PHYSICS
TEST – 6

11^{th} physics	**Unit 6**	**Work, Energy And Power**
	Unit 7	**MOTION OF SYSTEM OF PARTICLES AND RIGID BODIES**

Unit - 4
WORK, ENERGY AND POWER

INTRODUCTION

- The term work is used in diverse contexts in daily life. It refers to both physical as well as mental work. In fact, any activity can generally be called as work. But in Physics, the term work is treated as a physical quantity with a precise definition. Work is said to be done by the force when the force applied on a body displaces it. To do work, energy is required. In simple words, energy is defined as the ability to do work. Hence, work and energy are equivalents and have same dimension. Energy, in Physics exists in different forms such as mechanical, electrical, thermal, nuclear and so on. Many machines consume one form of energy and deliver energy in a different form. In this chapter we deal mainly with mechanical energy and its two types namely kinetic energy and potential energy. The next quantity in this sequence of discussion is the rate of work done or the rate of energy delivered. The rate of work done is called power. A powerful strike in cricket refers to a hit on the ball at a fast rate. This chapter aims at developing a good understanding of these three physical quantities namely work, energy and power and their physical significance.

WORK

- Let us consider a force ($\vec{F}$), acting on a body which moves it by a displacement in some direction ($d\vec{r}$)

- The expression for work done (w) by the force on the body is mathematically written as,

$$W=\vec{F}\,.\,d\vec{r}$$

- Here, the product $\vec{F}\,.\,d\vec{r}$ is a scalar product(or dot product). The scalar product of two vectors is a scalar. Thus, work done is a scalar quantity. It has only magnitude and no direction. In SI system, unit of work done is N m (or) joule (J). Its dimensional formula is $[ML^2T^{-2}]$.

The equation (4.1) is,

$$W= Fdr\cos\theta$$

- which can be realised using (as $\vec{a}.\vec{b}$ ab$\cos\theta$) where, θ is the angle between applied force and the displacementof the body.

- The work done by the force depends on the force (F), displacement (dr) and the angle (θ) between themWork done is zero in the following cases.

- **When the force is zero (F = 0)**. For example, a body moving on a horizontal smooth frictionless surface will continue to do so as no force (not even friction) is acting along the plane. (This is an ideal situation.)

- **When the displacement is zero (dr= 0).** For example, when force is applied ona rigid wall it does not produce anydisplacement. Hence, the work done iszero

- **When the force and displacement are perpendicular (θ = 90o) to each other.** when a body moves on a horizontaldirection, the gravitational force (mg)does no work on the body, since it acts atright angles to the displacement as shownin Figure 4.3(b). In circular motion thecentripetal force does not do work on theobject moving on a circle as it is alwaysperpendicular to the displacement.

- For a given force (F) and displacement (dr), the angle (θ) between them decides the value of work done as consolidated.

- There are many examples for the negative work done by a force. In a football game, the goalkeeper catches the ball coming towards him by applying a force such that the force is applied in a direction opposite to that of the motion of the ball till it comes to rest in his hands. During the time of applying the force, he does a negative work on the ball. We will discuss many more situations of negative work further in this unit.

- A box is pulled with a force of 25 N to produce a displacement of 15 m. If the angle between the force and displacement is 30o, find the work done by the force.

- Force, F = 25 N
- Displacement, dr = 15 m
- Angle between F and dr, θ = 30°

Angle (θ)	cosθ	Work
θ =0 °	1	Positive, Maximum
θ θ<9 °(actı e)	θ c osθ<1	Positive
θ =9 °(rit a t e)	0	Zero
9 °<θ<θ °	-1 c osθ<0	Negative
θ =θ °	-1	Negative, Maximum

Work done, $W = Fdr\cos\theta$

$$W = 25\times15\times\cos30 = 25\times15\times\frac{\sqrt{3}}{2}$$

$$W = 324.76\ J$$

Work done by a constant force

- When a constant force F acts on a body, the small work done (dW) by the force in producing a small displacement dr is given by the relation,

$$dW = (F\cos\theta)\ dr$$

- The total work done in producing a displacement from initial position $\mathbf{r_i}$ to final position $\mathbf{r_f}$ is,

$$W = \int_{r_1}^{r_f} dW$$

$$W = \int_{r_f}^{r_f} (F\cos\theta)dr = (F\cos\theta)\int_{r_f}^{r_f} dr$$

$$= (F\cos\theta)(r_f - r_i)$$

- The graphical representation of the work done by a constant force . The area under the graph shows the work done by the constant force.

