expressed in terms of rectangular coordinates x, y and z. When these coordinates change with time, then the particle is said to be in motion. However, it is not necessary that all the three coordinates should together change with time.

Even if one or two coordinates changes with time, the particle is said to be in motion. Then we have the following classification.

Motion in one dimension

One dimensional motion is the motion of a particle moving along a straight line.

This motion is sometimes known as rectilinear or linear motion.

In this motion, only one of the three rectangular coordinates specifying the position of the object changes with time.

For example, if a car moves from position A to position B along x–direction, as shown in Figure 2.8, then the variation in x–coordinate alone is noticed.

Examples

1. Motion of a train along a straight railway track.
2. An object falling freely under gravity close to Earth

Motion in two dimensions

If a particle is moving along a curved path in a plane, then it is said to be in two dimensional motion.

In this motion, two of the three rectangular coordinates specifying the position of object change with time.

Examples

1. Motion of a coin on a carrom board.
2. An insect crawling over the floor of a room.

Motion in three dimensions

A particle moving in usual three dimensional space has three dimensional motion.

In this motion, all the three coordinates specifying the position of an object change with respect to time. When a particle moves in three dimensions, all the three coordinates x, y and z will vary.

Examples

1. A bird flying in the sky.
2. Random motion of a gas molecule.

3. Flying of a kite on a windy day.

ELEMENTARY CONCEPTS OF VECTOR ALGEBRA

In physics, some quantities possess only magnitude and some quantities possess both magnitude and direction. To understand these physical quantities, it is very important to know the properties of vectors and scalars.

Scalar

It is a property which can be described only by magnitude. In physics a number of quantities can be described by scalars.

Examples

Distance, mass, temperature, speed and energy

Vector

It is a quantity which is described by both magnitude and direction. Geometrically a vector is a directed line segment which is shown in Figure 2.10. In physics certain quantities can be described only by vectors.

Examples

Force, velocity, displacement, position vector, acceleration, linear momentum and angular momentum.

Magnitude of a Vector

The length of a vector is called magnitude of the vector. It is always a positive quantity. Sometimes the magnitude of a vector is also called 'norm' of the vector. For a vector $\vec{A}$ the magnitude or norm is denoted by $|\vec{A}|$ or simply 'A'.

Different types of Vectors
Equal vectors:

Two vectors $\vec{A}$ and $\vec{B}$ said to be equal when they have equal magnitude and same direction and represent the same physical quantity

Collinear vectors: Collinear vectors are those which act along the same line. The angle between them can be 0° or 180°.

Parallel Vectors:

If two vectors A and B act in the same direction along the same line or on parallel lines, then the angle between them is 0^0

Anti–parallel vectors:

Two vectors $\vec{A}$ and $\vec{B}$ are said to be anti–parallel when they are in opposite directions along the same line or on parallel lines. Then the angle between them is 180^0

Unit vector:

A vector divided by its magnitude is a unit vector. The unit vector for $\vec{A}$ is denoted by $\hat{A}$ (read as A cap or A hat). It has a magnitude equal to unity or one.

$$\text{Since, } \hat{A} = \frac{\vec{A}}{A}, \text{ we can write } \vec{A} = A\hat{A}$$

Thus, we can say that the unit vector specifies only the direction of the vector quantity.

Orthogonal unit vectors: Let $\hat{i}, \hat{j}$ and $\hat{k}$ be three unit vectors which specify the directions along positive x–axis, positive y–axis and positive z–axis respectively.These three unit vectors are directedperpendicular to each other, the anglebetween any two of them is 90°. $\hat{i}, \hat{j}$ and $\hat{k}$ are examples of orthogonal vectors.Two vectors which are perpendicular toeach other are called orthogonal vectors.

Addition of Vectors

Since vectors have both magnitude and direction they cannot be added by the method of ordinary algebra. Thus, vectors can be added geometrically or analytically using certain rules called 'vector algebra'. In order to find the sum (resultant) of two vectors, which are inclined to each other, we use

1. Triangular law of addition method
2. Parallelogram law of vectors.

Triangular Law of addition method

Let us consider two vectors $\vec{A}$ and $\vec{B}$

To find the resultant of the two vectors we apply the triangular law of addition as follows:

Represent the vectors $\vec{A}$ and $\vec{B}$ by the two adjacent sides of a triangle taken in the same order. Then the resultant is given by the third side of the triangle.

To explain further, the head of the first vector $\vec{A}$ is connected to the tail of the second vector $\vec{B}$. Let θ be the angle between $\vec{A}$ and $\vec{B}$. Then $\vec{R}$ is the resultant vector connecting the tail of the first vector $\vec{A}$ to the head of the second vector $\vec{B}$. The magnitude of $\vec{R}$ (resultant) is given geometrically by the length of $\vec{R}$ (OQ) and the direction of the resultant vector is the angle between $\vec{R}$ and $\vec{A}$. Thus we write

$\vec{R} = \vec{A} + \vec{B}$.

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$

Magnitude of resultant vector

The magnitude and angle of the resultant vector are determined as follows.

From consider thetriangle ABN, which is obtained byextending the side OA to ON. ABN is aright angled triangle.

$$\cos\theta = \frac{AN}{B} \therefore AN = B\cos\theta \ and$$

$$\sin\theta = \frac{BN}{B} \quad \therefore BN = B\sin\theta$$

For ΔOBN, we have $OB^2 = ON^2 + BN^2$

$$\Rightarrow R^2 = (A + B\cos\theta)^2 + (B\ \sin\theta)^2$$

$$\Rightarrow R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\Rightarrow R^2 = A^2 + B^2\left(\cos^2\theta + \sin^2\theta\right) + 2AB\cos\theta$$

$$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

which is the magnitude of the resultant of $\vec{A}$ and $\vec{B}$

Direction of resultant vectors:

If θ is the angle between $\vec{A}$ and $\vec{B}$, then

$$\left|\vec{A} + \vec{B}\right| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

If $\vec{R}$ makes an angle α with $\vec{A}$, then in ΔOBN,

$$\tan\alpha = \frac{BN}{ON} = \frac{BN}{OA + AN}$$
$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$
$$\Rightarrow \alpha = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$$

Two vectors

$\vec{A}$ and $\vec{B}$ of magnitude 5 units and 7 units respectively make an angle 60° with each other as shown below. Find the magnitude of the resultant vector and its direction with respect to the vector $\vec{A}$.

By following the law of triangular addition, the resultant vector is given by

$$\vec{R} = \vec{A} + \vec{B}$$

The magnitude of the resultant vector $\vec{R}$ is given by

$$R = \left|\vec{R}\right| = \sqrt{5^2 + 7^2 + 2 \times 5 \times 7 \cos 60^\circ}$$
$$R = \sqrt{25 + 49 + \frac{70 \times 1}{2}} = \sqrt{109}\,units$$

The angle α between $\vec{R}$ and $\vec{A}$ is given by

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

$$\tan\alpha = \frac{7 \times \sin 60}{5 + 7\cos 60} = \frac{7\sqrt{3}}{10 + 7} = \frac{7\sqrt{3}}{17} \cong 0.713$$
$$\therefore \alpha \cong 35^\circ$$

Subtraction of vectors

Since vectors have both magnitude and direction two vectors cannot be subtracted from each other by the method of ordinary algebra. Thus, this subtraction can be done either geometrically or analytically. We shall now discuss subtraction of two vectors geometrically.

For two non-zero vectors $\vec{A}$ and $\vec{B}$ which are inclined to each other at an angle θ, the difference $\vec{A} - \vec{B}$ is obtained as follows. First obtain $-\vec{B}$. The angle between $\vec{A}$ and $-\vec{B}$ is 180–θ.

The difference $\vec{A}$ - $\vec{B}$ is the same as the resultant of $\vec{A}$ and -$\vec{B}$

We can $\vec{A}$ - $\vec{B}$ = $\vec{A}$ +(- $\vec{B}$) write and using the equation.

$$\left|\vec{A}-\vec{B}\right| = \sqrt{A^2 + B^2 + 2AB\cos(180-\theta)}$$

Since, cos(180 − θ) = −cos θ we, get

$$\Rightarrow \left|\vec{A}-\vec{B}\right| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

$$\tan\alpha_2 = \frac{B\sin(180^\circ - \theta)}{A + B\cos(180^\circ - \theta)}$$

But sin(180° − θ) = sinθ hence we get

$$\Rightarrow \tan\alpha_2 = \frac{B\sin\theta}{A - B\cos\theta}$$

Thus the difference $\vec{A}$ - $\vec{B}$ is a vector with magnitude and direction given by equations.

Two vectors

$\vec{A}$ and $\vec{B}$ of magnitude 5 units and 7 units make an angle 60° with each other. Find the magnitude of the difference vector $\vec{A}$ - $\vec{B}$ and its direction with respect to the vector $\vec{A}$.

$$\left|\vec{A}-\vec{B}\right| = \sqrt{5^2 + 7^2 - 2\times5\times7\cos60^\circ}$$
$$= \sqrt{25+49-35} = \sqrt{39}\ units$$

The angle that $\vec{A}$ - $\vec{B}$ makes with the vector $\vec{A}$ is given by

$$\tan\alpha_2 = \frac{7\sin60}{5-7\cos60} = \frac{7\sqrt{3}}{10-7} = \frac{7}{\sqrt{3}} = 4.041$$
$$\alpha_2 = \tan^{-1}(4.041) \cong 76^\circ$$

COMPONENTS OF A VECTOR

In the Cartesian coordinate system any vector $\vec{A}$ can be resolved into three components along x, y and z directions.

Consider a 3–dimensional coordinate system. With respect to this a vector can be written in component form as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

.

Here A_x is the x–component of $\vec{A}$, A_y is the y–component of $\vec{A}$ and A_z is the z component of $\vec{A}$.

In a 2–dimensional Cartesian coordinate system.

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

If $\vec{A}$ makes an angle θ with x axis, and A_x and A_y are the components of $\vec{A}$ along x–axis and y–axis respectively.

$$A_x = A\cos\theta,\ A_y = A\sin\theta$$

where 'A' is the magnitude (length) of the vector $\vec{A}$, $A = \sqrt{A_x^2 + A_y^2}$

What are the unit vectors along the negative x–direction, negative y–direction, and negative z– direction?

The unit vectors along the negative directions can be shown as in the following figure.

1. The unit vector along the negative x direction $= -\hat{i}$
2. The unit vector along the negative y direction $= -\hat{j}$
3. The unit vector along the negative z direction $= -\hat{k}$

Vector addition using components

In the previous section we have learnt about addition and subtraction of two vectors using geometric methods. But once we choose a coordinate system, the addition and subtraction of vectors becomes much easier to perform.

The two vector $\vec{A}$ and $\vec{B}$ in a Cartesian coordinate system can be expressed as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Then the addition of two vectors is equivalent to adding their corresponding x, y and z components.

$$\vec{A}+\vec{B}=\left(A_x+B_x\right)\hat{i}+(A_y+B_y)\hat{j}+\left(A_z+B_z\right)\hat{k}$$

Similarly the subtraction of two vectors is equivalent to subtracting the corresponding x, y and z components.

$$\vec{A}-\vec{B}=\left(A_x-B_x\right)\hat{i}+(A_y-B_y)\hat{j}+\left(A_z-B_z\right)\hat{k}$$

The above rules form an analytical way of adding and subtracting two vectors.

Two vectors $\vec{A}$ and $\vec{B}$ are given in the component form as $\vec{A}=5\hat{i}+7\hat{j}-4\hat{k}$ and $\vec{B}=6\hat{i}+3\hat{j}+2\hat{k}$. Find $\vec{A}+\vec{B}$, $\vec{B}+\vec{A}$, $\vec{A}-\vec{B}$, $\vec{B}-\vec{A}$

$$\begin{aligned}
\vec{A}+\vec{B} &= (5\hat{i}+7\hat{j}-4\hat{k})+(6\hat{i}+3\hat{j}+2\hat{k}) \\
&= 11\hat{i}+10\hat{j}-2\hat{k} \\
\vec{B}+\vec{A} &= (6\hat{i}+3\hat{j}+2\hat{k})+(5\hat{i}+7\hat{j}-4\hat{k}) \\
&= (6+5)\hat{i}+(3+7)\hat{j}+(2-4)\hat{k} \\
&= 11\hat{i}+10\hat{j}-2\hat{k} \\
\vec{A}-\vec{B} &= (5\hat{i}+7\hat{j}-4\hat{k})-(6\hat{i}+3\hat{j}+2\hat{k}) \\
&= -\hat{i}+4\hat{j}-6\hat{k} \\
\vec{B}-\vec{A} &= \hat{i}-4\hat{j}+6\hat{k}
\end{aligned}$$

Note that the vectors $\vec{A}+\vec{B}$ and $\vec{B}+\vec{A}$ are same and the vectors $\vec{A}-\vec{B}$ and $\vec{B}-\vec{A}$ are opposite to each other.

A vector $\vec{A}$ multiplied by a scalar λ results in another vector, λ $\vec{A}$. If λ is a positive number then λ $\vec{A}$ is also in the direction of $\vec{A}$. If λ is a negative number, λ $\vec{A}$ is in the opposite direction to the vector $\vec{A}$.

Given the vector $\vec{A}=2\hat{i}+3\hat{j}$, what is 3 $\vec{A}$?

$$3\vec{A}=3\left(2\hat{i}+3\hat{j}\right)=6\hat{i}+9\hat{j}$$

The vector $3\vec{A}$ is in the same direction as vector $\vec{A}$.

A vector $\vec{A}$ is given as in the following Figure. Find $4\vec{A}$ and $-4\vec{A}$

Solution

In physics, certain vector quantities canbe defined as a scalar times another vector quantity.

For example

Force. $\vec{F} = m\vec{a}$ Here mass 'm' is a scalar, and $\vec{a}$ is the acceleration. Since 'm' is always a positive scalar, the direction of force is always in the direction of acceleration.

Linear momentum $\vec{p} = m\vec{v}$ .Here $\vec{v}$ is the velocity. The direction of linear momentum is also in the direction of velocity.

Force $\vec{F} = q\vec{E}$ Here the electric charge ' q' is a scalar, and $\vec{E}$ is the electric field. Since charge can be positive or negative, the direction of force $\vec{F}$ is correspondingly either in the direction of $\vec{E}$ or opposite to the direction of $\vec{E}$.

Scalar Product of Two Vectors
Definition

The scalar product (or dot product) of two vectors is defined as the product of the magnitudes of both the vectors and the cosine of the angle between them.

Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having an angle θ between them, then their scalar product is defined as $\vec{A} . \vec{B}$ = AB cosθ. Here, A and B are magnitudes of $\vec{A}$ and $\vec{B}$.

Properties

The product quantity $\vec{A} . \vec{B}$ is always a scalar. It is positive if the angle between the vectors is acute (i.e., < 90°) and negative if the angle between them is obtuse (i.e. 90°<θ< 180°).

The scalar product is commutative,

$$\vec{A} . \vec{B} = \vec{B} . \vec{A}$$

The vectors obey distributive law i.e.

$$\vec{A}.\left(\vec{B}+\vec{C}\right) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$$

The angle between the vectors

$$\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{AB}\right]$$

The scalar product of two vectors will be maximum when cos θ = 1, i.e. θ = 0°, i.e., when the vectors are parallel;

$$(\vec{A} \cdot \vec{B})_{max} = AB$$

The scalar product of two vectors will be minimum, when cos θ = –1, i.e. θ = 180°($\vec{A} . \vec{B}$)$_{max}$=AB when the vectors are anti-parallel.

If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other then their scalar product $\vec{A} . \vec{B} = 0$, because cos 90°=0. Then the vectors $\vec{A}$ and $\vec{B}$ are said to be mutually orthogonal.

The scalar product of a vector with itself is termed as self–dot product and is given by $(\vec{A})^2 = \vec{A} . \vec{A}$ =AA Cosθ=A^2.Here angle θ = 0°

The magnitude or norm of the vector

$$\vec{A} \text{ is } |\vec{A}| = A = \sqrt{\vec{A} \cdot \vec{A}}$$

In case of a unit vector $\hat{n}$

$$\hat{n} . \hat{n} = 1 \times 1 \times \cos 0 = 1. \text{ For example } \hat{i} . \hat{i} = \hat{j} . \hat{j} = \hat{k} . \hat{k} = 1$$

In the case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 1 \cdot 1 \cos 90° = 0$$

In terms of components the scalar product of $\vec{A}$ and $\vec{B}$ can be written as

$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot \left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$
$$= A_xB_x + A_yB_y + A_zB_z, \text{ with all other terms zero.}$$

The magnitude of vector $|\vec{A}|$ is given by

$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Given two vectors $\vec{A} = 2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{B} = \hat{i} + 3\hat{j} + 6\hat{k}$ Find the product $\vec{A}.\vec{B}$ and the magnitudes of $\vec{A}$ and $\vec{B}$. What is the angle between them?

Solution

$$\vec{A}.\vec{B} = 2 + 12 + 30 = 44$$

$$\text{Magnitude } A = \sqrt{4+16+25} = \sqrt{45} \text{ units}$$
$$\text{Magnitude } B = \sqrt{1+9+36} = \sqrt{46} \text{ units}$$

The angle between the two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right)$$
$$= \cos^{-1}\left(\frac{44}{\sqrt{45 \times 46}}\right) = \cos^{-1}\left(\frac{44}{45.49}\right)$$
$$= \cos^{-1}(0.967)$$
$$\therefore \theta \cong 15^\circ$$

Check whether the following vectors are orthogonal.

1. $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 4\hat{i} - 5\hat{j}$
2. $\vec{C} = 5\hat{i} + 2\hat{j}$ and $\vec{D} = 2\hat{i} - 5\hat{j}$

Solution

$$\vec{A}.\vec{B} = 8 - 15 = -7 \neq 0$$

Hence $\vec{A}$ and $\vec{B}$ are not orthogonal to each other

$$\vec{C}.\vec{D} = 10 - 10 = 0$$

Hence, $\vec{C}$ and $\vec{D}$ are orthogonal to each other.

It is also possible to geometrically show that the vectors $\vec{C}$ and $\vec{D}$ are orthogonal to each other. This is shown in the following Figure.

In physics, the work done by a force $\vec{F}$ to move an object through a small displacement d$\vec{r}$ is defined as,

$$W = \vec{F}.d\vec{r}$$
$$W = F\ dr\ \cos\theta$$

The work done is basically a scalar product between the force vector and the displacement vector. Apart from work done, there are other physical quantities which are also defined through scalar products.

The Vector Product of Two Vectors

Definition

The vector product or cross product of two vectors is defined as another vector having a magnitude equal to the product of the magnitudes of two vectors and the sine of the angle between them. The direction of the product vector is perpendicular to the plane containing the two vectors, in accordance with the right hand screw rule or right hand thumb rule

Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product is written as $\vec{A} \times \vec{B}$ which is a vector $\vec{C}$ defined by

$$\vec{C} = \vec{A} \times \vec{B} = (AB \sin \theta)\hat{n}$$

The direction $\hat{n}$ of $\vec{A} \times \vec{B}$ i.e., $\vec{C}$ is perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ and is in the sense of advancement of a right handed screw rotated $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus, if a right-handed screw whose axis is perpendicular to the plane formed by A and B, is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A} \times \vec{B}$ i.e. $\vec{C}$

Properties of vector (cross) product.

The vector product of any two vectors is always another vector whose direction is perpendicular to the plane containing these two vectors, i.e., orthogonal to both the vectors $\vec{A}$ and $\vec{B}$ even thoughthe vectors $\vec{A}$ and $\vec{B}$ may or maynot be mutually orthogonal.