- An object of mass 2 kg falls from a height of 5 m to the ground. What is the work done by the gravitational force on the object? (Neglect air resistance; Take g = 10 m s^{-2})

- In this case the force acting on the object is downward gravitational force $m\vec{g}$. This is a constant force. Work done by gravitational force is

$$W = \int_{r_1}^{r_f} \vec{F} d\vec{r}$$

$$W = (\cos\theta \int_{r_1}^{r_f} d\vec{r} = (mg.\cos\theta)(r_f - r_1)$$

The object also moves downward which is in the direction of gravitational force $\vec{F} = m\vec{g}$ as shown in figure. Hence, the angle between them is θ = 0°; cos θ =1 and the displacement, $(r_f - r_1)$ =5m.

$$W = mg(r_f - r_1)$$
$$W = 2 \times \phi \times 5 = \phi$$

The work done by the gravitational force on the object is positive.

- An object of mass *m*=1 kg is sliding from top to bottom in the frictionless inclined plane of inclination angle θ=30°and the length of inclined plane is 10 m as shown in the figure. Calculate the work done by gravitational force and normal force on the object. Assume acceleration due to gravity, g = 10 m s^{-2}

- We calculated in the previous chapter that the acceleration experienced by the object in the inclined plane as gsinθ.According to Newton's second law, the force acting on the mass along the inclined plane F = mgsinθ. Note that this force is constant throughout the motion of the mass. The work done by the parallel component of gravitational force (mg sinθ) is given by

$$W = \vec{F} d\vec{r} = F dr \cos\phi$$

- where ϕ is the angle between the force (mg sin θ**)** and the direction of motion (dr). In this case, force (mg sin θ**)** and the displacement $(d\vec{r})$ are in the same direction. Hence $\phi = and \cos\phi = 1$

W = F dr = (mgsinθ) (dr)

(dr = length of the inclined place)

$$W = 1 \times 10 \times \sin(30°) \times 10 = 100 \times \frac{1}{2} = 50J$$

- The component mg cosθand the normal force *N* are perpendicular to the direction of motion of the object, so they do notperform any work.
- If an object of mass 2 kg is thrown up from the ground reaches a height of 5 m and falls back to the Earth (neglect the air resistance). Calculate

1. The work done by gravity when the object reaches 5 m height
2. The work done by gravity when the object comes back to Earth
3. Total work done by gravity both in upward and downward motion and mention the physical significance of the result.

- When the object goes up, the displacement points in the upward direction whereas the gravitational force acting on the object points in downward direction. Therefore, the angle between gravitational force and displacement of the object is 180°.

The work done by gravitational force in the upward motion.

Given that Δr =5m and F =mg

$$W_{up} = F\Delta r \cos\theta = mg\Delta r \cos 180^{o}$$
$$W_{up} = 2 \times 10 \times 5 \times (-1) = -100\,joule.$$
$$[\cos 180^{o} = -1]$$

- When the object falls back, both the gravitational force and displacement of the object are in the same direction. This implies that the angle between gravitational force and displacement of the object is 0°.

$$W_{down} = F\Delta r \cos 0^{o}$$
$$W_{down} = 2 \times 10 \times 5 \times (1) = 100\,joule$$
$$[\cos 0^{o} = 1]$$

The total work done by gravity in the entire trip (upward and downward motion).

$$W_{total} = W_{up} + W_{down}$$
$$= -100\,joule + 100\,joule = 0$$

- It implies that the gravity does not transfer any energy to the object. When the object is thrown upwards, the energy is transferred to the object by the external agency, which means that the object gains some energy. As soon as it comes back and hits the Earth, the energygained by the object is transferred to the surfaceof the Earth (i.e., dissipated to the Earth).