The vector product of two vectors is not commutative, $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ But, $\vec{A}\times\vec{B} = -\left[\vec{B}\times\vec{A}\right]$.Here it is worthwhile to note that $\left|\vec{A}\times\vec{B}\right| = \left|\vec{B}\times\vec{A}\right| = AB\sin\theta$ in the case of the product vectors $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ the magnitudes are equal but directions are opposite to each other.

The vector product of two vectors will have maximum magnitude when $\sin\theta$ =1. θ =90°. when the vectors $\vec{A}$ and $\vec{B}$ are orthogonal to each other.

$$\left(\vec{A}\times\vec{B}\right)_{\max} = AB\hat{n}$$

The vector product of two non–zero vectors will be minimum when $|\sin\theta| = 0$, $\theta = 0^o$ or 180^o

$$\left(\vec{A}\times\vec{B}\right)_{\min} = 0$$

the vector product of two non–zero vectors vanishes, if the vectors are either parallel or antiparallel.

The self–cross product, i.e., product of a vector with itself is the null vector

$$\vec{A} \times \vec{A} = \text{AA} \sin\theta^\circ \hat{n} = \vec{0}$$

In physics the null vector $\vec{0}$ is simply denoted as zero.

The self–vector products of unit vectors are thus zero.

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

In the case of orthogonal unit vectors, $\hat{i}, \hat{j}, \hat{k}$ in accordance with the right hand screw rule:

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i} \text{ and } \hat{k} \times \hat{i} = \hat{j}$$

Also, since the cross product is not commutative,

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}$$

In terms of components, the vector product of two vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\left(A_y B_z - A_z B_y\right) + \hat{j}\left(A_z B_x - A_x B_z\right) + \hat{k}\left(A_x B_y - A_y B_x\right)$$

Note that in the $\hat{j}^{th}$ component the order of multiplicationis different than $\hat{i}^{th}$ and $\hat{k}^{th}$

If two vectors $\vec{A}$ and $\vec{B}$ form adjacent sides in a parallelogram, then the magnitude of $\vec{A} \times \vec{B}$ will give the area of the parallelogram as represented graphically

Triangle with $\vec{A}$ and $\vec{B}$ as sides is $\frac{1}{2}\left|\vec{A} \times \vec{B}\right|$.

A number of quantities used in Physics are defined through vector products. Particularly physical quantities representing rotational effects like torque, angular momentum, are defined through vector products.

Examples

1. Torque $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{F}$ is Force and $\vec{r}$ is position vector of a particle
2. Angular momentum $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{p}$ is the linear momentum
3. Linear Velocity $\vec{v} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is angular velocity

Two vectors are given as $\vec{r} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ and $\vec{F} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ Find the resultant vector $\vec{\tau} = \vec{r} \times \vec{F}$

Solution

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 3 & -2 & 4 \end{vmatrix}$$

$$\vec{\tau} = \left(12 - (-10)\right)\hat{i} + (15 - 8)\,\hat{j} + \left(-4 - 9\right)\right)\hat{k}$$

$$\vec{\tau} = 22\hat{i} + 7\hat{j} - 13\hat{k}$$

Properties of the components of vectors

If two vectors $\vec{A}$ and $\vec{B}$ are equal, then their individual components are also equal.

Let $\vec{A} = \vec{B}$
Then

$$A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$A_x = B_x, A_y = B_y, A_z = B_z$$

EXAMPLE

Compare the components for the following vector equations

- $\vec{F} = m\vec{a}$ Here m is positive number
- $\vec{P} = 0$

Solution

$\vec{F} = m\vec{a}$

$$F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = m_x\hat{i} + m_y\hat{j} + m_z\hat{k}$$

By comparing the components, we get

$$F_m = ma_x, F_y = ma_y, F_z = ma_z$$

This implies that one vector equation is equivalent to three scalar equations.

$$\vec{P} = 0$$

$$P_x\hat{i} + P_y\hat{j} + P_z\hat{k} = 0_x\hat{i} + 0_y\hat{j} + 0_z\hat{k}$$

By comparing the components, we get

$$P_x = 0, P_y = 0, P_z = 0$$

EXAMPLE

Determine the value of the *T* from the given vector equation.

$$5\hat{j} - T\hat{j} = 6\hat{j} = 3T\hat{j}$$

Solution

By comparing the components both sides, we can write

$$5 - 6 = 3T + T$$
$$-1 = 4T$$
$$T = -\frac{1}{4}$$

EXAMPLE

Compare the components of vector equation $\vec{F_1} + \vec{F}_2 + \vec{F}_3 = \vec{F}_4$

Solution

We can resolve all the vectors in x, y and z components with respect to Cartesian coordinate system.

Once we resolve the components we can separately equate the x components on both sides, y components on both sides, and z components on both the sides of the equation, we then get

$$\overrightarrow{F_{1x}} + \vec{F}_{2x} + \vec{F}_{3x} = \vec{F}_{4x}$$
$$\overrightarrow{F_{1y}} + \vec{F}_{2y} + \vec{F}_{3y} = \vec{F}_{4y}$$
$$\overrightarrow{F_{1z}} + \vec{F}_{2z} + \vec{F}_{3z} = \vec{F}_{4z}$$

POSITION VECTOR

It is a vector which denotes the position of a particle at any instant of time, with respect to some reference frame or coordinate system.

The position vector $\vec{r}$ of the particle at a point P is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where x, y and z are components of $\vec{r}$

Determine the position vectors for the following particles which are located at points P, Q, R, S.

Solution

The position vector for the point P is

$$\vec{r}_p = 3\hat{i}$$

The position vector for the point Q is

$$\vec{r}_Q = 5\hat{i} + 4\hat{j}$$

The position vector for the point R is

$$\vec{r}_R = -2\hat{i}$$

The position vector for the point S is

$$\vec{r}_s = 3\hat{i} - 6\hat{j}$$

Example:

A person initially at rest starts to walk 2 m towards north, then 1 m towards east, then 5 m towards south and then 3 m towards west. What is the position vector of the person at the end of the trip?

Solution

As shown in the Figure, the positive x axis is taken as east direction, positive y direction is taken as north.

After the trip, the person reaches the point P whose position vector given by

$$\vec{r} = 2\hat{i} - 3\hat{j}$$

DISTANCE AND DISPLACEMENT

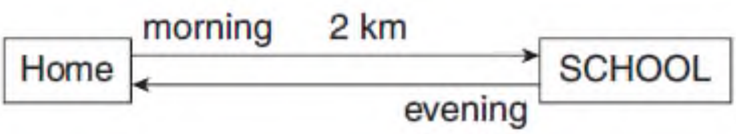
Distance is the actual path length travelled by an object in the given interval of time during the motion. It is a positive scalar quantity.

Displacement is the difference between the final and initial positions of the object in a given interval of time. It can also be defined as the shortest distance between these two positions of the object and its direction is from the initial to final position of the object, during the given interval of time. It is a vector quantity.

Example:

Assume your school is located 2 km away from your home. In the morning you are going to school and in the evening you come back home. In this entire trip what is the distance travelled and the displacement covered?

Solution



The displacement covered is zero. It is because your initial and final positions are the same. But the distance travelled is 4 km.

EXAMPLE

An athlete covers 3 rounds on a circular track of radius 50 m. Calculate the total distance and displacement travelled by him.

Solution

The total distance the athlete covered =3x circumference of track

Distance = $3 \times 2\pi \times 50$ m

$= 300\pi$ m (or)

Distance ≈ 300 x 3.14 ≈ 942 m

The displacement is zero, since the athlete reaches the same point A after three rounds from where he started.

Displacement Vector in Cartesian Coordinate System

In terms of position vector, the displacement vector is given as follows. Let us consider a particle moving from a point P1 having position vector $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ to a point P_2 where its position vector is $\vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

The displacement vector is given by

$$\Delta\vec{r} = \vec{r_2} - \vec{r_1}$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

EXAMPLE

Calculate the displacement vector for a particle moving from a point P to Q as shown below. Calculate the magnitude of displacement.

Solution

The displacement vector $\Delta\vec{r} = \vec{r_2} - \vec{r_1}$ with

$$\vec{r}_1 = \hat{i} + \hat{j} and \vec{r}_2 = 4\hat{i} + 2\hat{j}$$
$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (4\hat{i} + 2\hat{j}) - (\hat{i} + \hat{j})$$
$$= (4-1)\hat{i} + (2-1)\hat{j}$$
$$\Delta\vec{r} = 3\hat{i} + \hat{j}$$

The magnitude of the displacement vector

$$\Delta r = \sqrt{3^2 + 1^2} = \sqrt{10} unit$$

DIFFERENTIAL CALCULUS
The Concept of a function

Any physical quantity is represented by a "function" in mathematics. Take the example of temperature T. We know that the temperature of the surroundings is changing throughout the day. It increases till noon and decreases in the evening. At any time "t" the temperature T has a unique value. Mathematically this variation can be represented by the notation 'T (t)' and it should be called "temperature as a function of time". It implies that if the value of 't' is given, then the function "T (t)" will give the value of the temperature at that time't'. Similarly, the position of a bus in motion along the x direction can be represented by x(t) and this is called 'x' as a function of time'. Here 'x' denotes the x coordinate.

Example

Consider a function f(x) = x^2. Sometimes it is also represented as y = x^2. Here y is called the dependent variable and x is called independent variable. It means as x changes, y also changes. Once a physical quantity is represented by a function, one can study the variation of the function over time or over the independent variable on which the quantity depends. Calculus is the branch of mathematics used to analyse the change of any quantity.

If a function is represented by y = f(x), then dy/dx represents the derivative of y with respect to x. Mathematically this represents the variation of y with respect to change in x, for various continuous values of x.

Mathematically the derivative dy/dx is defined as follows

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$\frac{dy}{dx}$ represents the limit that the quantity $\frac{\Delta y}{\Delta x}$ attains, as Δx tends to zero.

EXAMPLE

Consider the function y =x^2 . Calculatethe derivative $\frac{dy}{dx}$ using the concept of limit.

Solution

Let us take two points given by

x_1 = 2 and x_2 = 3, then y_1= 4 and y_2 = 9

Here $\Delta x = 1$ and $\Delta y = 5$

Then

$$\frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5$$

If we take $x_1 = 2$ and $x_2 = 2.5$, then $y_1 = 4$ and $y_2 = (2.5)2 = 6.25$

Here $\Delta x = 0.5 = \frac{1}{2}$ and $\Delta y = 2.25$

Then

$$\frac{\Delta y}{\Delta x} = \frac{6.25-4}{0.5} = 4.5$$

If we take $x_1 = 2$ and $x_2 = 2.25$, then $y_1 = 4$ and $y_2 = 5.0625$

Here $\Delta x = 0.25 = \frac{1}{4}$, $\Delta y = 1.0625$

$$\frac{\Delta y}{\Delta x} = \frac{5.0625-4}{0.25} = \frac{(5.0625-4)}{\frac{1}{4}}$$

$$= 4(5.0625-4) = 4.25$$

If we take $x_1 = 2$ and $x_2 = 2.1$, then $y_1 = 4$ and $y_2 = 4.41$

Here $\Delta x = 0.1 = \frac{1}{10}$ and

$$\frac{\Delta y}{\Delta x} = \frac{(4.41-4)}{\frac{1}{10}} = 10(4.41-4) = 4.1$$

x_1	x_2	Δx	y_1	y_2	$\frac{\Delta y}{\Delta x}$
2	2.25	0.25	4	5.0625	4.25
2	2.1	0.1	4	4.41	4.1
2	2.01	0.01	4	4.0401	4.01
2	2.001	0.001	4	4.004001	4.001
2	2.0001	0.0001	4	4.00040001	4.0001

From the above table, the following inferences can be made.

As Δx tends to zero $\frac{\Delta y}{\Delta x}$, x approaches the limit given by the number 4.

At a point x = 2, the derivative $\frac{dy}{dx}$=4.

It should also be mentioned here that Δx→0 does not mean that Δx = 0.

This is because, if we substitute Δx = 0, $\frac{\Delta y}{\Delta x}$ becomes indeterminate.

In general, we can obtain the derivative of the function y = x^2, as follows:

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

The table below shows the derivatives of some common functions used in physics

Function	Derivative
y = x	dy/dx=1
y = x^2	dy/dx=2x
y = x^3	dy/dx =$3x^2$
y = x^n	dy/dx=nx^{n-1}
y = sinx	dy/dx= cosx
y = cosx	dy/dx= −sinx
y = constant	dy/dx= 0
y = AB	$\frac{dy}{dx} = A\left(\frac{dB}{dx}\right) + \left(\frac{dA}{dx}\right)B$

In physics, velocity, speed and acceleration are all derivatives with respect to time‘t’. This will be dealt with in the next section.

Example:

Find the derivative with respect to t, of the function x= $A_0 + A_1t + A_2\,t^2$ where A_0, A_1 and A_2 are constants.

Solution

Note that here the independent variable is ‘t’ and the dependent variable is ‘x’

The requived derivative is $dx/dt = 0+ A_1+2A_2t$

The second derivative is $d^2x/d^2t = 2A_2$

INTEGRAL CALCULUS

Integration is basically an area finding process. For certain geometric shapes we can directly find the area. But for irregular shapes the process of integration is used. Consider for example the areas of a rectangle and an irregularly shaped curve.

The area of the rectangle is simply given by A = length ×breadth = (b–a) c

But to find the area of the irregular shaped curve given by f(x), we divide the area into rectangular strips.

The area under the curve is approximately equal to sum of areas of each rectangular strip.

This is given by $A \approx f(a)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x.$

Where f (a) is the value of the functionf (x) at x = a, f (x_1) is the value of f (x) forx = x_1 and so on.

As we increase the number of strips, the area evaluated becomes more accurate. If the area under the curve is divided into N strips, the area under the curve is given by

$$A = \sum_{n=1}^{N} f(x_n)\Delta x_n$$

As the number of strips goes to infinity, N →∞, the sum becomes an integral,

$$A = \int_a^b f(x)dx$$

(Note: As N →∞, Δx→0)

The integration will give the total area under the curve f (x).

Examples

In physics the work done by a force F(x) on an object to move it from point a to point b in one dimension is given by

$$W = \int_a^b F(x)dx$$

(No scalar products is required here, since motion here is in one dimension)

1. The work done is the area under the force displacement graph

2. The impulse given by the force in an interval of time is calculated between the interval from time t = 0 to time t = t_1 as

$$\text{Impulse } I = \int_0^{t_1} F dt$$

The impulse is the area under the force function F (t) - t graph

Consider a particle located initially at point P having position vector $\vec{r_1}$. In a time interval Δt the particle is moved to the point Q having position vector $\vec{r_2}$. The displacement vector is $\Delta\vec{r} = \vec{r_2} - \vec{r_1}$.

The average velocity is defined as ratio of the displacement vector to the corresponding time interval

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$$

It is a vector quantity. The direction of average velocity is in the direction of the displacement vector $(\Delta\vec{r})$.

Average speed

The average speed is defined as the ratio of total path length travelled by the particle in a time interval.

Average speed = total path length / total time

EXAMPLE

Consider an object travelling in a semicircular path from point O to point P in 5 second, as is shown in the Figure. Calculate the average velocity and average speed.

Solution

Average velocity $\vec{v}_{avg} = \dfrac{\vec{r}_p - \vec{r}_o}{\Delta t}$

Here $\Delta t = 5\, s$

$\vec{r}_o = 0, \vec{r}_p = 10\hat{i}$

$\vec{v}_{avg} = \dfrac{10\hat{i}}{5\,\text{sec}} = 2\hat{i}\, cm\, s^{-1}$

The average velocity is in the positive x direction.
The average speed = total path length / time taken (the path is semi-circular)

$$= \frac{5\pi cm}{5\ s} = \pi cm\, s^{-1} \approx 3.14\ cm\, s^{-1}$$

Note that the average speed is greater than the magnitude of the average velocity.

Instantaneous velocity or velocity

The instantaneous velocity at an instant t or simply 'velocity' at an instant t is defined as limiting value of the average velocity as $\Delta t \to 0$, evaluated at time t.

In other words, velocity is equal to rate of change of position vector with respect to time. Velocity is a vector quantity.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

In component form, this velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\text{Here} \frac{dx}{dt} = v_x = x - \text{component of velocity}$$

$$\frac{dy}{dt} = v_y = y - \text{component of velocity}$$

$$\frac{dz}{dt} = v_z = z - \text{component of velocity}$$

The magnitude of velocity v is called speed and is given by

$$v=\sqrt{v_x^2+v_y^2+v_z^2}$$

Speed is always a positive scalar. The unit of speed is also meter per second.

EXAMPLE

The position vector of a particle is given $\vec{r}=2t\hat{i}+3t^2\hat{j}-5\hat{k}$.

a. Calculate the velocity and speed of the particle at any instant t
b. Calculate the velocity and speed of the particle at time t = 2 s

Solution

$$\text{The velocity } \vec{v}=\frac{d\vec{r}}{dt}=2\hat{i}+6t\hat{j}$$

$$\text{The speed v(t)} = \sqrt{2^2+(6t)^2}\, ms^{-1}$$

The velocity of the particle at t = 2 s

$$\vec{v}(2\sec)=2\hat{i}+12\hat{j}$$

The speed of the particle at t = 2 s

$$v(2s)=\sqrt{2^2+12^2}=\sqrt{4+144}$$
$$=\sqrt{148}\approx 12.16\ m\ s^{-1}$$

Note that the particle has velocity components along x and y direction. Along the z direction the position has constant value (−5) which is independent of time. Hence there is no z-component for the velocity.

EXAMPLE

The velocity of three particles A, B, C are given below. Which particle travels at the greatest speed?

$$\overrightarrow{v_A}=3\hat{i}-5\hat{j}+2\hat{k}$$
$$\overrightarrow{v_B}=\hat{i}+2\hat{j}+3\hat{k}$$
$$\overrightarrow{v_C}=5\hat{i}+3\hat{j}+4\hat{k}$$

Solution

We know that speed is the magnitude of the velocity vector. Hence,

$$\text{Speed of A} = \left|\vec{v_A}\right| = \sqrt{(3)^2 + (-5)^2 + (2)^2}$$

$$= \sqrt{9 + 25 + 4} = \sqrt{38} m\,s^{-1}$$

$$\text{Speed of B} = \left|\vec{v_B}\right| = \sqrt{(1)^2 + (2)^2 + (3)^2}$$

$$= \sqrt{1 + 4 + 9} = \sqrt{14}\, m\,s^{-1}$$

$$\text{Speed of C} = \left|\vec{v_C}\right| = \sqrt{(5)^2 + (3)^2 + (4)^2}$$

$$= \sqrt{25 + 9 + 16} = \sqrt{50}\, m\,s^{-1}$$

The particle C has the greatest speed.

$$\sqrt{50} > \sqrt{38} > \sqrt{14}$$

EXAMPLE

Two cars are travelling with respective velocities $\vec{v}_1 = 10\, m\,s^{-1}$ along east and $\vec{v}_2 = 10\, m\,s^{-1}$ along west. What are the speeds of the cars?

Solution

Both cars have the same magnitude of velocity. This implies that both cars travel at the same speed even though they have velocities in different directions. Speed will not give the direction of motion.

Momentum The linear momentum or simply momentum of a particle is defined as product of mass with velocity. It is denoted as `$\vec{p}$`. Momentum is also a vector quantity.

Solution

We use p = mv
For the mass of 10 g, m = 0.01 kg

$$p = 0.01 \times 10 = 0.1\, kg\ m\,s^{-1}$$

For the mass of 1 kg

$$p = 1 \times 10 = 10\, kg\ m\,s^{-1}$$

Thus even though both the masses have the same speed, the momentum of the heavier mass is 100 times greater than that of the lighter mass.

MOTION ALONG ONE DIMENSION

Average velocity

If a particle moves in one dimension, say for example along the x direction, then

$$\text{The average velocity} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

The average velocity is also a vector quantity. But in one dimension we have only two directions (positive and negative x direction), hence we use positive and negative signs to denote the direction.

The instantaneous velocity or velocity is defined as $v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$

Graphically the slope of the position-time graph will give the velocity of the particle. At the same time, if velocity time graph is given, the distance and displacement are determined by calculating the area under the curve. This is explained below.

We know that velocity is given by $\frac{dx}{dt}$ =v

Therefore, we can write dx =vdt

By integrating both sides, we get $\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\, dt.$

As already seen, integration is equivalent to area under the given curve. So the term $\int_{t_1}^{t_2} v\, dt$ represents the area under the curve vas a function of time.

Since the left hand side of the integration represents the displacement travelled by the particle from time t_1 to t_2, the area under the velocity time graph will give the displacement of the particle. If the area is negative, it means that displacement is negative, so the particle has travelled in the negative direction.

EXAMPLE

A particle moves along the x-axis in such a way that its coordinates x varies with time 't' according to the equation x = 2 - 5t + 6t^2. What is the initial velocity of the particle?

Solution

x t t

$$\text{Velocity, } v = \frac{dx}{dt} = \frac{d}{dt}(2 - 5t + 6t^2)$$

$$\text{or } v = -5 + 12t$$

$$\text{For initial velocity, } t = 0$$

$$\text{Initial velocity } = -5\,ms^{-1}$$

The negative sign implies that at t = 0 the velocity of the particle is along negative x direction.

Average speed

The average speed is defined as the ratio of the total path length traveled by the particle in a time interval, to the time interval

Average speed = total path length / total time period

Relative Velocity in One and Two Dimensional Motion

When two objects A and B are moving with different velocities, then the velocity of one object A with respect to another object B is called relative velocity of object A with respect to B.

Case 1

Consider two objects A and B moving with uniform velocities V_A and V_B, as shown, along straight tracks in the same direction $\overrightarrow{V_A}, \overrightarrow{V_B}$ with respect to ground.

The relative velocity of object A with respect to object B is $\overrightarrow{V_{AB}} = \overrightarrow{V_A} - \overrightarrow{V_B}$

The relative velocity of object B with respect to object A is $\overrightarrow{V_{BA}} = \overrightarrow{V_B} - \overrightarrow{V_A}$

Thus, if two objects are moving in the same direction, the magnitude of relative velocity of one object with respect to another is equal to the difference in magnitude of two velocities.

EXAMPLE

Suppose two cars A and B are moving with uniform velocities with respect to ground along parallel tracks and in the same direction. Let the velocities of A and B be 35 km h^{-1} due east and 40 km h^{-1} due east respectively. What is the relative velocity of car B with respect to A?

Solution

The relative velocity of B with respect to A, $\overrightarrow{V_{BA}} = \overrightarrow{V_B} - \overrightarrow{V_A}$ =5 km h^{-1}due east

Similarly, the relative velocity of A with respect to B i.e., $\overrightarrow{V_{AB}} = \overrightarrow{V_A} - \overrightarrow{V_B}$ =5 km h^{-1}due west.

To a passenger in the car A, the car B will appear to be moving east with a velocity 5 km h^{-1}. To a passenger in train B, the train A will appear to move westwards with a velocity of 5 km h^{-1}

Case 2

Consider two objects A and B moving with uniform velocities V_A and V_B along the same straight tracks but opposite in direction

$$\underset{\overrightarrow{V_A}}{\longrightarrow} \quad \underset{\overrightarrow{V_B}}{\longleftarrow}$$

The relative velocity of object A with respect to object B is

$$\vec{V}_{AB} = \vec{V}_A - (-\overrightarrow{V_B}) = \vec{V}_A + \vec{V}_B$$

The relative velocity of object B with respect to object A is

$$\vec{V}_{BA} = -\overrightarrow{V_B} - \vec{V}_A = -(\vec{V}_A + \vec{V}_B)$$

Thus, if two objects are moving in opposite directions, the magnitude of relative velocity of one object with respect to other is equal to the sum of magnitude of their velocities.

Case 3

Consider the velocities $\vec{V}_A$ and $\vec{V}_B$ at an angle θ between their directions.

The relative velocity of A with respect to B, $\overrightarrow{V_{AB}} = \overrightarrow{V_A} - \overrightarrow{V_B}$

Then, the magnitude and direction of $\overrightarrow{V_{AB}}$ is given by $V_{AB} = \sqrt{V_A^2 + V_B^2 - 2V_AV_B\cos\theta}$

and $\tan\beta = \dfrac{V_B\sin\theta}{V_A - V_B\cos\theta}$ (Here β is angle between $\vec{V}_{AB}$ and $\overrightarrow{V_B}$)

1. When θ = 0°, the bodies move along parallel straight lines in the same direction, We have $V_{AB} = (V_A - V_B)$ in the direction of $\vec{V}_A$. Obviously $V_{BA} = (V_B + V_A)$ in the direction of $\overrightarrow{V_B}$.

2. When θ = 180°, the bodies move along parallel straight lines in opposite directions,

We have $V_{AB} = (V_A + V_B)$ in the direction of $\vec{V}_A$.Similarly, $V_{BA} = (V_B - V_A)$ in the direction of $\overrightarrow{V_B}$.

3. If the two bodies are moving at right angles to each other, then θ = 90°. The magnitude of the relative velocity of A with respect to $B = v_{AB} = \sqrt{v_A^2 + v_B^2}$.

4. Consider a person moving horizontally with velocity $\vec{V}_M$. Let rain fall vertically with velocity $\vec{V}_R$. An umbrella is held to avoid the rain. Then the relative velocity of the rain with respect to the person is,

$$\vec{V}_{RM} = \vec{V}_R . - \vec{V}_M .$$

which has magnitude

$$V_{RM} = \sqrt{V_R^2 + V_M^2}$$

and direction $\theta = \tan^{-1}\left(\frac{V_M}{V_R}\right)$

In order to save himself from the rain, he should hold an umbrella at an angle θ with the vertical.

EXAMPLE

Suppose two trains A and B are moving with uniform velocities along parallel tracks but in opposite directions. Let the velocity of train A be 40 km h−1 due east and that of train B be 40 km h^{-1} due west. Calculate the relative velocities of the trains

Solution

Relative velocity of A with respect to B, v_{AB} = 80 km h^{-1} due east

Thus to a passenger in train B, the train A will appear to move east with a velocity of 80 km h^{-1}.

The relative velocity of B with respect to A, V_{BA} = 80 km h_{-1} due west

To a passenger in train A, the train B will appear to move westwards with a velocity of 80 km h^{-1}.

EXAMPLE

Consider two trains A and B moving along parallel tracks with the same velocity in the same direction. Let the velocity of each train be 50 km h^{-1} due east. Calculate the relative velocities of the trains.

Solution

Relative velocity of B with respect to A, $v_{BA} = v_B - v_A$

= 50 km h^{-1} + (−50) km h^{-1}
= 0 km h^{-1}

Similarly, relative velocity of A with respect to B i.e., v_{AB} is also zero.
Thus each train will appear to be at rest with respect to the other.

EXAMPLE

How long will a boy sitting near the window of a train travelling at 36 km h^{-1} see a train passing by in the opposite direction with a speed of 18 km h^{-1}. The length of the slowmoving train is 90 m.

Solution

The relative velocity of the slow-moving train with respect to the boy is = (36 + 18) km h^{-1} = 54 km h^{-1} = $= 54\times\frac{5}{18} m\,s^{-1} = 15\,m\,s^{-1}$

Since the boy will watch the full length of the other train, to find the time taken to watch the full train:

$$15 = \frac{90}{t}\; or\; t = \frac{90}{15} = 6\,s$$

EXAMPLE

A swimmer's speed in the direction of flow of a river is 12 km h^{-1}. Against the direction of flow of the river the swimmer's speed is 6 km h^{-1}. Calculate the swimmer's speed in still water and the velocity of the river flow.

Solution

Let v_s and vr, represent the velocities of the swimmer and river respectively with respect to ground.

$$v_s + v_r = 12 \qquad (1)$$
$$\text{and } v_s - v_r = 6 \qquad (2)$$

Adding the both equations (1) and (2) $2v_s = 12 + 6 = 18$ km h^{-1} or vs = 9 km h^{-1}.

From Equation (1),

$$9 + v_r = 12 \text{ or}$$
$$v_r = 3 \text{ km h}^{-1}$$

When the river flow and swimmer move in the same direction, the net velocity of swimmer is 12 km h^{-1}.

Accelerated Motion

During non-uniform motion of an object, the velocity of the object changes from instant to instant i.e., the velocity of the object is no more constant but changes with time. Such a motion is said to be an accelerated motion.

1. In accelerated motion, if the change in velocity of an object per unit time is same (constant) then the object is said to be moving with uniformly accelerated motion.
2. On the other hand, if the change in velocity per unit time is different at different times, then the object is said to be moving with non-uniform accelerated motion.

Average acceleration

If an object changes its velocity from $\vec{v}_1$ $\vec{v}_2$ to in a time interval $\Delta t = t_1 - t_2$, then the average acceleration is defined as the ratio of change in velocity over the time interval $\Delta t = t_1 - t_2$.

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta \vec{v}}{\Delta t}$$

Average acceleration is a vector quantity in the same direction as the vector Δv .

Instantaneous acceleration

Usually, the average acceleration will give the change in velocity only over the entire time interval. It will not give value of the acceleration at any instant time t.

Instantaneous acceleration or acceleration of a particle at time 't' is given by the ratio of change in velocity over Δt, as Δt approaches zero.

$$\text{Acceleration}\, \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

In other words, the acceleration of the particle at an instant t is equal to rate of change of velocity.

Acceleration is a vector quantity. Its SI unit is ms^{-2} and its dimensional formula is $M^0L^1T^{-2}$

Acceleration is positive if its velocity is increasing, and is negative if the velocity is decreasing. The negative acceleration is called retardation or deceleration.

In terms of components, we can write

$$\vec{a}_x = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = \frac{d\vec{v}}{dt}$$

Thus $a_x = \frac{dv_x}{dt}, a_y = \frac{dv_y}{dt}, a_z = \frac{dv_z}{dt}$ are the components of instantaneous acceleration.

Since each component of velocity is the derivative of the corresponding coordinate, we can express the components a_x, a_y, and a_z, as

$$a_x = \frac{d^2x}{dt^2}, a_y = \frac{d^2y}{dt^2}, a_z = \frac{d^2z}{dt^2}$$

Then the acceleration vector $\vec{a}$ itself is

$$\vec{a}_x = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} = \frac{d^2\vec{r}}{dt^2}$$

Thus acceleration is the second derivative of position vector with respect to time.

Graphically the acceleration is the slope in the velocity-time graph. At the same time if the acceleration-time graph is given, then the velocity can be found from the area under the acceleration-time graph.

From $\frac{dv}{dt} = \text{a, we have dv} = \text{adt; hence v} = \int_{t_1}^{t_2} a\,dt$

For an initial time t_1 and final time t_2

EXAMPLE

A velocity–time graph is given for a particle moving in x direction, as below

1. Describe the motion qualitatively in the interval 0 to 55 s .
2. Find the distance and displacement travelled from 0 s to 40 s .
3. Find the acceleration at t = 5 s and at t 20 s

Solution

From O to A: (0 s to 10 s)

At t = 0 s the particle has zero velocity. At t > 0, particle has positive velocity and moves in the positive x direction. From 0 s to 10 s the slope $(\frac{dv}{dt})$ is positive, implying the particle is accelerating. Thus the velocity increases during this time interval.

From A to B: (10 s to 15 s)

From 10 s to 15 s the velocity stays constant at 60 m s-1. The acceleration is 0 during this period. But the particle continues to travel in the positive x-direction.

From B to C : (15 s to 30 s)

From the 15 s to 30 s the slope is negative, implying the velocity is decreasing. But the particle is moving in the positive x direction. At t = 30 s the velocity becomes zero, and the particle comes to rest momentarily at t = 30 s .

From C to D: (30 s to 40 s)

From 30 s to 40 s the velocity is negative. It implies that the particle starts to move in the negative x direction. The magnitude of velocity increases to a maximum 40 m s^{-1}.

From D to E: (40 s to 55 s)

From 40 s to 55 s the velocity is still negative, but starts increasing from –40 m s^{-1}At t = 55 s the velocity of the particle is zero and particle comes to rest.

The total area under the curve from 0 s to 40 s will give the displacement. Here the area from O to C represents motion along positive x–direction and the area under the graph from C to D represents the particle's motion along negative x–direction.

The displacement travelled by the particle from 0 s to 10 s $= \frac{1}{2} \times 10 \times 60 = 300$m

The displacement travelled from 10 s to 15 s = 60 × 5 = 300 m

The displacement travelled from 15 s to 30 s = $\frac{1}{2} \times 15 \times 60 = 450$m

The displacement travelled from 30 s to 40 s = $\frac{1}{2}$ ×10×(−40) = −200m.

Here the negative sign implies that the particle travels 200 m in the negative x direction.

The total displacement from 0 s to 40 s is given by

300 m + 300 m + 450 m − 200 m = +850 m.

Thus the particle's net displacement is along the positive x-direction.

The total distance travelled by the particle from 0 s to 40 s = 300 + 300 + 450 + 200 = 1250 m.

The acceleration is given by the slope in the velocity-time graph. In the first 10 seconds the velocity has constant slope (constant acceleration). It implies that the acceleration a is from v1 = 0 to v2 = 60 m s^{-1}.

Hence $a = \frac{v_2 - v_1}{t_2 - t_1}$ gives

$$a = \frac{60-0}{10-0} = 6\,m\,s^{-2}$$

Next, the particle has constant negative slope from 15 s to 30 s . In this case v_2=0 and v_1=60 m s^{-1}. Thus the acceleration at t = 20 s is given by $a = \frac{0-60}{30-15} = -4m\,s^{-2}$ Here the negative sign implies that the particle has negative acceleration.

EXAMPLE2.32

If the position vector of the particle is given by $\vec{r} = 3t^2\hat{i} + 5t\hat{j} + 4\hat{k}$ Find the

a. The velocity of the particle at t = 3 s
b. Speed of the particle at t = 3 s
c. acceleration of the particle at time t = 3 s

Solution

a. The velocity $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ We obtain, $\vec{v}(t) = 6t\hat{i} + 5\hat{j}$ The velocity has only two components v_x=6t, depending on time t and v_y=5 which is independent of time.

The velocity at $t = 3\,s \ \ is\, \vec{v}(3) = 18\hat{i} + 5\hat{j}$

b. The speed at t = 3 s is $v = \sqrt{18^2 + 5^2} = \sqrt{349} \approx 18.68\,m\,s^{-1}$

c. The acceleration $\vec{a}$ is, $\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6\hat{i}$ The acceleration has only the x-component. Note that acceleration here is independent of t, which means $\vec{a}$ is constant. Even at t = 3 s it has same value $\vec{a} = 6\hat{i}$. The velocity is non-uniform, but the acceleration is uniform (constant) in this case.

EXAMPLE

An object is thrown vertically downward. What is the acceleration experienced by the object?

Solution

We know that when the object falls towards the Earth, it experiences acceleration due to gravity g = 9.8 m s^{-2} downward. We can choose the coordinate system as shown in the figure.

The acceleration is along the negative y direction.

$$\vec{a} = g(-j) = -g\hat{j}$$

Equations of Uniformly Accelerated Motion by Calculus Method

Consider an object moving in a straight line with uniform or constant acceleration 'a'.

Let u be the velocity of the object at time t = 0, and v be velocity of the body at a later time t.

Velocity - time relation

a. The acceleration of the body at any instant is given by the first derivative of the velocity with respect to time,

$$a = \frac{dv}{dt}\, or\, dv = a\,dt$$

Integrating both sides with the condition that as time changes from 0 to t, the velocity changes from u to v. For the constant acceleration,

$$\int_u^v dv = \int_0^t a\ dt = a\int_0^t dt \Rightarrow [v]_u^v = a[t]_0^t$$
$$v - u = at \quad (or) \quad v = u + at \quad \rightarrow (2.7)$$

Displacement – time relation

b. The velocity of the body is given by the first derivative of the displacement with respect to time.

$$v = \frac{ds}{dt} \; or \; ds = vdt$$

and since $v = u + at$,

We get $ds = (u + at)dt$

Assume that initially at time t = 0, the particle started from the origin. At later time t, the particle displacement is s. Further assuming that acceleration is time independent, we have

$$\int_0^s ds = \int_0^t u\,dt + \int_o^t at\,dt \;\; (or) \;\; s = ut + \frac{1}{2}at^2 \;\; (2.8)$$

Velocity – displacement relation

c. The acceleration is given by the first derivative of velocity with respect to time.

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

[since ds/dt = v] where s is displacement traversed.

This is rewritten as $a = \frac{1}{2}\frac{dv^2}{ds}$

or $ds = \frac{1}{2a}d(v^2)$

Integrating the above equation, using the fact when the velocity changes from u^2 to v^2, displacement changes from 0 to s, we get

$$\int_0^s ds = \int_u^v \frac{1}{2a}d\left(v^2\right)$$

$$\therefore s = \frac{1}{2a}\left(v^2 - u^2\right)$$

$$\therefore v^2 = u^2 + 2as$$

We can also derive the displacement s in terms of initial velocity u and final velocity v. From the equation (2.7) we can write,

$$at = v - u$$

Substitute this in equation (2.8), we get

$$s = ut + \frac{1}{2}(v - u)t$$

$$s = \frac{(u + v)t}{2}$$

The equations (2.7), (2.8), (2.9) and (2.10) are called kinematic equations of motion, and have a wide variety of practical applications.

Kinematic equations

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{(u + v)t}{2}$$

It is to be noted that all these kinematic equations are valid only if the motion is in a straight line with constant acceleration. For circular motion and oscillatory motion these equations are not applicable.

Equations of motion under gravity

A practical example of a straight line motion with constant acceleration is the motion of an object near the surface of the Earth. We know that near the surface of the Earth, the acceleration due to gravity 'g' is constant. All straight line motions under this acceleration can be well understood using the kinematic equations given earlier

Case (1): A body falling from a height h

Consider an object of mass m falling from a height h. Assume there is no air resistance. For convenience, let us choose the downward direction as positive y-axis as shown in the Figure 2.37. The object experiences acceleration 'g' due to gravity which is constant near the surface of the Earth. We can use kinematic equations to explain its motion. We have

The acceleration $\vec{a} = g\hat{j}$

By comparing the components, we get

$$a_x = 0, a_z = 0, a_y = g$$

Let us take for simplicity, $a_y = a = g$

If the particle is thrown with initial velocity 'u' downward which is in negative y axis, then velocity and position at of the particle any time t is given by

$$v = u + gt$$

$$y = ut + \frac{1}{2}gt^2$$

The square of the speed of the particle when it is at a distance y from the hill-top, is

$$v^2 = u^2 + 2gy$$

Suppose the particle starts from rest.
Then u = 0

Then the velocity v, the position of the particle and v2 at any time t are given by (for a point y from the hill-top)

$$v = gt$$

$$y = \frac{1}{2}gt^2$$

$$v^2 = 2gy$$

The time (t = T) taken by the particle to reach the ground (for which y = h), is given by using equation

$$h = \frac{1}{2} g T^2$$

$$T = \sqrt{\frac{2h}{g}}$$

The equation (2.18) implies that greater the height(h), particle takes more time(T) to reach the ground. For lesser height(h), it takes lesser time to reach the ground.

The speed of the particle when it reaches the ground (y = h) can be found using equation (2.16), we get

$$v_{ground} = \sqrt{2gh}$$

The above equation implies that the body falling from greater height(h) will have higher velocity when it reaches the ground.