A weight lifter lifts a mass of 250 kg with a force 5000 N to the height of 5 m.

1. What is the workdone by the weight lifter?
2. What is the workdone by the gravity?
3. What is the net workdone on the object?

- When the weight lifter lifts the mass, force and displacement are in the same direction, which means that the angle between them $\theta = 0^o$. Therefore, the work done by the weight lifter,

$$W_{\text{weight lifter}} = F_w h \cos\theta = F_w h(\cos 0^0)$$
$$= 5000 \times 5 \times (1) = 25,000\, joule = 25kJ$$

- When the weight lifter lifts the mass, the gravity acts downwards which means that the force and displacement are in opposite direction. Therefore, the angle between them $\theta = 180^o$

$$W_{\text{gravity}} = F_g h \cos\theta = mgh(\cos 180^o)$$
$$= 250 \times 10 \times 5 \times (-1)$$
$$= -12,500\, joule = -12.5kJ$$

- The net work done (or total work done) on the object

$$W_{\text{net}} = W_{\text{weight lifter}} + W_{\text{gravity}}$$
$$= 25kJ - 12.5kJ = +12.5kJ$$

Work done by a variable force

- When the component of a variable force F acts on a body, the small work done (dW) by the force in producing a small displacement dr is given by the relation

dW=F cos θ dr

[F cos θ is the component of the variable force F]

- where, F and θ are variables. The total work done for a displacement from initial positionr_i to final position r_f is given by the relation,

$$W = \int_{r_i}^{r_f} dW = \int_{r_i}^{r_f} F \cos\theta dr$$

- A graphical representation of the work done by a variable force.The area under the graph is the work done by the variable force.

- A variable force $F = k x^2$acts on a particle which is initially at rest. Calculate the work done by the force during the displacement of the particle from $x = 0$ m to $x = 4$ m. (Assume the constant $k = 1$ N m^{-2})

Work done,

$$W = \int_{x_i}^{x_f} F(x)dx = k\int_0^4 x^2 dx = \frac{64}{3} Nm$$

- Energy is defined as the capacity to do work. In other words, work done is the manifestation of energy. That is why work and energy have the same dimension (ML^2T^{-2})

- The important aspect of energy is that for an isolated system, the sum of all forms of energy i.e., the total energy remains the same in any process irrespective of whatever internal changes may take place. This means that the energy disappearing in one form reappears in another form. This is known as the law of conservation of energy. In this chapter we shall take up only the mechanical energy for discussion.

  In a broader sense, mechanical energy is classified into two types

1. Kinetic energy
2. Potential energy

- The energy possessed by a body due to its motion is called kinetic energy. The energy possessed by the body by virtue of its position is called potential energy.
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- The SI unit of energy is the same as that of work done i.e., N m (or) joule (J). The dimension of energy is also the same as that of work done. It is given by [ML^2T^{-2}]. The other units of energy and their SI equivalent values.

SI equivalent of other units of energy

Unit	Equivalent in joule
1 erg (CGS unit)	10^{-7} J
1 electron volt (eV)	1.6×10^{-19} J

1 calorie (cal)	4.186 J
1 kilowatt hour (kWh)	3.6×10^6 J

Kinetic energy

- Kinetic energy is the energy possessed by a body by virtue of its motion. All moving objects have kinetic energy. A body that is in motion has the ability to do work. For example a hammer kept at rest on a nail does not push the nail into the wood. Whereas the same hammer when it strikes the nail, draws the nail into the wood. Kinetic energy is measured by the amount of work that the body can perform before it comes to rest. The amount of work done by a moving body depends both on the mass of the body and the magnitude of its velocity. A body which is not in motion does not have kinetic energy.