The motion of a body falling towards the Earth from a small altitude (h << R), purely under the force of gravity is called free fall. (Here R is radius of the Earth)

EXAMPLE

An iron ball and a feather are both falling from a height of 10 m.

a. What are the time taken by the iron ball and feather to reach the ground?
b. What are the velocities of iron ball and feather when they reach the ground? (Ignore air resistance and take g = 10 m s^{-2})

Solution

Since kinematic equations are independent of mass of the object, according to equation (2.8) the time taken by both iron ball and feather to reach the ground are the same. This is given by

$$T = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 10}{10}} = \sqrt{2}\, s \approx 1.414\, s$$

Thus, both feather and iron ball reach ground at the same time.

By following equation (2.19) both iron ball and feather reach the Earth with the same speed. It is given by

$$v = \sqrt{2gh} = \sqrt{2 \times 10 \times 10}$$
$$= \sqrt{200}\, m\, s^{-1} \approx 14.14\, m\, s^{-1}$$

EXAMPLE

Is it possible to measure the depth of a well using kinematic equations?

Consider a well without water, of some depth d. Take a small object (for example lemon) and a stopwatch. When you drop the lemon, start the stop watch. As soon as the lemon touches the bottom of the well, stop the watch. Note the time taken by the lemon to reach the bottom and denote the time as t.

Since the initial velocity of lemon u=0 and the acceleration due to gravity g is constant over the well, we can usethe equations of motion for constant acceleration.

$$s = ut + \frac{1}{2}at^2$$

Since u=0, s=d,a=g (Since we choose the y axis downwards), Then

$$d = \frac{1}{2}gt^2$$

Substituting g=9.8 m s^{-2} we get the depth of the well.

To estimate the error in our calculation we can use another method to measure the depth of the well. Take a long rope and hang the rope inside the well till it touches the bottom. Measure the length of the rope which is the correct depth of the well ($d_{correct}$). Then

$$error = d_{correct} - d$$

$$relative\ error = \frac{d_{correct} - d}{d_{correct}}$$

$$percentage\ of\ relative\ error = \frac{d_{correct} - d}{d_{correct}} \times 100$$

What would be the reason for an error, if any?

Repeat the experiment for different masses and compare the result with $d_{correct}$ every time.

Case (ii): A body thrown vertically upwards

Consider an object of mass m thrown vertically upwards with an initial velocity u. Let us neglect the air friction. In this case we choose the vertical direction as positive y axis as shown in the Figure 2.38, then the acceleration a = −g (neglect air friction) and g points towards the negative y axis. The kinematic equations for this motion are,

The velocity and position of the object at any time t are,

$$v = u - gt$$

$$s = ut - \frac{1}{2}gt^2$$

The velocity of the object at any position y (from the point where the object is thrown) is

$$v^2 = u^2 - 2gy$$

EXAMPLE

A train was moving at the rate of 54 km h^{-1} when brakes were applied. It came to rest within a distance of 225 m. Calculate the retardation produced in the train.

Solution

The final velocity of the particle v = 0 The initial velocity of the particle

$$u = 54 \times \frac{5}{18} m s^{-1} = 15\ m s^{-1}$$

$$S = 225\ m$$

Retardation is always against the velocity of the particle.

$$v^2 = u^2 - 2aS$$

$$0 = (10)^2 - 2a\,(225)$$

$$-450\ a = 100$$

$$a = -\frac{1}{2} ms^{-2} = 0.5\ \ ms^{-2}$$

Hence, retardation $= 0.5\ m\ s^{-2}$

PROJECTILE MOTION

Introduction

When an object is thrown in the air with some initial velocity (NOT just upwards), and then allowed to move under the action of gravity alone, the object is known as a projectile. The path followed by the particle is called its trajectory.

Examples of projectile are

1. An object dropped from window of a moving train.
2. A bullet fired from a rifle.
3. A ball thrown in any direction.
4. A javelin or shot put thrown by an athlete.
5. A jet of water issuing from a hole near the bottom of a water tank.

It is found that a projectile moves under the combined effect of two velocities.

i. A uniform velocity in the horizontal direction, which will not change provided there is no air resistance.

ii. A uniformly changing velocity (i.e., increasing or decreasing) in the vertical direction.

There are two types of projectile motion:

i. Projectile given an initial velocity in the horizontal direction (horizontal projection)

ii. Projectile given an initial velocity at an angle to the horizontal (angular projection)

To study the motion of a projectile, let us assume that,

i. Air resistance is neglected.

ii. The effect due to rotation of Earth and curvature of Earth is negligible.

iii. The acceleration due to gravity is constant in magnitude and direction at all points of the motion of the projectile

Projectile in horizontal projection

Consider a projectile, say a ball, thrown horizontally with an initial velocity $\vec{u}$ from the top of a tower of height h

As the ball moves, it covers a horizontal distance due to its uniform horizontal velocity u, and a vertical downward distance because of constant acceleration due to gravity g. Thus, under the combined effect the ball moves along the path OPA. The motion is in a 2-dimensional plane. Let the ball take time t to reach the ground at point A, Then the horizontal distance travelled by the ball is x (t) = x, and the vertical distance travelled is y (t) = y

We can apply the kinematic equations along the x direction and y direction separately. Since this is two-dimensional motion, the velocity will have both horizontal component u_x and vertical component u_y.

Motion along horizontal direction

The particle has zero acceleration along x direction. So, the initial velocity u_x remains constant throughout the motion.

The distance traveled by the projectile at a time t is given by the equation $x = u_x t + \frac{1}{2} a t^2$.

Since a = 0 along x direction, we have

$$x = u_x t$$

Motion along downward direction

Here $u_y = 0$ (initial velocity has no downward component), a = g (we choose the +ve y-axis in downward direction), and distance y at time t

$\therefore$ From equation, $y = u_y t + \frac{1}{2} a t^2$, we get

$$y = \frac{1}{2} g t^2 \qquad (2.24)$$

Substituting the value of t from equation

$$y = \frac{1}{2} g \frac{x^2}{u_x^2} = \left(\frac{g}{2u_x^2}\right) x^2$$

$$y = Kx^2$$

where $K = \frac{g}{2u_x^2}$ is constant

The equation of a parabola. Thus, the path followed by the projectile is a parabola.

Time of Flight:

The time taken for the projectile to complete its trajectory or time taken by the projectile to hit the ground is called time of flight.

Consider the example of a tower and projectile. Let h be the height of a tower. Let T be the time taken by the projectile to hit the ground, after being thrown horizontally from the tower.

We know that $s_y = u_y t + \frac{1}{2}at^2$ for vertical motion. Here $s_y = h$, $t = T$, $u_y = 0$ (i.e., no initial vertical velocity). Then

$$h = \frac{1}{2}gT^2 \quad \text{or} \quad T = \sqrt{\frac{2h}{g}}$$

Thus, the time of flight for projectile motion depends on the height of the tower, but is independent of the horizontal velocity of projection. If one ball falls vertically and another ball is projected horizontally with some velocity, both the balls will reach the bottom at the same time.

Horizontal range:

The horizontal distance covered by the projectile from the foot of the tower to the point where the projectile hits the ground is called horizontal range. For horizontal motion, we have

$$s_x = u_x t + \frac{1}{2}at^2.$$

Here, s_x = R (range), u_x = u, a = 0 (no horizontal acceleration) T is time of flight. Then horizontal range = uT.

Since the time of flight $T = \sqrt{\frac{2h}{g}}$ we substitute this and we get the horizontal range of the particle as $R = u\sqrt{\frac{2h}{g}}$.

The above equation implies that the range R is directly proportional to the initial velocity u and inversely proportional to acceleration due to gravity g.

Resultant Velocity (Velocity of projectile at any time):

At any instant t, the projectile has velocity components along both x-axis and y-axis. The resultant of these two components gives the velocity of the projectile at that instant t,

The velocity component at any t along horizontal (x-axis) is $v_x = u_x + a_x t$
Since, $u_x = u$, $a_x = 0$, we get

$$v_x = u$$

The component of velocity along vertical direction (y-axis) is $v_y = u_y + a_y t$
Since, $u_y = 0$, $a_y = g$, we get

$$v_y = gt$$

Hence the velocity of the particle at any instant is

$$\vec{v} = u\hat{i} + gt\hat{j}$$

The speed of the particle at any instant t is given by

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2}$$
$$v = \sqrt{u^2 + g^2t^2}$$

Speed of the projectile when it hits the ground:

When the projectile hits the ground after initially thrown horizontally from the top of tower of height h, the time of flight is

$$t = \sqrt{\frac{2h}{g}}$$

The horizontal component velocity of the projectile remains the same i.e $v_x = u$

The vertical component velocity of the projectile at time T is

$$v_y = gT = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}$$

The speed of the particle when it reaches the ground is

$$v = \sqrt{u^2 + 2gh}$$

Projectile under an angular projection

This projectile motion takes place when the initial velocity is not horizontal, but at some angle with the vertical,

(Oblique projectile)
Examples:

- Water ejected out of a hose pipe held obliquely.
- Cannon fired in a battle ground.

Consider an object thrown with initial velocity $\vec{u}$ at an angle θ with the horizontal.

$$\vec{u} = u_x\hat{i} + u_y\hat{j}$$

where $u_x = u\cos\theta$ is the horizontal component and $u_y = u\sin\theta$ the vertical component of velocity.

Since the acceleration due to gravity is in the direction opposite to the direction of vertical component u_y, this component will gradually reduce to zero at the maximum height of the projectile. At this maximum height, the same gravitational force will push the projectile to move downward and fall to the ground. There is no acceleration along the x direction throughout the motion. So, the horizontal component of the velocity ($u_x = u\cos\theta$) remains the same till the object reaches the ground.

Hence after the time t, the velocity along horizontal motion $v_x = u_x + a_xt = u_x = u\cos\theta$

The horizontal distance travelled by projectile in time t is $S_x = u_xt + \frac{1}{2}a_xt^2$

Here, $s_x = x$, $u_x = u\cos\theta$, $a_x = 0$

$$\text{Thus, } x = u\cos\theta . t \text{ or } t = \frac{x}{u\cos\theta}$$

Next, for the vertical motion $v_y = u_y + a_yt$

Here $u_y = u\sin\theta$, $a_y = -g$ (acceleration due to gravity acts opposite to the motion). Thus

$$\text{Thus, } v_y = u\sin\theta - gt$$

The vertical distance travelled by the projectile in the same time t is $S_y = u_yt + \frac{1}{2}a_yt^2$

Here, $s_y = y$, $u_y = u\sin\theta$, $a_x = -g$. Then

$$y = u \sin\theta\, t - \frac{1}{2} g t^2$$

Substitute the value of t from equation

$$y = u \sin\theta \frac{x}{u\cos\theta} - \frac{1}{2} g \frac{x^2}{u^2 \cos^2\theta}$$

$$y = x \tan\theta - \frac{1}{2} g \frac{x^2}{u^2 \cos^2\theta} \qquad (2.31)$$

Thus the path followed by the projectile is an inverted parabola.

Maximum height (h_{max})

The maximum vertical distance travelled by the projectile during its journey is called maximum height. This is determined as follows:

For the vertical part of the motion,

$$v_y^2 = u_y^2 + 2a_y s$$

Here, $u_y = u \sin\theta$, $a = -g$, $s = h_{max}$, and at the maximum height $v_y = 0$
Hence,

$$(0)^2 = u^2 \sin^2\theta = 2gh_{max}$$

$$\text{Or } h_{max} = \frac{u^2 \sin^2\theta}{2g}$$

Time of flight (T_f)

The total time taken by the projectile from the point of projection till it hits the horizontal plane is called time of flight.

This time of flight is the time taken by the projectile to go from point O to B via point A

We know that $S_y = u_y t + \frac{1}{2} a_y t^2$

Here, $s_y = y = 0$ (net displacement in y-direction is zero), $u_y = u \sin\theta$, $a_y = -g$, $t = T_f$
Then

$$0 = u\sin\theta\, T_f - \frac{1}{2} g T_f^2$$
$$T_f = 2u\frac{\sin\theta}{g}$$

Horizontal range (R)

The maximum horizontal distance between the point of projection and the point on the horizontal plane where the projectile hits the ground is called horizontal range (R). This is found easily since the horizontal component of initial velocity remains the same. We can write

Range R = Horizontal component of velocity x time of flight $= u\cos\theta \times T_f$

$$R = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g}$$
$$\therefore R = \frac{u^2\sin 2\theta}{g} \qquad (2.33)$$

The horizontal range directly depends on the initial speed (u) and the sine of angle of projection (θ). It inversely depends on acceleration due to gravity 'g'

For a given initial speed u, the maximum possible range is reached when sin2θ is maximum, sin2θ = 1. This implies 2θ = $\pi/2$

$$\theta = \frac{\pi}{4}$$

This means that if the particle is projected at 45 degrees with respect to horizontal, it attains maximum range, given by.

$$R_{max} = \frac{u^2}{g}$$

EXAMPLE

Suppose an object is thrown with initial speed 10 m s^{-1} at an angle $\pi/4$ with the horizontal, what is the range covered? Suppose the same object is thrown similarly in the Moon, will there be any change in the range? If yes, what is the change? (The acceleration due to gravity in the Moon $g_{moon} = \frac{1}{6}g$)

Solution

In projectile motion, the range of particle is given by,

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\theta = \pi/4 \quad u = v_0 = 10 m\ s^{-1}$$

$$\therefore R_{earth} = \frac{(10)^2 \ \sin \pi/2}{9.8} = 100/9.8$$

$$R_{earth} = 10.20\ m \text{ (Approximately 10 m)}$$

If the same object is thrown in the Moon, the range will increase because in the Moon, the acceleration due to gravity is smaller than g on Earth,

$$g_{moon} = \frac{g}{6}$$

$$R_{moon} = \frac{u^2 \sin 2\theta}{g_{moon}} = \frac{v_0^{\ 2} \sin 2\theta}{g/6}$$

$$\therefore R_{moon} = 6R_{earth}$$

$$R_{moon} = 6 \times 10.20 = 61.20\ m$$

$$\text{(Approximately 60 m)}$$

The range attained on the Moon is approximately six times that on Earth.

EXAMPLE

In the cricket game, a batsman strikes the ball such that it moves with the speed 30 m s^{-1} at an angle 300 with the horizontal as shown in the figure. The boundary line of the cricket ground is located at a distance of 75 m from the batsman? Will the ball go for a six? (Neglect the air resistance and take acceleration due to gravity g = 10 m s^{-2}).

Solution

The motion of the cricket ball in air is essentially a projectile motion. As we have already seen, the range (horizontal distance) of the projectile motion is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

The initial speed $u = 30\,m\,s^{-1}$

The projection angle $\theta = 30°$

The horizontal distance travelled by the cricket ball

$$R = \frac{(30)^2 \times sin 60°}{10} = \frac{900 \times \frac{\sqrt{3}}{2}}{10} = 77.94\,m$$

This distance is greater than the distance of the boundary line. Hence the ball will cross this line and go for a six.

Introduction to Degrees and Radians

In measuring angles, there are several possible units used, but the most common units are degrees and radians. Radians are used in measuring area, volume, and circumference of circles and surface area of spheres.

Radian describes the planar angle subtended by a circular arc at the center of a circle. It is defined as the length of the arc divided by the radius of the arc. One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle.

Degree is the unit of measurement which is used to determine the size of an angle. When an angle goes all the way around in a circle, the total angle covered is equivalent to 360°. Thus, a circle has 360°. In terms of radians, the full circle has 2π radian.

$$\text{Hence we write } 360° = 2\pi \ \textit{radians}$$

$$\text{or 1 radians} = \frac{180}{\pi} \text{ degrees}$$

$$\text{which means 1 rad} = 57.295^0$$

EXAMPLE

Calculate the angle θ subtended by the two adjacent wooden spokes of a bullock cart wheel is shown in the figure. Express the angle in both radian and degree.

Solution

The full wheel subtends 2π radians at the center of the wheel. The wheel is divided into 12 parts (arcs).

So one part subtends an angle $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ radian at the center

Since, π rad =180°, $\frac{\pi}{6}$ radian is equal to 30 degree.

The angle subtended by two adjacentwooden spokes is 30 degree at the center.

Angular displacement

Consider a particle revolving around a point O in a circle of radius r (Figure 2.45). Let the position of the particle at time t = 0 be A and after time t, its position is B.

Then,

The angle described by the particle about the axis of rotation (or center O) in a given time is called angular displacement.
angular displacement = ∠AOB =θ
The unit of angular displacement is radian.

The angular displacement (θ) in radian is related to arc length S (AB) and radius r as

$$\theta = \frac{S}{r}, \quad \text{or} \quad S = r\theta$$

Angular velocity (ϖ)

The rate of change of angular displacement is called angular velocity.

If θ is the angular displacement in time t, then the angular velocity $\omega$is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The unit of angular velocity is radian per second (rad s^{-1}). The direction of angular velocity is along the axis of rotation following the right hand rule.

Angular acceleration (α)

The rate of change of angular velocity is called angular acceleration.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

The angular acceleration is also a vector quantity which need not be in the same direction as angular velocity.

Tangential acceleration

Consider an object moving along a circle of radius r. In a time Δt, the object travels an arc distance Δs as shown in Figure 2.47. The corresponding angle subtended is Δθ
The Δs can be written in terms of Δθ as,

$$\Delta s = r\Delta\theta$$

In a time Δt, we have

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

In the limit Δt →0, the above equation becomes

$$\frac{ds}{dt} = r\omega$$

Here $\frac{ds}{dt}$ linear speed (v) which is tangential to the circle and ω is angular speed. So equation (2.37) becomes

$$v = r\omega$$

which gives the relation between linear speed and angular speed.

Equation (2.38) is true only for circular motion. In general the relation between linear and angular velocity is given by

$$\vec{v} = \vec{\omega} \times \vec{r}$$

For circular motion equation (2.39) reduces to equation (2.38) since ϖ and $\vec{r}$ are perpendicular to each other.

Differentiating the equation (2.38) with respect to time, we get (since r is constant)

$$\frac{dv}{dt} = \frac{rd\omega}{dt} = r\alpha$$

Here $\frac{dv}{dt}$ is the tangential acceleration and is denoted as a_t $\frac{d\omega}{dt}$ is the angular acceleration α. Then eqn. (2.39) becomes

$$a_t = r\alpha$$

The tangential acceleration a_t experienced by an object is circular motion

Circular Motion

When a point object is moving on a circular path with a constant speed, it covers equal distances on the circumference of the circle in equal intervals of time. Then the object is said to be in uniform circular motion.

In uniform circular motion, the velocity is always changing but speed remains the same. Physically it implies that magnitude of velocity vector remains constant and only the direction changes continuously.

If the velocity changes in both speed and direction during the circular motion, we get non uniform circular motion.

Centripetal acceleration

As seen already, in uniform circular motion the velocity vector turns continuously without changing its magnitude (speed),

Note that the length of the velocity vector (blue) is not changed during the motion, implying that the speed remains constant. Even though the velocity is tangential at every point in the circle, the acceleration is acting towards the center of the circle. This is called centripetal acceleration. It always points towards the center of the circle.

The centripetal acceleration is derived from a simple geometrical relationship between position and velocity vectors

Let the directions of position and velocity vectors shift through the same angle θ in a small interval of time Δt, as shown in Figure 2.52. For uniform circular motion, $r = |\vec{r}_1| = |\vec{r}_2|$ and $v = |\vec{v}_1| = |\vec{v}_2|$. If the particle moves from position vector $\vec{r}_1$ to $\vec{r}_2$ the displacement is given by $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ and the change in velocity from $\vec{v}_1$ to $\vec{v}_2$ is given by $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ The magnitudes of the displacement Δr and of Δv satisfy the following relation

$$\frac{\Delta r}{r} = -\frac{\Delta v}{v} = \theta$$

Here the negative sign implies that Δv pointsradially inward, towards the center of the circle.

$$\Delta v = -v\left(\frac{\Delta r}{r}\right)$$

$$\text{Then,} \quad a = \frac{\Delta v}{\Delta t} = \frac{v}{r}\left(\frac{\Delta r}{\Delta t}\right) = -\frac{v^2}{r}$$

For uniform circular motion v =ωr, where ω is the angular velocity of the particle about the center. Then the centripetal acceleration can be written as

$$a = -\omega^2 r$$

Non uniform circular motion

If the speed of the object in circular motion is not constant, then we have non-uniform circular motion. For example, when the bob attached to a string moves in vertical circle, the speed of the bob is not the same at all time. Whenever the speed is not same in circular motion, the particle will have both centripetal and tangential acceleration.

The resultant acceleration is obtained by vector sum of centripetal and tangential acceleration.

Since centripetal acceleration is $\frac{v^2}{r}$ the magnitude of this resultant acceleration is given by

$$a_R = \sqrt{a_t^2 + \left(\frac{v^2}{r}\right)^2}$$

This resultant acceleration makes an angle θ with the radius vector

This angle is given by $\tan\theta = \frac{a_t}{\left(\frac{v^2}{r}\right)}$

EXAMPLE

A particle moves in a circle of radius 10 m. Its linear speed is given by v =3t where t is in second and v is in m s^{-1}.

1. Find the centripetal and tangential acceleration at t = 2 s.
2. Calculate the angle between the resultant acceleration and the radius vector.

Solution

The linear speed at t = 2 s

$$v = 3t = 6\,m\,s^{-1}$$

The centripetal acceleration at t = 2 s is

$$a_c = \frac{v^2}{r} = \frac{(6)^2}{10} = 3.6\,m\,s^{-2}$$

The tangential acceleration is $a_t = \frac{dv}{dt} = 3\,m\,s^{-2}$

The angle between the radius vector with resultant acceleration is given by

$$\tan\theta = \frac{a_t}{a_c} = \frac{3}{3.6} = 0.833$$

$$\theta = \tan^{-1}(0.833) = 0.69 \text{ radian}$$

$$\text{In terms of degree } \theta = 0.69 \times 57.17^\circ \approx 40^\circ$$

Kinematic Equations of circular motion

If an object is in circular motion with constant angular acceleration α, we can derive kinematic equations for this motion, analogous to those for linear motion.

Let us consider a particle executing circular motion with initial angular velocity ω_0. After a time interval t it attains a final angular velocity ω. During this time, it covers an angular displacement θ. Because of the change in angular velocity there is an angular acceleration α .

The kinematic equations for circular motion are easily written by following the kinematic equations for linear motion in section 2.4.3

The linear displacement (s) is replaced by the angular displacement(θ).

The velocity (v) is replaced by angular velocity (ω).

The acceleration (a) is replaced by angular acceleration (α).

The initial velocity (u) is replaced by the initial angular velocity (ω_0).

By following this convention, kinematic equations for circular motion are as in the table given below.

Kinematic equations for linear motion	Kinematic equations for angular motion
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = \frac{(v+u)t}{2}$	$\theta = \frac{(\omega_0 + \omega)t}{2}$

EXAMPLE

A particle is in circular motion with an acceleration $\alpha = 0.2 \text{rad s}^{-2}$.

1. What is the angular displacement made by the particle after 5 s?
2. What is the angular velocity at t = 5 s?. Assume the initial angular velocity is zero.

Solution

Since the initial angular velocity is zero ($\omega_0 = 0$).

The angular displacement made by the particle is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = \frac{1}{2} \times 2 \times 10^{-1} \times 25 = 2.5\ rad$$

In terms of degree

$$\theta = 2.5 \times 57.17^\circ \approx 143^\circ$$

UNIT – 3 LAWS OF MOTION

INTRODUCTION

Each and every object in the universe interacts with every other object. The cool breeze interacts with the tree. The tree interacts with the Earth. In fact, all species interact with nature. But, what is the difference between a human's interaction with nature and that of an animal's. Human's interaction has one extra quality. We not only interact with nature but also try to understand and explain natural phenomena scientifically.

In the history of mankind, the most curiosity driven scientific question asked was about motion of objects–'How things move?' and 'Why things move?' Surprisingly, these simple questions have paved the way for development from early civilization to the modern technological era of the 21st century.

Objects move because something pushes or pulls them. For example, if a book is at rest, it will not move unless a force is applied on it. In other words, to move an object a force must be applied on it. About 2500 years ago, the famous philosopher, Aristotle, said that 'Force causes motion'. This statement is based on common sense. But any scientific answer cannot be based on common sense. It must be endorsed with quantitative experimental proof.

In the 15th century, Galileo challenged Aristotle's idea by doing a series of experiments. He said force is not required to maintain motion.

Galileo demonstrated his own idea using the following simple experiment. When a ball rolls from the top of an inclined plane to its bottom, after reaching the ground it moves some distance and continues to move on to another inclined plane of same angle of inclination as shown in the Figure 3.1(a). By increasing the smoothness of both the inclined planes, the ball reach almost the same height(h) from where it was released (L1) in the second plane (L2) (Figure 3.1(b)). The motion of the ball is then observed by varying the angle of inclination of the second plane keeping the same smoothness. If the angle of inclination is reduced, the ball travels longer distance in the second plane to reach the same height (Figure 3.1 (c)). When the angle of inclination is made zero, the ball moves forever in the horizontal direction (Figure 3.1(d)). If the Aristotelian idea were true, the ball would not have moved in the second plane even if its smoothness is made maximum since no force acted on it in the horizontal direction. From this simple experiment, Galileo proved that force is not required to maintain motion. An object can be in motion even without a force acting on it.

In essence, Aristotle coupled the motion with force while Galileo decoupled the motion and force.

NEWTON'S LAWS

Newton analysed the views of Galileo, and other scientist like Kepler and Copernicus on motion and provided much deeper insights in the form of three laws.

Newton's First Law

Every object continues to be in the state of rest or of uniform motion (constant velocity) unless there is external force acting on it.

This inability of objects to move on its own or change its state of motion is called inertia. Inertia means resistance to change its state. Depending on the circumstances, there can be three types of inertia.

Inertia of rest:

When a stationary bus starts to move, the passengers experience a sudden backward push. Due to inertia, the body (of a passenger) will try to continue in the state of rest, while the bus moves forward. This appears as a backward push as shown in Figure 3.2. The inability of an object to change its state of rest is called inertia of rest.

Inertia of motion: When the bus is in motion, and if the brake is applied suddenly, passengers move forward and hit against the front seat. In this case, the bus comes to a stop, while the body (of a passenger) continues to move forward due to the property of inertia as shown in Figure 3.3. The inability of an object to change its state of uniform speed (constant speed) on its own is called inertia of motion.

Inertia of direction:

When a stone attached to a string is in whirling motion, and if the string is cut suddenly, the stone will not continue to move in circular motion but moves tangential to the circle as illustrated in Figure 3.4. This is because the body cannot change its direction of motion without any force acting on it. The inability of an object to change its direction of motion on its ownis called inertia of direction.

When we say that an object is at rest or in motion with constant velocity, it has a meaning only if it is specified with respect to some reference frames. In physics, any motion has to be stated with respect to a reference frame. It is to be noted that Newton's fi rst law is valid only in certain special reference frames called inertial frames. In fact, Newton's first law defines an inertial frame.

Inertial Frames

If an object is free from all forces, then it moves with constant velocity or remains at rest when seen from inertial frames. Thus, there exists some special set of frames in which if an object experiences no force it moves with constant velocity or remains at rest. But how do we know whether an object is experiencing a force or not? All the objects in the Earth experience Earth's gravitational force. In the ideal case, if an object is in deep space (very far away from any other object), then Newton's first law will be certainly valid. Such deep space can be treated as an inertial frame. But practically it is not possible to reach such deep space and verify Newton's first law.

For all practical purposes, we can treat Earth as an inertial frame because an object on the table in the laboratory appears to be at rest always. This object never picks up acceleration in the horizontal direction since no force acts on it in the horizontal direction. So the laboratory can be taken as an inertial frame for all physics experiments and calculations. For making these conclusions, we analyse only the horizontal motion of the object as there is no horizontal force that acts on it. We should notanalyse the motion in vertical direction as the two forces (gravitational force in the downward direction and normal force in upward direction) that act on it makes the net force is zero in vertical direction. Newton's first law deals with the motion of objects in the absence of any force and not the motion under zero net force. Suppose a train is moving with constant velocity with respect to an inertial frame, then an object at rest in the inertial frame (outside the train) appears to move with constant velocity with respect to the train (viewed from within the train). So the train can be treated as an inertial frame. All inertial frames are moving with constant velocity relative to each other. If an object appears to be at rest in one inertial frame, it may appear to move with constant velocity with respect to another inertial frame. For example, in Figure 3.5, the car is moving with uniform velocity v with respect to a person standing (at rest) on the ground. As the car is moving with constant velocity with respect to ground to the person is at rest on the ground, both frames (with respect to the car and to the ground) are inertial frames.

Suppose an object remains at rest on a smooth table kept inside the train, and if the train suddenly accelerates (which we may not sense), the object appears to accelerate backwards even without any force acting on it. It is a clear violation of Newton's first law as the object gets accelerated without being acted upon by a force. It implies that the train is not an inertial frame when it is accelerated. For example, Figure 3.6 shows that car 2 is a non-inertial frame since it moves with acceleration $\vec{a}$ with respect to the ground.

These kinds of accelerated frames are callednon-inertial frames. A rotating frame is alsoa non inertial frame since rotation requiresacceleration. In this sense, Earth is not reallyan inertial frame since it has self-rotationand orbital motion. But these rotationaleffects of Earth can be ignored for the motioninvolved in our day-to-day life. For example,when an object is thrown, or the timeperiod of a simple pendulum is measuredin the physics laboratory, the Earth's selfrotationhas very negligible effect on it. Inthis sense, Earth can be treated as an inertialframe. But at the same time, to analysethe motion of satellites and wind patternsaround the Earth, we cannot treat Earth asan inertial frame since its self-rotation hasa strong influence on wind patterns andsatellite motion.

Newton's Second Law

This law states that

The force acting on an object is equal to the rate of change of its momentum

$$\vec{F} = \frac{d\vec{p}}{dt}$$

In simple words, whenever the momentum of the body changes, there must be a force acting on it. The momentum of the object is defined as $\vec{p} = m\vec{v}$. In most cases, the mass of the object remains constant during the motion. In such cases, the above equation gets modified into a simpler form

$$\vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}.$$

$$\vec{F} = m\vec{a}.$$

The above equation conveys the fact that if there is an acceleration $\vec{a}$ on the body, then there must be a force acting on it. This implies that if there is a change in velocity, then there must be a force acting on the body. The force and acceleration are always in the same direction. Newton's second law was a paradigm shift from Aristotle's idea of motion. According to Newton, the force need not cause the motion but only a change in motion. It is to be noted that Newton's second law is valid only in inertial frames. In non-inertial frames Newton's second law cannot be used in this form. It requires some modification.

In the SI system of units, the unit of force is measured in newtons and it is denoted by symbol 'N'.

One Newton is defined as the force which acts on 1 kg of mass to give an acceleration 1 m s^{-2} in the direction of the force.

Aristotle vs. Newton's approach on sliding object

Newton's second law gives the correct explanation for the experiment on the inclined plane that was discussed in section 3.1. In normal cases, where friction is not negligible, once the object reaches the bottom of the inclined plane (Figure 3.1), it travels some distance and stops. Note that it stops because there is a frictional force acting in the direction opposite to its velocity. It is this frictional force that reduces the velocity of the object to zero and brings it to rest. As per Aristotle's idea, as soon as the body reaches the bottom of the plane, it can travel only a small distance and stops because there is no force acting on the object. Essentially, he did not consider the frictional force acting on the object.

Newton's Third Law

Consider Figure 3.8(a) whenever an object 1 exerts a force on the object $2(\vec{F}_{21})$, then object 2 must also exert equal and opposite force on the object $1(\vec{F}_{12})$. These forces must lie along the line joining the two objects.

$$\vec{F}_{12} = -\vec{F}_{21}$$

Newton's third law assures that the forces occur as equal and opposite pairs. An isolated force or a single force cannot exist in nature. Newton's third law states that for every action there is an equal and opposite reaction. Here, action and reaction pair of forces do not act on the same body but on two different bodies. Any one of the forces can be called as an action force and the other the reaction force. Newton's third law is valid in both inertial and non-inertial frames.

These action-reaction forces are not cause and effect forces. It means that when the object 1 exerts force on the object 2, the object 2 exerts equal and opposite force on the body 1 at the same instant.

Discussion on Newton's Laws

Newton's laws are vector laws. The equation $\vec{F} = m\vec{a}$ is a vector equation and essentially it is equal to three scalar equations. In Cartesian coordinates, this equation can be written as $F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = ma_x\hat{i} + ma_y\hat{j} + ma_z\hat{k}$. By comparing both sides, the three scalar equations are

$F_x = ma_x$ The acceleration along the x direction depends only on the component of force acting along the x-direction.

The acceleration along the y direction depends only on the component of force acting along the y-direction.

$F_z = ma_z$ The acceleration along the z direction depends only on the component of force acting along the z-direction.

From the above equations, we can infer that the force acting along y direction cannot alter the acceleration along x direction. In the same way, F_z cannot affect a_y and a_x. This understanding is essential for solving problems.

The acceleration experienced by the body at time t depends on the force which acts on the body at that instant of time. It does not depend on the force which acted on the body before the time t. This can be expressed as

$$\vec{F}(t) = m\vec{a}(t)$$

Acceleration of the object does not depend on the previous history of the force. For example, when a spin bowler or a fast bowler throws the ball to the batsman, once the ball leaves the hand of the bowler, it experiences only gravitational force and air frictional force. The acceleration of the ball is independent of how the ball was bowled (with a lower or a higher speed).

In general, the direction of a force may be different from the direction of motion. Though in some cases, the object may move in the same direction as the direction of the force, it is not always true. A few examples are given below.

Case 1: Force and motion in the same direction

When an apple falls towards the Earth, the direction of motion (direction of velocity) of the apple and that of force are in the same downward direction

Case 2: Force and motion not in the same direction

The Moon experiences a force towards the Earth. But it actually moves in elliptical orbit. In this case, the direction of the force is different from the direction of motion

Case 3: Force and motion in opposite direction

If an object is thrown vertically upward, the direction of motion is upward, but gravitational force is downward as

Case 4: Zero net force, but there is motion

When a raindrop gets detached from the cloud it experiences both downward gravitational force and upward air drag force. As it descends towards the Earth, the upward air drag force increases and after a certain time, the upward air drag force cancels the downward gravity. From then on the raindrop moves at constant velocity till it touches the surface of the Earth. Hence the raindrop comes with zero net force, therefore with zero acceleration but with non-zero terminal velocity.

If multiple forces $\vec{F}_1, \vec{F}_2, \vec{F}_3 .... \vec{F}_n$ act on the same body, then the total force $\left(\vec{F}_{net}\right)$ is equivalent to the vectorial sum of the individual forces. Their net force provides the acceleration.

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n$$

Newton's second law for this case is

$$\vec{F}_{net} = m\vec{a}$$

In this case the direction of acceleration is in the direction of net force.

Newton's second law can also be written in the following form. Since the acceleration is the second derivative of position vector of the body $\left(\vec{a} = \frac{d^2\vec{r}}{dt^2}\right)$ the force on the body is

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}.$$

From this expression, we can infer that Newton's second law is basically a second order ordinary differential equation and whenever the second derivative of position vector is not zero, there must be a force acting on the body.

If no force acts on the body then Newton's second law, $m\frac{d\vec{v}}{dt} = 0$.

It implies that $\vec{v}$ constant. It is essentially Newton's first law. It implies that the second law is consistent with the first law. However, it should not be thought of as the reduction of second law to the first when no force acts on the object. Newton's first and second laws are independent laws. They can internally be consistent with each other but cannot be derived from each other.

Newton's second law is cause and effect relation. Force is the cause and acceleration is the effect. Conventionally, the effect should be written on the left and cause on the right hand side of the equation. So the correct way of writing Newton's second law is $m\vec{a} = \vec{F}$ or $\frac{d\vec{p}}{dt} = \vec{F}$.

APPLICATION OF NEWTON'S LAWS
Free Body Diagram

Free body diagram is a simple tool to analyse the motion of the object using Newton's laws.

The following systematic steps are followed for developing the free body diagram:

1. Identify the forces acting on the object.
2. Represent the object as a point.
3. Draw the vectors representing the forces acting on the object.

When we draw the free body diagram for an object or a system, the forces exerted by the object should not be included in the free body diagram.

EXAMPLE

A book of mass m is at rest on the table. (1) What are the forces acting on the book? (2) What are the forces exerted by the book? (3) Draw the free body diagram for the book.

Solution

There are two forces acting on the book.

I. Gravitational force (mg) acting downwards on the book
II. Normal contact force (N) exerted by the surface of the table on the book. It acts upwards as shown in the figure.

According to Newton's third law, there are two reaction forces exerted by the book.

I. The book exerts an equal and opposite force (mg) on the Earth which acts upwards.
II. The book exerts a force which is equal and opposite to normal force on the surface of the table (N) acting downwards.

EXAMPLE

If two objects of masses 2.5 kg and 100 kg experience the same force 5 N, what is the acceleration experienced by each of them?

Solution

From Newton's second law (in magnitude form), F = ma

For the object of mass 2.5 kg, theacceleration is $a = \frac{F}{m} = \frac{5}{2.5} = 2\, m\, s^{-2}$

For the object of mass 100 kg, the acceleration is $a = \frac{F}{m} = \frac{5}{100} = 0.05\, m\, s^{-2}$

When an apple falls, it experiences Earth's gravitational force. According to Newton's third law, the apple exerts equal and opposite force on the Earth. Even though both the apple and Earth experience the same force, their acceleration is different. The mass of Earth is enormous compared to that of an apple. So an apple experiences larger acceleration and the Earth experiences almost negligible acceleration. Due to the negligible acceleration, Earth appears to be stationary when an apple falls.

EXAMPLE

Which is the greatest force among the three force $\vec{F}_1, \vec{F}_2, \vec{F}_3$ shown below

Solution

Force is a vector and magnitude of the vector is represented by the length of the vector. Here $\vec{F}_1$ has greater length compared to other two. So $\vec{F}_1$ is largest of the three.

EXAMPLE

Apply Newton's second law to a mango hanging from a tree. (Mass of the mango is 400 gm)

Solution

Note: Before applying Newton's laws, the following steps have to be followed:

1. Choose a suitable inertial coordinate system to analyse the problem. For most of the cases we can take Earth as an inertial coordinate system.
2. Identify the system to which Newton's laws need to be applied. The system can be a single object or more than one object.
3. Draw the free body diagram.
4. Once the forces acting on the system are identified, and the free body diagram is drawn, apply Newton's second law. In the left hand side of the equation, write the forces acting on the system in vector notation and equate it to the right hand side of equation which is the product of mass and acceleration. Here, acceleration should also be in vector notation.
5. If acceleration is given, the force can be calculated. If the force is given, acceleration can be calculated.

By following the above steps:

We fix the inertial coordinate system on the ground as shown in the figure.