Work–Kinetic Energy Theorem

- Work and energy are equivalents. This is true in the case of kinetic energy also. To prove this, let us consider a body of mass m at rest on a frictionless horizontal surface.

- The work (W) done by the constant force (F) for a displacement (s) in the same\ direction is,

$$W=Fs$$

The constant force is given by the equation,

$$F=ma$$

$$V^2=u^2+2as$$

$$a=\frac{V^2-u^2}{2s}$$

Substituting for a in equation

$$F=m\left(\frac{V^2-u^2}{2s}\right)$$

$$W=m\left(\frac{V^2}{2s}S\right)-m\left(\frac{u^2}{2s}S\right)$$

$$W=\frac{1}{2}mv^2-\frac{1}{2}mu^2$$

The term $\left(\frac{1}{2}mv^2\right)$ in the above equation is the kinetic energy of the body of mass (m) moving with velocity (v).

$$\text{KE}=\frac{1}{2}mv^2$$

Kinetic energy of the body is always positive. From equations

$$\Delta\text{KE}=\frac{1}{2}mv^2-\frac{1}{2}mu^2$$

Thus, $\text{W}=\Delta\text{KE}$

- The expression on the right hand side (RHS) of equation (4.12) is the change in kinetic energy (ΔKE) of the body.
- This implies that the work done by the force on the body changes the kinetic energy of the body. This is called work-kinetic energy theorem.

The work-kinetic energy theorem implies the following.

1. If the work done by the force on thebody is positive then its kinetic energy increases.
2. If the work done by the force on the body is negative then its kinetic energy decreases.
3. If there is no work done by the force on the body then there is no change in its kinetic energy, which means that the body has moved at constant speed provided its mass remains constant.

Relation between Momentum and Kinetic Energy

- Consider an object of mass m moving with a velocity $\vec{v}$. Then its linear momentum is $p=m\vec{v}$ and its kinetic energy, $\text{KE}=\frac{1}{2}mv^2$.

$$\text{KE}=\frac{1}{2}mv^2=\frac{1}{2}m\left(\vec{v}.\vec{v}\right)$$

Multiplying both the numerator and denominator of equation

$$\text{KE}=\frac{1}{2}\frac{m^2\left(\vec{v}.\vec{v}\right)}{m}$$

$$= \frac{1}{2}\frac{(m\vec{v}).(m\vec{v})}{m}\left[\vec{p} = m\vec{v}\right]$$

$$= \frac{1}{2}\frac{\vec{p}.\vec{p}}{m}$$

$$= \frac{p^2}{2m}$$

$$KE = \frac{p^2}{2m}$$

- where $|\vec{p}|$ is the magnitude of the momentum. The magnitude of the linear momentum can be obtained by

$$|\vec{p}| = p = \sqrt{2m(KE)}$$

- Note that if kinetic energy and mass are given, only the magnitude of the momentum can be calculated but not the direction of momentum. It is because the kinetic energy and mass are scalars.

Two objects of masses 2 kg and 4 kg are moving with the same momentum of 20 kg m s^{-1}.

1. Will they have same kinetic energy?
2. Will they have same speed?

The kinetic energy of the mass is given by $KE = \frac{p^2}{2m}$

For the object of mass 2 kg, kineticenergy is

$$KE_1 = \frac{(20)^2}{2\times 2} = \frac{400}{4} = 100J$$

For the object of mass 4 kg, kinetic energy is

$$KE_2 = \frac{(20)^2}{2\times 4} = \frac{400}{8} = 50J$$

- Note that $KE_1 \neq KE_2$ i.e., even though both are having the same momentum, the kinetic energy of both masses is not the same. The kinetic energy of the heavier object has lesser kinetic energy than smaller mass. It is because the kinetic energy is inversely proportional to the mass $\left(KE \alpha \frac{1}{m} \right)$ for a given momentum.

As the momentum, $p = mv$, the two objects will not have same speed.

Potential Energy

- The potential energy of a body is associated with its position and configuration with respect to its surroundings. This is because the various forces acting on the body also depends on position and configuration.

- "Potential energy of an object at a point Pis defined as the amount of work done by an external force in moving the object at constant velocity from the point O (initial location) to the point P (final location). At initial point O potential energy can be taken as zero.

Mathematically, potential energy is defined as $U = \int \vec{F}_a . d\vec{r}$

- where the limit of integration ranges from initial location point O to final location point P.

We have various types of potential energies. Each type is associated with a particular force.

1. The energy possessed by the body due to gravitational force gives rise to gravitational potential energy
2. The energy due to spring force and other similar forces give rise to elastic potential energy.
3. The energy due to electrostatic force on charges gives rise to electrostatic potential energy.

We will learn more about conservativeforces in the section. Now, we continueto discuss more about gravitational potential energy and elastic potential energy.

Potential energy near the surface of theEarth

- The gravitational potential energy (U) at some height *h* is equal to the amount of work required to take the object from ground to that height *h* with constant velocity.

- Let us consider a body of mass *m* being moved from ground to the height h against the gravitational force.

- The gravitational force $\overrightarrow{F_g}$ acting on the body is, $\overrightarrow{F_g} = -mg\hat{j}$ (as the force is in y direction, unit vector $\hat{j}$ is used). Here, negative sign implies that the force is acting vertically downwards. In order to move the body without acceleration (or with constant velocity), an external applied force $\overrightarrow{F_a}$ equal in magnitude but opposite to that of gravitational force $\overrightarrow{F_g}$ has to be applied on the body i.e., $\overrightarrow{F_a} = \overrightarrow{F_g}$. This implies that = $\overrightarrow{F_a} = +mg\hat{j}$. The positive sign implies that the applied force is in vertically upward direction. Hence, when the body is lifted up its velocity remains unchanged and thus its kinetic energy also remains constant.

The gravitational potential energy (U) at some height *h* is equal to the amount of work required to take the object from the ground to that height *h*.

$$U = \int \overrightarrow{F_a} d\vec{r} = \int_0^h \left|\overrightarrow{F_a}\right| \left|d\vec{r}\right| \cos\theta$$

Since the displacement and the applied force are in the same upward direction, the angle between them, $\theta = 0^o$. Hence, cos0º=1 and $\left|\overrightarrow{F_a}\right| = mg$ and $\left|d\vec{r}\right| = dr$.

$$U = mg\int_0^h dr$$

$$U = mg\left[r\right]_0^h = mgh$$

- Note that the potential energy stored in the object is defined through work done by the external force which is positive.Physically this implies that the agencywhich is applying the external force is transferring the energy to the object which is then stored as potential energy. If the object is allowed to fall from a height *h* then the stored potential energy is converted into kinetic energy.

An object of mass 2 kg is taken to a height 5 m from the ground _*g* _10ms^{-2}.

1. Calculate the potential energy stored in the object.
2. Where does this potential energy come from?
3. What external force must act to bring the mass to that height?
4. What is the net force that acts on the object while the object is taken to the height 'h'?

- The potential energy *U* =*mg h* = 2×10×5 = 100 J Here the positive sign implies that the energy is stored on the mass

- This potential energy is transferred from external agency which applies the force on the mass.

The external applied force $\vec{F_a}$ which takes the object to the height 5 m is $\vec{F_a} = -\vec{F_g}$.

$$\vec{F_a} = -\left(-mg\hat{j}\right) = mg\hat{j}$$

where, $\hat{j}$ represents unit vector along vertical upward direction.