The forces acting on the mango are

1. Gravitational force exerted by the Earth on the mango acting downward along negative y axis

2. Tension (in the cord attached to the mango) acts upward along positive y axis.

The free body diagram for the mango is shown in the figure

$$\vec{F}_g = mg\left(-\hat{j}\right) = -mg\hat{j}$$

Here, mg is the magnitude of the gravitational force and $\left(-\hat{j}\right)$ represents the unit vector in negative y direction

$$\vec{T} = T\hat{j}$$

Here T is the magnitude of the tension force and $\hat{j}$ represents the unit vector in positive y direction

$$\vec{F}_{net} = \vec{F}_g + \vec{T} = -mg\hat{j} + T\hat{j} = (T - mg)\,\hat{j}$$

From Newton's second law $\vec{F}_{net} = m\vec{a}$

Since the mango is at rest with respect to us (inertial coordinate system) the acceleration is zero $(\vec{a} = 0)$

So $\vec{F}_{net} = m\vec{a} = 0$

$$(T - mg)\,\hat{j} = 0$$

By comparing the components on both sides of the above equation, we get T −mg = 0

So the tension force acting on the mango is given by T=mg

Mass of the mango m = 400g and g = 9.8 m s^{-2}

Tension acting on the mango is T = 0.4 × 9.8 = 3.92 N

EXAMPLE

A person rides a bike with a constant velocity $\vec{v}$ with respect to ground and another biker accelerates with acceleration $\vec{a}$ with respect to ground. Who can apply Newton's second law with respect to a stationary observer on the ground?

Solution

Second biker cannot apply Newton's second law, because he is moving with acceleration $\vec{a}$ with respect to Earth (he is not in inertial frame). But the first biker can apply Newton's second law because he is moving at constant velocity with respect to Earth (he is in inertial frame).

EXAMPLE

The position vector of a particle is given by $\vec{r} = 3t\hat{i} + 5t^2\hat{j} + 7\hat{k}$. Find the direction in which the particle experiences net force?

Solution

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(3t)\hat{i} + \frac{d}{dt}(5t^2)\hat{j} + \frac{d}{dt}(7)\hat{k}$$

$$\frac{d\bar{r}}{dt} = 3\hat{i} + 10t\hat{j}$$

Acceleration of the particle

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2} = 10\hat{j}$$

Here, the particle has acceleration only along positive y direction. According to Newton's second law, net force must also act along positive y direction. In addition, the particle has constant velocity in positive x direction and no velocity in z direction. Hence, there are no net force along x or z direction.

EXAMPLE

Consider a bob attached to a string, hanging from a stand. It oscillates as shown in the figure.

Solution

1. Identify the forces that act on the bob?
2. What is the acceleration experienced by the bob?

Two forces act on the bob.

1. Gravitational force (mg) acting downwards
2. Tension (T) exerted by the string on the bob, whose position determines the direction of T as shown in figure.

The bob is moving in a circular arc as shown in the above figure. Hence it has centripetal acceleration. At a point A and C, the bob comes to rest momentarily and then its velocity increases when it moves towards point B. Hence, there is a tangential acceleration along the arc. The gravitational force can be resolved into two components (mg cosθ, mg sinθ) as shown below

EXAMPLE

The velocity of a particle moving in a plane is given by the following diagram. Find out the direction of force acting on the particle?

Solution

The velocity of the particleis $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ As shown in the figure, the particle is moving in the xy plane, there is no motion in the z direction. So velocity in the z direction is zero (v_z=0). The velocity of the particle has x component (v_x) and y component (v_y). From fi gure, as time increases from t = 0 sec to t = 3 sec, the length of the vector in y direction is changing (increasing). It means y component of velocity (v_y) is increasing with respect to time. According to Newton's second law, if velocity changes with respect to time then there must be acceleration. In this case, the particle has acceleration in the y direction since the y component of velocity changes. So the particle experiences force in the y direction. The length of the vector in x direction does not change. It means that the particle has constant velocity in the x direction. So no force or zero net force acts in the x direction.

EXAMPLE

Apply Newton's second law for an object at rest on Earth and analyse the result.

Solution

The object is at rest with respect to Earth (inertial coordinate system). There are two forces that act on the object.

1. Gravity acting downward (negative y-direction)
2. Normal force by the surface of the Earth acting upward (positive y-direction)

The free body diagram for this object is

$$\vec{F}_g = -mg\hat{j}$$
$$\vec{N} = N\hat{j}$$

Net force $\vec{F}_{net} = -mg\hat{j} + N\hat{j}$

But there is no acceleration on the ball. So $\vec{a} = 0$.. By applying Newton's second law $\left(\vec{F}_{net} = m\vec{a}\right)$

Since $\vec{a} = 0, \vec{F}_{net} = -mg\hat{j} + N\hat{j}$

$$(-mg+N)\hat{j}=0$$

By comparing the components on both sides of the equation, we get

$$-mg + N = 0$$
$$N = mg$$

We can conclude that if the object is at rest, the magnitude of normal force is exactly equal to the magnitude of gravity.

EXAMPLE

A particle of mass 2 kg experiences two forces $\vec{F}_1 = 5\hat{i}+8\hat{j}+7\hat{k}$ and $\vec{F}_2 = 3\hat{i}-4\hat{j}+3\hat{k}$ What is the acceleration of the particle?

Solution

We use Newton's second law, $\vec{F}_{net} = m\vec{a}$ where $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$. From the above equations the acceleration is $\vec{a} = \frac{\vec{F}_{net}}{m}$ where

$$\vec{F}_{net} = (5+3)\hat{i} + (8-4)\hat{j} + (7+3)\hat{k}$$

$$\vec{F}_{net} = 8\hat{i} + 4\hat{j} + 10\hat{k}$$

$$\vec{a} = \left(\frac{8}{2}\right)\hat{i} + \left(\frac{4}{2}\right)\hat{j} + \left(\frac{10}{2}\right)\hat{k}$$

$$\vec{a} = 4\hat{i} + 2\hat{j} + 5\hat{k}$$

EXAMPLE

Identify the forces acting on blocks A, B and C shown in the figure.

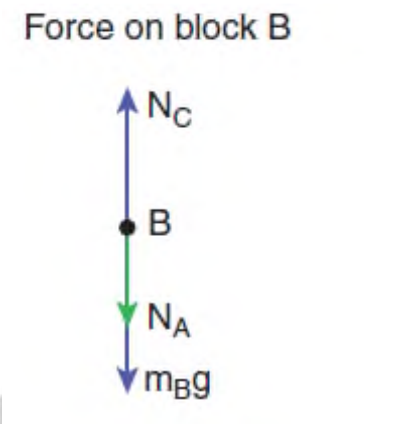
Solution
Forces on block A:

1. Downward gravitational force exerted by the Earth ($m_A g$)
2. Upward normal force (N_B) exerted by block B (N_B)

The free body diagram for block A is as shown in the following picture.

Forces on block B :

1. Downward gravitational force exerted by Earth (m_Bg)
2. Downward force exerted by block A (N_A)
3. Upward normal force exerted by block C (N_C)



Forces onblock C:

1. Downward gravitational force exerted by Earth (m_Cg)
2. Downward force exerted by block B (N_B)
3. Upward force exerted by the table (N_{table})

EXAMPLE

Consider a horse attached to the cart which is initially at rest. If the horse starts walking forward, the cart also accelerates in the forward direction. If the horse pulls the cart with force F_h in forward direction, then according to Newton's third law, the cart also pulls the horse by equivalent opposite force F_c=F_h in backward direction. Then total force on 'cart+horse' is zero. Why is it then the 'cart+horse' accelerates and moves forward?

Solution

This paradox arises due to wrongapplication of Newton's second and thirdlaws. Before applying Newton's laws, weshould decide 'what is the system?'.Oncewe identify the 'system', then it is possible toidentify all the forces acting on the system.We should not consider the force exertedby the system. If there is an unbalancedforce acting on the system, then it shouldhave acceleration in the direction of theresultant force. By following these steps wewill analyse the horse and cart motion.

If we decide on the cart+horse as a 'system', then we should not consider the force exerted by the horse on the cart or the force exerted by cart on the horse. Both are internal forces acting on each other. According to Newton's third law, total internal force acting on the system is zero and it cannot accelerate the system. The acceleration of the system is caused by some external force. In this case, the force exerted by the road on the system is the external force acting on the system. It is wrong to conclude that the total force acting on the system (cart+horse) is zero without including all the forces acting on the system. The road is pushing the horse
and cart forward with acceleration. As there is an external force acting on the system, Newton's second law has to be applied and not Newton's third law.

The following figures illustrates this.

If we consider the horse as the 'system', then there are three forces acting on the horse.

1. Downward gravitational force (m_gh)
2. Force exerted by the road (F_r)
3. Backward force exerted by the cart (F_c)

The force exerted by the road can be resolved into parallel and perpendicular components. The perpendicular component balances the downward gravitational force. There is parallel component along the forward direction. It is greater than the backward force (F_c). So there is net force along the forward direction which causes the forward movement of the horse.

If we take the cart as the system, then there are three forces acting on the cart.

1. Downward gravitational force (m_cg)
2. Force exerted by the road (F_r)
3. Force exerted by the horse (F_h)

The force exerted by the road$\left(\vec{F}_r\right)$ can be resolved into parallel and perpendicular components. The perpendicular component cancels the downward gravity (m_cg). Parallel component acts backwards and the force exerted by the horse $\left(\vec{F}_h\right)$acts forward. Force$\left(\vec{F}_h\right)$is greater than the parallel component acting in the opposite direction. So there is an overall unbalanced force in the forward direction which causes the cart to accelerate forward.

If we take the cart+horse as a system, then there are two forces acting on the system.

1. Downward gravitational force $(m_h+m_c)g$
2. The force exerted by the road (F_r) on the system.

3. In this case the force exerted by the road (F_r) on the system (cart+horse) is resolved in to parallel and perpendicular components. The perpendicular component is the normal force which cancels the downward gravitational force $(m_h+m_c)g$. The parallel component of the force is not balanced, hence the system (cart+horse) accelerates and moves forward due to this force.

The acceleration is given by $a = \frac{d^2 y}{dt^2}$

$$a = \frac{dv}{dt}$$

v =velocity of the particle in y direction

$$v = \frac{dy}{dt} = u - gt$$

The momentum of the particle = mv = m (u-gt).

$$a = \frac{dv}{dt} = -g$$

The force acting on the object is given by F =ma = −mg

The negative sign implies that the force is acting on the negative y direction. This is exactly the force that acts on the object in projectile motion.

Particle Moving in an Inclined Plane

When an object of mass m slides on a frictionless surface inclined at an angle θ as shown in the Figure 3.12, the forces acting on it decides the

1. acceleration of the object
2. speed of the object when it reaches the bottom

The force acting on the object is

1. Downward gravitational force (mg)
2. Normal force perpendicular to inclined surface (N)

To draw the free body diagram, the block is assumed to be a point mass (Figure 3.13 (a)). Since the motion is on the inclined surface, we have to choose the coordinate system parallel to the inclined surface.

The gravitational force mg is resolved in to parallel component mg $\sin\theta$ along the inclined plane and perpendicular component mg $\cos\theta$ perpendicular to the inclined surface.

Note that the angle made by the gravitational force (mg) with the perpendicular to the surface is equal to the angle of inclination θ.

There is no motion(acceleration) along the y axis. Applying Newton's second law in the y direction

$$-mg\cos\theta\hat{j} + N\hat{j} = 0\left(No\ acceleration\right)$$

By comparing the components on both sides, $N - mg\cos\theta = 0$

$$N = mg\cos\theta$$

The magnitude of normal force (N) exerted by the surface is equivalent to mg $\cos\theta$.

The object slides (with an acceleration) along the x direction. Applying Newton's\ second law in the x direction

$$mg\sin\theta\hat{i} = ma\hat{i}$$

By comparing the components on both sides, we can equate

$$mg\sin\theta = ma$$

The acceleration of the sliding object is

$$a = g\sin\theta$$

Note that the acceleration depends on the angle of inclination θ . If the angle θ is 90 degree, the block will move vertically with acceleration a = g.

Newton's kinematic equation is used to find the speed of the object when it reaches the bottom. The acceleration is constant throughout the motion.

$$v^2 = u^2 + 2as \text{ along the x direction}$$

The acceleration a is equal to g sinθ. The initial speed (u) is equal to zero as it starts from rest. Here s is the length of the inclined surface.

The speed (v) when it reaches the bottom is (using equation (3.3))

$$v = \sqrt{2sg\ sin\theta}$$

Two Bodies in Contact on a Horizontal Surface

Consider two blocks of masses m1 and m2 (m1> m2) kept in contact with each other on a smooth, horizontal frictionless surface as shown in Figure 3.14.

By the application of a horizontal force F, both the blocks are set into motion with acceleration 'a' simultaneously in the direction of the force F.

To find the acceleration $\vec{a}$, Newton's second law has to be applied to the system (combined mass m = m1 + m2)

$$\vec{F} = m\vec{a}$$

If we choose the motion of the two masses along the positive x direction,

$$F\hat{i} = ma\hat{i}$$

By comparing components on both sides of the above equation

$$F = \text{ma} \quad \text{where m} = \text{m}_1 + \text{m}_2$$

The acceleration of the system is given by

$$\therefore a = \frac{F}{\text{m}_1 + \text{m}_2}$$

The force exerted by the block m1 on m_2 due to its motion is called force of contact $\left(\vec{f}_{21}\right)$. According to Newton's third law, the block m_2 will exert an equivalent opposite reaction force $\left(\vec{f}_{12}\right)$ on block m_1.

$$\therefore F\hat{i} - f_{12}\hat{i} = m_1 a\hat{i}$$

By comparing the components on both sides of the above equation, we get

$$F - f_{12} = m_1 a$$

$$f_{12} = F - m_1 a$$

Substituting the value of acceleration from equation

$$f_{12} = F - m_1\left(\frac{F}{m_1 + m_2}\right)$$

$$f_{12} = F\left[1 - \frac{m_1}{m_1 + m_2}\right]$$

$$f_{12} = \frac{Fm_2}{m_1 + m_2}$$

Equation (3.7) shows that the magnitude of contact force depends on mass m_2 which provides the reaction force. Note that this force is acting along the negative x direction.

In vector notation, the reaction force on mass m_1 is given by $\vec{f}_{12} = -\frac{Fm_2}{m_1 + m_2}$

For mass m_2 there is only one force acting on it in the x direction and it is denoted by $\vec{f}_{21}$. This force is exerted by mass m1. The free body diagram for mass m_2

Applying Newton's second law for mass m_2

$$f_{21}\hat{i} = m_2 a\hat{i}$$

By comparing the components on both sides of the above equation

$$f_{21} = m_2 a$$

Substituting for acceleration from equation (3.5) in equation (3.8), we get

$$f_{21} = \frac{Fm_2}{m_1 + m_2}$$

In this case the magnitude of the contact force is

$$f_{21} = \frac{Fm_2}{m_1 + m_2}$$

The direction of this force is along the positive x direction.

In vector notation, the force acting on mass m2exerted by mass $m_1$is

$$\vec{f}_{21} = \frac{Fm_2}{m_1 + m_2}$$

Note $\vec{f}_{12} = -\vec{f}_{21}$ which confirms Newton's third law.

Motion of Connected Bodies

When objects are connected by strings and a force F is applied either vertically or horizontally or along an inclined plane, it produces a tension T in the string, which affects the acceleration to an extent. Let us discuss various cases for the same.

Case 1: Vertical motion

Consider two blocks of masses m_1 and m_2 ($m_1 > m_2$) connected by a light and inextensible string that passes over a pulley as shown in Figure

Let the tension in the string be T and acceleration a. When the system is released, both the blocks start moving, m_2 vertically upward and m_1 downward with same acceleration a. The gravitational force m1g on mass m1 is used in lifting the mass m_2.

The upward direction is chosen as y direction.

Applying Newton's second law for mass m2

$$T\hat{j} - m_2 g\hat{j} = m_2 a\hat{j}$$

The left hand side of the above equation is the total force that acts on m_2 and the right hand side is the product of mass and acceleration of m2 in y direction.

By comparing the components on both sides, we get

$$T - m_2 g = m_2 a$$

Similarly, applying Newton's second law for mass m_1

$$T\hat{j} - m_1 g\hat{j} = -m_1 a\hat{j}$$

As mass m_1 moves downward $(-\hat{j})$ its acceleration is along $(-\hat{j})$ By comparing the components on both sides, we get

$$T - m_1 g = -m_1 a$$

$$m_1 g - T = m_1 a$$

$$m_1 g - m_2 g = m_1 a + m_2 a$$

$$(m_1 - m_2) g = (m_1 + m_2) a$$

From equation (3.11), the acceleration of both the masses is

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

If both the masses are equal (m1=m2), from equation

$$a=0$$

This shows that if the masses are equal, there is no acceleration and the system as a whole will be at rest.

To find the tension acting on the string, substitute the acceleration from the equation (3.12) into the equation (3.9).

$$T - m_2 g = m_2 \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$$

$$T = m_2 g + m_2 \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g$$

By taking m2g common in the RHS of equation (3.13)

$$T = m_2 g \left(1 + \frac{m_1 - m_2}{m_1 + m_2} \right)$$

$$T = m_2 g \left(\frac{m_1 + m_2 + m_1 - m_2}{m_1 + m_2} \right)$$

$$T = \left(\frac{2 m_1 m_2}{m_1 + m_2} \right) g$$

Equation (3.12) gives only magnitude of acceleration.

For mass m1, the acceleration vector is given by $\vec{a} = -\left(\frac{m_1 - m_2}{m_1 + m_2} \right) \hat{j}$

For mass m2, the acceleration vector is given b $\vec{a} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) \hat{j}$

Case 2: Horizontal motion

In this case, mass m2 is kept on a horizontal table and mass m1 is hanging through a small pulley as shown in Figure 3.17. Assume that there is no friction on the surface.

As both the blocks are connected to the unstretchable string, if m1 moves with an acceleration a downward then m2 also moves with the same acceleration a horizontally.

The forces acting on mass m_2 are

1. Downward gravitational force ($m_2 g$)
2. Upward normal force (N) exerted by the surface
3. Horizontal tension (T) exerted by the string

The forces acting on mass m_1 are

1. Downward gravitational force (m_1g)
2. Tension (T) acting upwards

The free body diagrams for both the masses

Applying Newton's second law for m_1

$$T\hat{j} - m_1 g\hat{j} = -m_1 a\hat{j}$$

By comparing the components on both sides of the above equation

$$T - m_1 g = -m_1 a$$

Applying Newton's second law for m_2

$$T\hat{i} = m_2 a\hat{i}$$

By comparing the components on both sides of above equation,

$$T = m_2 a$$

There is no acceleration along y direction for m_2.

$$N\hat{j} - m_2 g\hat{j} = 0$$

By comparing the components on both sides of the above equation

$$N - m_2 g = 0$$
$$N = m_2 g$$

By substituting equation (3.15) in equation (3.14), we can find the tension T

$$m_2a - m_1g = -m_1a$$

$$m_2a + m_1a = m_1g$$

$$a = \frac{m_1}{m_1 + m_2} g$$

Tension in the string can be obtained by substituting equation (3.17) in equation (3.15)

$$T = \frac{m_1 m_2}{m_1 + m_2} g$$

Comparing motion in both cases, it is clear that the tension in the string for horizontal motion is half of the tension for vertical motion for same set of masses and strings.

This result has an important application in industries. The ropes used in conveyor belts (horizontal motion) work for longer duration than those of cranes and lifts (vertical motion).

Concurrent Forces and Lami's Theorem

A collection of forces is said to be concurrent, if the lines of forces act at a common point. Figure 3.19 illustrates concurrent forces.

Concurrent forces need not be in the same plane. If they are in the same plane, they are concurrent as well as coplanar forces.

LAMI'S THEOREM

If a system of three concurrent and coplanar forces is in equilibrium, then Lami's theorem states that the magnitude of each force of the system is proportional to sine of the angle between the other two forces. The constant of proportionality is same for all three forces.