From the definition of potential energy, the object must be moved at constant velocity. So the net force acting on the object is zero.

$$\vec{F_g} + \vec{F_a} = 0$$

Elastic Potential Energy

- When a spring is elongated, it develops a restoring force. The potential energy possessed by a spring due to a deforming force which stretches or compresses the spring is termed as elastic potential energy. The work done by the applied force against the restoring force of the spring is stored as the elastic potential energy in the spring.
- Consider a spring-mass system. Let us assume a mass, m lying on a smooth horizontal. Here, $x = 0$ is the equilibrium position. One end of the spring is attached to a rigid wall and the other end to the mass.
- As long as the spring remains in equilibrium position, its potential energy is zero. Now an external force $\vec{F_a}$ is applied so that it is stretched by a distance (x) in the direction of the force.
- There is a restoring force called spring force $\vec{F_s}$ developed in the spring which tries to bring the mass back to its original position. This applied force and the spring force are equal in magnitude but opposite in direction i.e., $\vec{F_a} = -\vec{F_s}$ According Hooke's law, the restoring force developed in the spring is

$$\vec{F_s} = -k\vec{x}$$

- The negative sign in the above expression implies that the spring force is always opposite to that of displacement $\vec{x}$ and k is the force constant. Therefore applied force is

$\overrightarrow{F_a} = +k\vec{x}$. The positive sign implies that the applied force is in the direction of displacement $\vec{x}$. The spring force is an example of variable force as it depends on the displacement $\vec{x}$. Let the spring be stretched to a small distance

$d\vec{x}$.The work done by the applied force on the spring to stretch it by a displacement $\vec{x}$ is stored as elastic potential energy.

$$U = \int \overrightarrow{F_a} . d\vec{r} = \int_0^x \left|\overrightarrow{F_a}\right| \left|d\vec{r}\right| \cos\theta$$

$$= \int_0^x F_a dx \cos\theta$$

- The applied force $\overrightarrow{F_a}$ and the displacement $d\vec{r}$ (i.e., here dx) are in the same direction. As, the initial position is taken as the equilibrium position or mean position, x=0 is the lower limit of integration.

$$U = \int_0^x kxdx$$

$$U = k\left[\frac{x^2}{2}\right]_0^x$$

$$U = \frac{1}{2}kx^2$$

- If the initial position is not zero, and if the mass is changed from position x_i to x_f ,then the elastic potential energy is

$$U = \frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

Force-displacement graph for a spring

- Since the restoring spring force and displacement are linearly related as $F = -k\,x$,and are opposite in direction, the graph between F and x is a straight line with dwelling only in the second and fourth quadrant as shown in Figure 4.10. Th e elastic potential energy can be easily calculatedby drawing a F - x graph. Th e shaded area (triangle) is the work done by the spring force.

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\times(x)\times(kx)$$

$$= \frac{1}{2}kx^2$$

Potential energy-displacement graph for a spring

- A compressed or extended spring will transfer its stored potential energy into kinetic energy of the mass attached to the spring.

In a frictionless environment, the energy gets transferred from kinetic to potential and potential to kinetic repeatedly such that the total energy of the system remains constant. At the mean position,

$$\Delta KE = \Delta U$$

- Let the two springs A and B be such that $k_A > k_B$. On which spring will more work has to be done if they are stretched by the same force?

$$F = K_A x_A = K_B x_B$$

$$x_A = \frac{F}{k_A}, x_B = \frac{F}{k_B}$$

- The work done on the springs are stored as potential energy in the springs.

$$U_A = \frac{1}{2}k_A x_A^2 \; ; \quad U_B = \frac{1}{2}k_B x_B^2$$

$$\frac{U_A}{U_B} = \frac{k_A x_A^2}{k_B x_B^2} = \frac{k_A\left(\frac{F}{k_A}\right)^2}{k_B\left(\frac{F}{k_B}\right)^2} = \frac{\frac{1}{k_A}}{\frac{1}{k_B}}$$

$$\frac{U_A}{U_B} = \frac{k_B}{k_A}$$

$k_A > k_B$ implies that $U_B > U_A$. Thus, more work is done on B than A.

- A body of mass m is attached to the spring which is elongated to 25 cm by an applied force from its equilibrium position.

1. Calculate the potential energy stored in the spring-mass system?
2. What is the work done by the spring force in this elongation?
3. Suppose the spring is compressed to the same 25 cm, calculate the potential energy stored and also the work done by the spring force during compression. (The spring constant, k = 0.1 N m^{-1}).