Let us consider three coplanar and concurrent forces $\vec{F}_1, \vec{F}_2$ *and* $\vec{F}_3$ which act at a common point O as shown in Figure 3.20. If the point is at equilibrium, then according to Lami's theorem

$$\left|\vec{F}_1\right| \propto \sin\alpha$$

$$\left|\vec{F}_2\right| \propto \sin\beta$$

$$\left|\vec{F}_3\right| \propto \sin\gamma$$

Therefore, $$\frac{\left|\vec{F}_1\right|}{\sin\alpha} = \frac{\left|\vec{F}_2\right|}{\sin\beta} = \frac{\left|\vec{F}_3\right|}{\sin\gamma}$$

Lami's theorem is useful to analyse the forces acting on objects which are in static equilibrium.

Application of Lami's Theorem

EXAMPLE

A baby is playing in a swing which is hanging with the help of two identical chains is at rest. Identify the forces acting on the baby. Apply Lami's theorem and find out the tension acting on the chain.

Solution

The baby and the chains are modeled as a particle hung by two strings as shown in the figure. There are three forces acting on the baby.

1. Downward gravitational force along negative y direction (mg)
2. Tension (T) along the two strings

These three forces are coplanar as well as concurrent as shown in the following figure.

$$\frac{T}{\sin(180-\theta)} = \frac{T}{\sin(180-\theta)} = \frac{mg}{\sin(2\theta)}$$

Since $\sin(180-\theta) = \sin\theta$ and $\sin(2\theta) =$
$2\sin\theta\cos\theta$

$$\frac{T}{\sin\theta} = \frac{mg}{2\sin\theta\cos\theta}$$

From this, the tension on each string is $T = \frac{mg}{2\cos\theta}$.

LAW OF CONSERVATION OF TOTAL LINEAR MOMENTUM

In nature, conservation laws play a very important role. The dynamics of motion of bodies can be analysed very effectively using conservation laws. There are three conservation laws in mechanics. Conservation of total energy, conservation of total linear momentum, and conservation of angular momentum. By combining Newton's second and third laws, we can derive the law of conservation of total linear momentum.

When two particles interact with each other, they exert equal and opposite forces on each other. The particle 1 exerts force $\vec{F}_{21}$ on particle 2 and particle 2 exerts an exactly equal and opposite force $\vec{F}_{12}$ on particle 1, according to Newton's third law.

$$\vec{F}_{21} = -\vec{F}_{12}$$

In terms of momentum of particles, the force on each particle (Newton's second law) can be written as

$$\vec{F}_{12} = \frac{d\vec{p}_1}{dt} \quad \text{and} \quad \vec{F}_{21} = \frac{d\vec{p}_2}{dt}.$$

Here $\vec{p}_1$ is the momentum of particle 1 which changes due to the force $\vec{F}_{12}$ exerted by particle 2. Further $\vec{p}_2$ is the momentum of particle 2. This changes due to $\vec{F}_{21}$ exerted by particle 1.

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0$$

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

It implies that $\vec{p}_1 + \vec{p}_2$ =constant vector (always).

$\vec{p}_1 + \vec{p}_2$ is the total linear momentum of the two particles $\left(\vec{P}_{tot} = \vec{p}_1 + \vec{p}_2\right)$. It is also called as total linear momentum of the system. Here, the two particles constitute the system. From this result, the law of conservation of linear momentum can be stated as follows.

If there are no external forces acting on the system, then the total linear momentum of the system ($\vec{P}_{tot}$)is always a constant vector. In other words, the total linear momentum of the system is conserved in time. Here the word 'conserve' means that $\vec{p}_1$ and $\vec{p}_2$ can vary, in such a way that $\vec{p}_1 + \vec{p}_2$ is a constant vector.

The forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are called the internal forces of the system, because they act only between the two particles. There is no external force acting on the two particles from outside. In such a case the total linear momentum of the system is a constant vector or is conserved.

EXAMPLE

Identify the internal and external forces acting on the following systems.

1. Earth alone as a system
2. Earth and Sun as a system
3. Our body as a system while walking
4. Our body + Earth as a system

Solution

Earth alone as a system

Earth orbits the Sun due to gravitational attraction of the Sun. If we consider Earth as a system, then Sun's gravitational force is an external force. If we take the Moon into account, it also exerts an external force on Earth.

(Earth + Sun) as a system

In this case, there are two internal forces which form an action and reaction pair the gravitational force exerted by the Sun on Earth and gravitational force exerted by the Earth on the Sun.

Our body as a system

While walking, we exert a force on the Earth and Earth exerts an equal and opposite force on our body. If our body alone is considered as a system, then the force exerted by the Earth on our body is external.

(Our body + Earth) as a system

In this case, there are two internal forces present in the system. One is the force exerted by our body on the Earth and the other is the equal and opposite force exerted by the Earth on our body.

Meaning of law of conservation of momentum

The Law of conservation of linear momentum is a vector law. It implies that both the magnitude and direction of total linear momentum are constant. In some cases, this total momentum can also be zero.

To analyse the motion of a particle, we can either use Newton's second law or the law of conservation of linear momentum. Newton's second law requires us to specify the forces involved in the process. This is difficult to specify in real situations. But conservation of linear momentum does not require any force involved in the process. It is covenient and hence important.

For example, when two particles collide, the forces exerted by these two particles on each other is difficult to specify. But it is easier to apply conservation of linear momentum during the collision process.

Examples

Consider the firing of a gun. Here the system is Gun+bullet. Initially the gun and bullet are at rest, hence the total linear momentum of the system is zero. Let $\vec{p}_1$ be the momentum of the bullet and $\vec{p}_2$ momentum of the gun before firing. Since initially both are at rest,

$$\vec{p}_1 = 0, \vec{p}_2 = 0.$$

Total momentum before fi ring the gun is zero, $\vec{p}_1 + \vec{p}_2 = 0.$

According to the law of conservation of linear momentum, total linear momemtum has to be zero after the fi ring also.

When the gun is fi red, a force is exerted by the gun on the bullet in forward direction. Now the momentum of the bullet changes from $\vec{p}_1$ to $\vec{p}_1$. To conserve the total linear momentum of the system, the momentum of the gun must also change from $\vec{p}_2$ to $\vec{p}_2$. Due to the conservation of linear momentum, $\vec{p}_1' + \vec{p}_2' = 0$. It implies that $\vec{p}_1' = -\vec{p}_2'$, the momentum of the gun is exactly equal, but in the opposite direction to the momentum of the bullet. This is the reason after firing, the gun suddenly moves backward with the momentum $\left(-\vec{p}_2\right)$. It is called 'recoil momemtum'. This is an example of conservation of total linear momentum.

Consider two particles. One is at rest and the other moves towards the first particle (which is at rest). They collide and after collison move in some arbitrary directions. In this case, before collision, the total linear momentum of the system is equal to the initial linear momentum of the moving particle. According to conservation of momentum, the total linear momentum after collision also has to be in the forward direction. The following figure explains this.

A more accurate calculation is covered in section 4.4. It is to be noted that the total momentum vector before and after collison points in the same direction. This simply means that the total linear momentum is constant before and after the collision. At the time of collision, each particle exerts a force on the other. As the two particles are considered as a system, these forces are only internal, and the total linear momentum cannot be altered by internal forces.

Impulse

If a very large force acts on an object for a very short duration, then the force is called impulsive force or impulse.

If a force (F) acts on the object in a very short interval of time (Δt), from Newton's second law in magnitude form

$$Fdt = dp$$

Integrating over time from an initial time t_i to a final time t_f, we get

$$\int_i^f dp = \int_{t_i}^{t_f} F\,dt$$

$$p_f - p_i = \int_{t_i}^{t_f} F\,dt$$

p_i= initial momentum of the object at timet_i
p_f=final momentum of the object at timet_f

$p_f - p_i = \Delta p =$ change in momentum of the object during the time interval $t_f - t_i = \Delta t$.

The integral $\int_{t_i}^{t_f} F\,dt = J$ is called the impulse and it is equal to change in momentum of the object.

If the force is constant over the time interval, then

$$\int_{t_i}^{t_f} F\,dt = F\int_{t_i}^{t_f} dt = F\left(t_f - t_i\right) = F\Delta t$$

$$F\Delta t = \Delta p$$

For a constant force, the impulse is denoted as J = FΔt and it is also equal to change in momentum ('p) of the object over the time interval 't.

Impulse is a vector quantity and its unit is Ns.

The average force acted on the object over the short interval of time is defined by

$$F_{avg} = \frac{\Delta p}{\Delta t}$$

From equation (3.25), the average force that act on the object is greater if 't is smaller. Whenever the momentum of the body changes very quickly, the average force becomes larger.

The impulse can also be written in terms of the average force. Since 'p is change in momentum of the object and is equal to impulse (J), we have

$$J = F_{avg}\Delta t$$

The graphical representation of constant force impulse and variable force impulse.

ilustration

When a cricket player catches the ball, he pulls his hands gradually in the direction of the ball's motion. Why?

If he stops his hands soon after catching the ball, the ball comes to rest very quickly. It means that the momentum of the ball is brought to rest very quickly. So the average force acting on the body will be very large. Due to this large average force, the hands will get hurt. To avoid getting hurt, the player brings the ball to rest slowly.

When a car meets with an accident, its momentum reduces drastically in a very short time. This is very dangerous for the passengers inside the car since they will experience a large force. To prevent this fatal shock, cars are designed with air bags in such a way that when the car meets with an accident, the momentum of the passengers will reduce slowly so that the average force acting on them will be smaller.

The shock absorbers in two wheelers play the same role as airbags in the car. When there is a bump on the road, a sudden force is transferred to the vehicle. The shock absorber prolongs the period of transfer of force on to the body of the rider. Vehicles without shock absorbers will harm the body due to this reason.

Jumping on a concrete cemented floor is more dangerous than jumping on the sand. Sand brings the body to rest slowly than the concrete floor, so that the average force experienced by the body will be lesser.

EXAMPLE

An object of mass 10 kg moving with a speed of 15 m s^{-1} hits the wall and comes to rest within

1. 0.03 second
2. 10 second

Calculate the impulse and average force acting on the object in both the cases.

Solution

Initial momentum of the object p_i =10 × 15 =150 k gm s^{-1}

Final momentum of the object p_f =0

$$\Delta p = 150 - 0 = 150\ kg\ m s^{-1}$$

Impulse J = Δp = 150N s.
Impulse J = Δp = 150N s

$$\text{Average force } F_{avg} = \frac{\Delta p}{\Delta t} = \frac{150}{0.03} = 5000\,N$$

$$\text{Average force } F_{avg} = \frac{150}{10} = 15\,N$$

We see that, impulse is the same in both cases, but the average force is different.

FRICTION

Introduction

If a very gentle force in the horizontal direction is given to an object at rest on the table, it does not move. It is because of the opposing force exerted by the surface on the object which resists its motion. This force is called the frictional force which always opposes the relative motion between an object and the surface where it is placed. If the force applied is increased, the object moves after a certain limit.

Relative motion: when a force parallel to the surface is applied on the object, the force tries to move the object with respect to the surface. This 'relative motion' is opposed by the surface by exerting a frictional force on the object in a direction opposite to applied force. Frictional force always acts on the object parallel to the surface on which the object is placed. There are two kinds of friction namely 1) Static friction and 2) Kinetic friction.

Static Friction $\left(\vec{\mathbf{f}}_s\right)$

Static friction is the force which opposes the initiation of motion of an object on the surface. When the object is at rest on the surface, only two forces act on it. They are the downward gravitational force and upward normal force. The resultant of these two forces on the object is zero. As a result the object is at rest as shown in Figure 3.23

some external force F_{ext} is applied on the object parallel to the surface on which the object is at rest, the surface exerts exactly an equal and opposite force on the object to resist its motion and tries to keep the object at rest. It implies that external force and frictional force are exactly equal and opposite. Therefore, no motion parallel to the surface takes place. But if the external force is increased, after a particular limit, the surface cannot provide sufficient opposing frictional force to balance the external force on the object. Then the object starts to slide. This is the maximal static friction that can be exerted by the surface. Experimentally, it is found that the magnitude of static frictional force f_s satisfies the following empirical relation.

$$0 \le f_s \le \mu_s N,$$

where μs is the coefficient of static friction. It depends on the nature of the surfaces in contact. N is normal force exerted by the surface on the body and sometimes it is equal to mg. But it need not be equal to mg always.

Equation (3.27) implies that the force of static friction can take any value from zero to $\mu_s N$.

If the object is at rest and no external force is applied on the object, the static friction acting on the object is zero (f_s=0).

If the object is at rest, and there is an external force applied parallel to the surface, then the force of static friction acting on the object is exactly equal to the external force applied on the object $\left(f_s = F_{ext}\right)$. But still the static friction f_s is less than $\mu_s N$.

When object begins to slide, the static friction (f_s) acting on the object attains maximum,

The static and kinetic frictions (which we discuss later) depend on the normal force acting on the object. If the object is pressed hard on the surface then the normal force acting on the object will increase. As a consequence it is more difficult to move the object. This is shown in Figure 3.23 (a) and (b). The static friction does not depend upon the area of contact.

EXAMPLE

Consider an object of mass 2 kg resting on the floor. The coefficient of static friction between the object and the floor is μ_s = 0.8. What force must be applied on the object to move it?

Solution

Since the object is at rest, the gravitational force experienced by an object is balanced by normal force exerted by floor.

$$N = mg$$

The maximum static frictional force $f_s^{\max} = \mu_s N = \mu_s mg$

$$f_s^{max} = 0.8 \times 2 \times 9.8 = 15.68\,N$$

Therefore to move the object the external force should be greater than maximum static friction.

$$F_{ext} > 15.68\,N$$

EXAMPLE

Consider an object of mass 50 kg at rest on the floor. A Force of 5 N is applied on the object but it does not move. What is the frictional force that acts on the object?

Solution

When the object is at rest, the external force and the static frictional force are equal and opposite

The magnitudes of these two forces are equal, $f_s = F_{ext}$

Therefore, the static frictional force acting on the object is

$$f_s = 5N\ .$$

The direction of this frictional force is opposite to the direction of F_{ext}.

EXAMPLE

Two bodies of masses 7 kg and 5 kg are connected by a light string passing over a smooth pulley at the edge of the table as shown in the figure. The coefficient of static friction between the surfaces (body and table) is 0.9. Will the mass m_1 = 7 kg on the surface move? If not what value of m_2 should be used so that mass 7 kg begins to slide on the table?

Solution

As shown in the figure, there are four forces acting on the mass m_1

1. Downward gravitational force along the negative y-axis (m_1g)
2. Upward normal force along the positive y axis (N)
3. Tension force due to mass m2 along the positive x axis
4. Frictional force along the negative x axis

Since the mass m_1 has no vertical motion, $m_1g = N$

To determine whether the mass m_1 moves on the surface, calculate the maximum static friction exerted by the table on the mass m1. If the tension on the mass m_1 is equal to or greater than this maximum static friction, the object will move.

$$f_s^{max} = \mu_s N = \mu_s m_1 g$$
$$f_s^{max} = 0.9 X 7 X 9.8 = 61.74\ N$$

$$T = m_2 g = 5 X 9.8 = 49\ N$$
$$T < f_s^{max}$$

The tension acting on the mass m1 is less than the maximum static friction. So the mass m1 will not move.

To move the mass m_1, T> f_s^{max} where T = m_2g

$$m_2 = \frac{\mu_s m_1 g}{g} = \mu_s m_1$$
$$m_2 = 0.9 X 7 = 6.3\ kg$$

If the mass m_2 is greater than 6.3 kg then the mass m1 will begin to slide. Note that if there is no friction on the surface, the mass m_1 will move for m_2 even for just 1 kg.

The values of coefficient of static friction for pairs of materials are presented in Table 3.1. Note that the ice and ice pair have very low coefficient of static friction. This means a block of ice can move easily over another block of ice.

Material	Coefficient of Static Friction
Glass and glass	1.0
Ice and ice	0.10
Steel and steel	0.75
Wood and wood	0.35
Rubber tyre and dry concrete road	1.0
Rubber tyre and wet road	0.7

Kinetic Friction

If the external force acting on the object is greater than maximum static friction, the objects begin to slide. When an object slides, the surface exerts a frictional force called kinetic friction $\vec{f}_k$ (also called sliding friction or dynamic friction). To move an object at constant velocity we must apply a force which is equal in magnitude and opposite to the direction of kinetic friction.

Experimentally it was found that the magnitude of kinetic friction satisfies the relation

$$f_k = \mu_k N$$

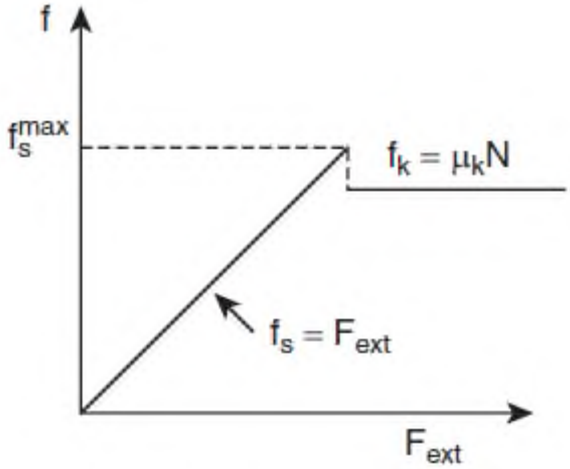
where μ_k is the coefficient of kinetic friction and N the normal force exerted by the surface on the object,

$$\mu_k < \mu_s$$

This implies that starting of a motion is more difficult than maintaining it. The salient features of static and kinetic friction

Static friction	Kinetic friction
It opposes the starting of motion	It opposes the relative motion of the object with respect to the surface
Independent of surface of contact	Independent of surface of contact
μ_s depends on the nature of materials in mutual contact	μ_k depends on nature of materials and temperature of the surface
Depends on the magnitude of applied force	Independent of magnitude of applied force
It can take values from zero to $\mu_s N$	It can never be zero and always equals to $\mu_k N$ whatever be the speed (true <10 ms^{-1})
$f_s^{max} > f_k$	It is less than maximal value of static friction
$\mu_s > \mu_k$	Coefficient of kinetic friction is less than coefficient of static friction

The variation of both static and kinetic frictional forces with external applied force



The Figure 3.25 shows that static friction increases linearly with external applied force till it reaches the maximum. If the object begins to move then the kinetic friction is slightly lesser than the maximum static friction. Note that the kinetic friction is constant and it is independent of applied force.

To Move an Object - Push or pull? Which is easier?

When a body is pushed at an arbitrary angle θ$\left(0 \ to \ \frac{\pi}{2}\right)$the applied force F can be resolved into two components as F sinθ parallel to the surface and F cosθ perpendicular to the surface as shown in Figure 3.26. Th e total downward force acting on the body is mg + Fcosθ. It implies that the normal force acting on the body increases. Since there is no acceleration along the vertical direction the normal force N is equal to

$$N_{push} = mg + Fcos\theta$$

As a result the maximal static friction also increases and is equal to

$$f_s^{max} = \mu_s N_{push} = \mu_s\left(mg + F\,cos\theta\right)$$

Equation (3.30) shows that a greater force needs to be applied to push the object into motion.

When an object is pulled at an angle θ, the applied force is resolved into two components as shown in Figure 3.27 The total downward force acting on the object is

$$N_{pull} = mg - Fcos\theta$$

Equation (3.31) shows that the normal force is less than N_{push}. From equations (3.29) and (3.31), it is easier to pull an object than to push to make it move.

Angle of Friction

The angle of friction is defined as the angle between the normal force (N) and the resultant force (R) of normal force and maximum friction force $f_s^{\max}$

In Figure 3.28 the resultant force is

$$R = \sqrt{\left(f_s^{max}\right)^2 + N^2}$$

$$\tan\theta = \frac{f_s^{max}}{N}$$

But from the frictional relation, the object begins to slide when $f_s^{\max} = \mu_s N$

$$\text{or when } \frac{f_s^{max}}{N} = \mu_s$$

From equations (3.32) and (3.33) the coefficient of static friction is

$$\mu_s = tan\theta$$

Angle of Repose

Consider an inclined plane on which an object is placed, as shown in Figure 3.30. Let the angle which this plane makes with the horizontal be θ. For small angles of θ , the object may not slide down. As θ is increased, for a particular value of θ, the object begins to slide down. This value is called angle of repose. Hence, the angle of repose is the angle of inclined plane with the horizontal such that an object placed on it begins to slide.

Let us consider the various forces in action here. The gravitational force mg is resolved into components parallel (mg sinθ) and perpendicular (mg cosθ) to the inclined plane.

The component of force parallel to the inclined plane (mg sinθ) tries to move the object down.

The component of force perpendicular to the inclined plane (mg $\cos\theta$) is balanced by the Normal force (N).

$$N = mg\cos\theta$$

When the object just begins to move, the static friction attains its maximum value

$$f_s = f_s^{max} = \mu_s N = \mu_s mg\cos\theta$$

This friction also satisfies the relation

$$f_s^{max} = mg\sin\theta$$

Equating the right hand side of equations (3.35) and (3.36), we get

$$\mu_s = \sin\theta / \cos\theta$$

From the definition of angle of friction, we also know thatin which θ is the angle of friction.

$$\tan\theta = \mu_s,$$

Thus the angle of repose is the same as angle of friction. But the difference is that the angle of repose refers to inclined surfaces and the angle of friction is applicable to any type of surface

EXAMPLE

A block of mass m slides down the plane inclined at an angle 60° with an acceleration $\frac{g}{2}$. Find the coefficient of kinetic friction?

Solution

Kinetic friction comes to play as the block is moving on the surface.

The forces acting on the mass are the normal force perpendicular to surface, downward gravitational force and kinetic friction f_k along the surface

$$mg\sin\theta - f_k = \text{ma}$$

But a = g/2

$$mg\sin 60^\circ - f_k = \text{mg}/2$$

$$\frac{\sqrt{3}}{2}\text{mg} - f_k = \text{mg}/2$$

$$f_k = mg\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

$$f_K = \left(\frac{\sqrt{3}-1}{2}\right)\text{mg}$$

There is no motion along the y-direction as normal force is exactly balanced by the mg $\cos\theta$.

$$\text{mg}\cos\theta = \text{N} = \text{mg}/2$$

$$f_K = \mu_K \text{N} = \mu_K \text{mg}/2$$

$$\mu_K = \frac{\left(\frac{\sqrt{3}-1}{2}\right)mg}{\frac{mg}{2}}$$

$$\mu_K = \sqrt{3} - 1$$

Application of Angle of Repose

Antlions make sand traps in such a way that when an insect enters the edge of the trap, it starts to slide towards the bottom where the antilon hide itself. The angle of inclination of sand trap is made to be equal to angle of repose.

Children are fond of playing on sliding board (Figure 3.31). Sliding will be easier when the angle of inclination of the board is greater than the angle of repose. At the same time if inclination angle is much larger than the angle of repose, the slider will reach the bottom at greater speed and get hurt.

Rolling Friction

The invention of the wheel plays a crucial role in human civilization. One of the important applications is suitcases with rolling on coasters. Rolling wheels makes it easier than carrying luggage. When an object moves on a surface, essentially it is sliding on it. But wheels move on the surface through rolling motion. In rolling motion when a wheel moves on a surface, the point of contact with surface is always at rest. Since the point of contact is at rest, there is no relative motion between the wheel and surface. Hence the frictional force is very less. At the same time if an object moves without a wheel, there is a relative motion between the object and the surface. As a result frictional force is larger. This makes it difficult to move the object. The Figure 3.32 shows the difference between rolling and kinetic friction.

Ideally in pure rolling, motion of the point of contact with the surface should be at rest, but in practice it is not so. Due to the elastic nature of the surface at the point of contact there will be some deformation on the object at this point on the wheel or surface as shown in Figure 3.33. Due to this deformation, there will be minimal friction between wheel and surface. It is called 'rolling friction'. In fact, 'rolling friction' is much smaller than kinetic friction.

Methods to Reduce Friction

Frictional force has both positive and negative effects. In some cases it is absolutely necessary. Walking is possible because of frictional force. Vehicles (bicycle, car) can move because of the frictional force between the tyre and the road. In the braking system, kinetic friction plays a major role. As we have already seen, the frictional force comes into effect whenever there is relative motion between two surfaces. In big machines used in industries, relative motion between different parts of the machine produce unwanted heat which reduces its efficiency. To reduce this kinetic friction lubricants are used as shown in Figure 3.34.

Ball bearings provides another effective way to reduce the kinetic friction (Figure 3.35) in machines. If ball bearings are fixed between two surfaces, during the relative motion only the rolling friction comes to effect and not kinetic friction. As we have seen earlier, the rolling friction is much smaller than kinetic friction; hence the machines are protected from wear and tear over the years.

During the time of Newton and Galileo, frictional force was considered as one of the natural forces like gravitational force. But in the twentieth century, the understanding on atoms, electron and protons has changed the perspective. The frictional force is actually the electromagnetic force between the atoms on the two surfaces. Even well polished surfaces have irregularities on the surface at the microscopic level as seen in the Figure 3.36.

EXAMPLE

Consider an object moving on a horizontal surface with a constant velocity. Some external force is applied on the object to keep the object moving with a constant velocity. What is the net force acting on the object?

Solution

If an object moves with constant velocity, then it has no acceleration. According to Newton's second law there is no net force acting on the object. The external force is balanced by the kinetic friction.

DYNAMICS OF CIRCULAR MOTION

In the previous sections we have studied how to analyse linear motion using Newton's laws. It is also important to know how to apply Newton's laws to circular motion, since circular motion is one of the very common types of motion that we come across in our daily life. A particle can be in linear motion with or without any external force. But when circular motion occurs there must necessarily be some force acting on the object. There is no Newton's first law for circular motion. In other words without a force, circular motion cannot occur in nature. A force can change the velocity of a particle in three different ways.

1. The magnitude of the velocity can be changed without changing the direction of the velocity. In this case the particle will move in the same direction but with acceleration.

Examples

Particle falling down vertically, bike moving in a straight road with acceleration

2. The direction of motion alone can be changed without changing the magnitude (speed). If this happens continuously then we call it 'uniform circular motion
3. Both the direction and magnitude (speed) of velocity can be changed. If this happens non circular motion occurs. For example oscillation of a swing or simple pendulum, elliptical motion of planets around the Sun.

In this section we will deal with uniform circular motion and non-circular motion.

Centripetal force

If a particle is in uniform circular motion, there must be centripetal acceleration towards the center of the circle. If there is acceleration then there must be some force acting on it with respect to an inertial frame. This force is called centripetal force.

As we have seen in chapter 2, the centripetal acceleration of a particle in the circular motion is given by $a = \frac{v^2}{r}$ and it acts towards center of the circle. According to Newton's second law, the centripetal force is given by

$$F_{cp} = m a_{cp} = \frac{mv^2}{r}$$

The word Centripetal force means center seeking force. In vector notation

$$\vec{F}_{cp} = -\frac{mv^2}{r}\hat{r}$$

For uniform circular motion

$$\vec{F}_{cp} = -m\omega^2 r\,\hat{r}$$

The direction –r^ points towards the center of the circle which is the direction of centripetal force as shown in Figure 3.38.

It should be noted that 'centripetal force' is not other forces like gravitational force or spring force. It can be said as 'force towards center'. The origin of the centripetal force can be gravitational force, tension in the string, frictional force, Coulomb force etc. Any of these forces can act as a centripetal force.

1. In the case of whirling motion of a stone tied to a string, the centripetal force on the particle is provided by the tensional force on the string. In circular motion in an amusement park, the centripetal force is provided by the tension in the iron ropes.
2. In motion of satellites around the Earth, the centripetal force is given by Earth's gravitational force on the satellites. Newton's second law for satellite motion is

$$F = \text{earth's gravitational force} = \frac{mv^2}{r}$$

Where r- distance of the planet from the center of the Earth.

3. When a car is moving on a circular track the centripetal force is given by the frictional force between the road and the tyres Newton's second law for this case is

$$\text{Frictional force} = \frac{mv^2}{r}$$

m-mass of the car

v-speed of the car

r-radius of curvature of track

Even when the car moves on a curved track, the car experiences the centripetal force which is provided by frictional force between the surface and the tyre of the car. This is shown in the Figure 3.41.

4. When the planets orbit around the Sun, they experience centripetal force towards the center of the Sun. Here gravitational force of the Sun acts as centripetal force on the planets as shown in Figure 3.42

Newton's second law for this motion Gravitational force of Sun on the planet $= \frac{mv^2}{r}$

EXAMPLE

If a stone of mass 0.25 kg tied to a string executes uniform circular motion with a speed of 2 m s^{-1} of radius 3 m, what is the magnitude of tensional force acting on the stone?

Solution

$$F_{cp} = \frac{\frac{1}{4}\times(2)^2}{3} = 0.333\ \text{N}.$$

EXAMPLE

The Moon orbits the Earth once in 27.3 days in an almost circular orbit. Calculate the centripetal acceleration experienced by the Moon? (Radius of the Earth is 6.4×10^6 m)

Solution

The centripetal acceleration is given by $a = \frac{v^2}{r}$. This expression explicitly depends on Moon's speed which is non trivial. We can work with the formula

$$\omega^2 R_m = a_m$$

a_m is centripetal acceleration of the Moon due to Earth's gravity.

ω is angular velocity.

R_m is the distance between Earth and the Moon, which is 60 times the radius of the Earth

$$R_m = 60R = 60 \times 6.4 \times 10^6 = 384 \times 10^6\ m$$

As we know the angular velocity $\omega = \frac{2\pi}{T}$ and T = 27.3 days= 27.3 × 24 × 60 × 60 second = 2.358 × 106 sec

By substituting these values in the formula for acceleration

$$a_m = \frac{(4\pi^2)(384 \times 10^6)}{(2.358 \times 10^6)^2} = 0.00272\ \mathrm{m\ s^{-2}}$$

The centripetal acceleration of Moon towards the Earth is 0.00272 m s^{-2}

Vehicle on a levelled circular road

When a vehicle travels in a curved path, there must be a centripetal force acting on it. This centripetal force is provided by the frictional force between tyre and surface of the road. Consider a vehicle of mass 'm' moving at a speed 'v' in the circular track of radius 'r'. There are three forces acting on the vehicle when it moves as shown in the Figure 3.43

1. Gravitational force (mg) acting downwards
2. Normal force (mg) acting upwards
3. Frictional force (F_s) acting horizontally inwards along the road

Suppose the road is horizontal then the normal force and gravitational force are exactly equal and opposite. The centripetal force is provided by the force of static friction Fs between the tyre and surface of the road which acts towards the center of the circular track,

$$\frac{mv^2}{r} = F_s$$

As we have already seen in the previous section, the static friction can increase from zero to a maximum value

$$F_s \le \mu_s mg.$$

There are two conditions possible:

$$\text{If } \frac{mv^2}{r} \le \mu_s mg, \text{ or } \mu_s \ge \frac{v^2}{rg} \text{ or } \sqrt{\mu_s rg} \ge v$$

The static friction would be able to provide necessary centripetal force to bend the car on the road. So the coefficient of static friction between the tyre and the surface of the road determines what maximum speed the car can have for safe turn.

$$\text{If } \frac{mv^2}{r} > \mu_s mg, \text{ or } \mu_s < \frac{v^2}{rg} \text{ (skid)}$$

If the static friction is not able to provide enough centripetal force to turn, the vehicle will start to skid.

EXAMPLE

Consider a circular leveled road of radius 10 m having coefficient of static friction 0.81. Three cars (A, B and C) are travelling with speed 7 m s^{-1}, 8 m s^{-1} and 10 ms^{-1} respectively. Which car will skid when it moves in the circular level road? (g =10 m s^{-2})

Solution

From the safe turn condition the speed of the vehicle (v) must be less than or equal to $\sqrt{\mu_s rg}$

$$v \le \sqrt{\mu_s rg}$$

$$\sqrt{\mu_s rg} = \sqrt{0.81 \times 10 \times 10} = 9\, m s^{-1}$$

For Car C, $\sqrt{\mu_s rg}$ is less than v

The speed of car A, B and C are 7 m s^{-1}, 8 m s−1 and 10 m s^{-1} respectively. The cars A and B will have safe turns. But the car C has speed 10 m s^{-1} while it turns which exceeds the safe turning speed. Hence, the car C will skid.

Banking of Tracks

In a leveled circular road, skidding mainly depends on the coefficient of static friction ms The coefficient of static friction depends on the nature of the surface which has a maximum limiting value. To avoid this problem, usually the outer edge of the road is slightly raised compared to inner edge as shown in the Figure 3.44. This is called banking of roads or tracks. This introduces an inclination, and the angle is called banking angle.

Let the surface of the road make angle θ with horizontal surface. Then the normal force makes the same angle θ with the vertical. When the car takes a turn, there are two forces acting on the car:

1. Gravitational force mg (downwards)
2. Normal force N (perpendicular to surface)

We can resolve the normal force into two components. N cos θ and N sin θ as shown in Figure 3.46. The component N cos θ balances the downward gravitational force 'mg' and component N sin θ will provide the necessary centripetal acceleration. By using Newton second law

$$N\cos\theta = mg$$

$$N\sin\theta = \frac{mv^2}{r}$$

By dividing the equations we get $\tan\theta = \frac{v^2}{rg}$

$$v = \sqrt{rg\tan\theta}$$

The banking angle θ and radius of curvature of the road or track determines the safe speed of the car at the turning. If the speed of car exceeds this safe speed, then it starts to skid outward but frictional force comes into effect and provides an additional centripetal force to prevent the outward skidding. At the same time, if the speed of the car is little lesser than safe speed, it starts to skid inward and frictional force comes into effect, which reduces centripetal force to prevent inward skidding. However if the speed of the vehicle is sufficiently greater than the correct speed, then frictional force cannot stop the car from skidding.

EXAMPLE

Consider a circular road of radius 20 meter banked at an angle of 15 degree. With what speed a car has to move on the turn so that it will have safe turn?

Solution

$$v = \sqrt{(rg\tan\theta)} = \sqrt{20\times 9.8\times \tan 15^\circ}$$

$$= \sqrt{20\times 9.8\times 0.26} = 7.1\ \text{m s}^{-1}$$

Centrifugal Force

Circular motion can be analysed from two different frames of reference. One is the inertial frame (which is either at rest or in uniform motion) where Newton's laws are obeyed. The other is the rotating frame of reference which is a non-inertial frame of reference as it is accelerating. When we examine the circular motion from these frames of reference the situations are entirely different. To use Newton's first and second laws in the rotational frame of reference, we need to include a pseudo force called 'centrifugal force'. This 'centrifugal force' appears to act on the object with respect to rotating frames. To understand the concept of centrifugal force, we can take a specific case and discuss as done below.

Consider the case of a whirling motion of a stone tied to a string. Assume that the stone has angular velocity ω in the inertial frame (at rest). If the motion of the stone is observed from a frame which is also rotating along with the stone with same angular velocity ω then, the stone appears to be at rest. This implies that in addition to the inward centripetal force $-m\omega^2 r$ there must be an equal and opposite force that acts on the stone outward with value $+m\omega^2 r$. So the total force acting on the stone in a rotating frame is equal to zero ($-m\omega^2 r + m\omega^2 r = 0$). This outward force $+m\omega^2 r$ is called the centrifugal force. The word 'centrifugal' means 'flee from center'. Note that the 'centrifugal force' appears to act on the particle, only when we analyse the motion from a rotating frame. With respect to an inertial frame there is only centripetal force which is given by the tension in the string. For this reason centrifugal force is called as a 'pseudo force'. A pseudo force has no origin. It arises due to the non inertial nature of the frame considered. When circular motion problems are solved from a rotating frame of reference, while drawing free body diagram of a particle, the centrifugal force should necessarily be included as shown in the Figure 3.45.

Effects of Centrifugal Force

Although centrifugal force is a pseudo force, its effects are real. When a car takes a turn in a curved road, person inside the car feels an outward force which pushes the person away. This outward force is also called centrifugal force. If there is sufficient friction between the person and the seat, it will prevent the person from moving outwards. When a car moving in a straight line suddenly takes a turn, the objects not fixed to the car try to continue in linear motion due to their inertia of direction. While observing this motion from an inertial frame, it appears as a straight line as shown in Figure 3.46. But, when it is observed from the rotating frame it appears to move outwards.

A person standing on a rotating platform feels an outward centrifugal force and is likely to be pushed away from the platform. Many a time the frictional force between the platform and the person is not sufficient to overcome outward push. To avoid this, usually the outer edge of the platform is little inclined upwards which exerts a normal force on the person which prevents the person from falling as illustrated in Figures 3.47.

Centrifugal Force due to Rotation of the Earth

Even though Earth is treated as an inertial frame, it is actually not so. Earth spins about its own axis with an angular velocity ω. Any object on the surface of Earth (rotational frame) experiences a centrifugal force. The centrifugal force appears to act exactly in opposite direction from the axis of rotation. It is shown in the Figure 3.48.

The centrifugal force on a man standing on the surface of the Earth is $F_c = m\omega^2 r$

where r is perpendicular distance of the man from the axis of rotation. By using right angle triangle as shown in the Figure 3.48, the distance $r = R\cos\theta$

Here R = radius of the Earth

and θ = latitude of the Earth where the man is standing.

EXAMPLE

Calculate the centrifugal force experienced by a man of 60 kg standing at Chennai? (Given: Latitude of Chennai is 13°

Solution

The centrifugal force is given by $F_c = m\,\omega^2 R\cos\theta$

The angular velocity (ω) of Earth $= \frac{2\pi}{r}$ where T is time period of the Earth (24 hours)

$$\omega = \frac{2\pi}{24\times 60\times 60} = \frac{2\pi}{86400}$$
$$= 7.268\times 10^{-5}\,rad sec^{-1}$$

The radius of the Earth R = 6400 Km = 6400×10^3 m

Latitude of Chennai = 13°

$$F_{cf} = 60\times\left(7.268\times 10^{-5}\right)^2 \times 6400\times 10^3$$
$$\times\cos\left(13°\right) = 1.9678\,N$$

A 60 kg man experiences centrifugal force of approximately 2 Newton. But due to Earth's gravity a man of 60 kg experiences a force = mg = 60 × 9.8 = 588N. This force is very much larger than the centrifugal force.

Centripetal Force Versus Centrifugal Force

Salient features of centripetal and centrifugal forces are compared in Table 3.4.

Centripetal force	Centrifugal force
It is a real force which is exerted on the body by the external agencies like gravitational force, tension in the string, normal force etc.	It is a pseudo force or fictitious force which cannot arise from gravitational force, tension force, normal force etc.
Acts in both inertial and non-inertial frames	Acts only in rotating frames (non-inertial frame)
It acts towards the axis of rotation or center of the circle in circular motion	It acts outwards from the axis of rotation or radially outwards from the center of the circular motion
$\lvert F_{cp} \rvert = m\omega^2 r = \frac{mv^2}{r}$	$\lvert F_{cf} \rvert = m\omega^2 r = \frac{mv^2}{r}$
Real force and has real effects	Pseudo force but has real effects
Origin of centripetal force is interaction between two objects.	Origin of centrifugal force is inertia. It does not arise from interaction. In an inertial frame the object's inertial motion appears as centrifugal force in the rotating frame.
In inertial frames centripetal force has to be included when free body diagrams are drawn.	In inertial frames there is no centrifugal force. In rotating frames, both centripetal and centrifugal force have to be included when free body diagrams are drawn.