The spring constant, k = 0.1 N m^{-1}
The displacement, x = 25 cm = 0.25 m

The potential energy stored in the spring is given by

$$U = \frac{1}{2}kx^2 = \frac{1}{2} \times 0.1 \times (0.25)^2 = 0.0031 J$$

The work done W_s by the spring force $\vec{F_s}$ is given by,

$$W_s = \int_0^x \vec{F}_s \cdot d\vec{r} = \int_0^x \left(-kx\hat{i}\right) \cdot (dx\,\hat{i})$$

The spring force $\vec{F_s}$ acts in the negative x direction while elongation acts in thpositive x direction.

$$W_s = \int_0^x (-kx)dx = -\frac{1}{2}kx^2$$

$$W_s = -\frac{1}{2} \times 0.1 \times (0.25)^2 = -0.0031 J$$

- Note that the potential energy is defined through the work done by the external agency. The positive sign in the potential energy implies that the energy is transferred from the agency to the object. But the work done by the restoring force in this case is negative since restoring force is in the opposite direction to the displacement direction.

- During compression also the potential energy stored in the object is the same.

$$U = \frac{1}{2}kx^2 = 0.0031J$$

Work done by the restoring spring force during compression is given by

$$W_s = \int_0^x \vec{F}_s . d\vec{r} = \int_0^x (k x \hat{i}).(-dx \hat{i})$$

In the case of compression, the restoring spring force acts towards positive *x*-axis and displacement is along negative *x* direction.

$$W_s = \int_0^x (-kx) dx = -\frac{1}{2}kx^2 = -0.0031J$$

Conservative and nonconservativeforces
Conservative force

- A force is said to be a conservative force if the work done by or against the force in moving the body depends only on the initial and final positions of the body and not on the nature of the path followed between the initial and final positions. Let us consider an object at point A on the Earth. It can be taken to another point B at a height *h* above the surface of the Earth by three paths.

- Whatever may be the path, the work done against the gravitational force is the same as long as the initial and final positions are the same. This is the reason why gravitational force is a conservative force. Conservative force is equal to the negative gradient of the potential energy. In one dimensional case,Examples for conservative forces are elastic spring force, electrostatic force, magnetic force, gravitational force, etc.

S.No	Conservative forces	Non-conservative forces
1.	Work done is independent of the path	Work done depends upon the path
2.	Work done in a round trip is zero	Work done in a round trip is not zero
3	Total energy remains constant	Energy is dissipated as heat energy
4	Work done is completely recoverable	Work done is not completely recoverable.
5	Force is the negative gradient of potential energy	No such relation exists.

Non-conservative force

- A force is said to be non-conservative if the work done by or against the force in moving a body depends upon the path between the initial and final positions. This means that the value of work done is different in different paths.

1. Frictional forces are non-conservative forces as the work done against friction depends on the length of the path moved by the body.
2. The force due to air resistance, viscous force are also non-conservative forces as the work done by or against these forces depends upon the velocity of motion.

Compute the work done by the gravitational force for the following cases

$$\text{Force } \vec{F} = mg\left(-\hat{j}\right) = -mg\hat{j}$$

Displacement vector $d\vec{r} = dx\hat{i} + dy\hat{j}$

(As the displacement is in two dimension; unit vectors $\hat{i}$ and $\hat{j}$ are used)

- Since the motion is only vertical, horizontal displacement component dx is zero. Hence, work done by the force along path 1 (of distance h).

$$W_{path\,1} = \int_A^B \vec{F}.d\vec{r} = \int_A^B \left(-mg\hat{j}\right).\left(dy\,\hat{j}\right)$$

$$= -mg\int_0^h dy = -mgh$$

Total work done for path 2 is

$$W_{path\,2} = \int_A^B \vec{F}.d\vec{r} = \int_A^C \vec{F}.d\vec{r} + \int_C^D \vec{F}.d\vec{r} + \int_D^B \vec{F}.d\vec{r}$$

But

$$\int_A^C \vec{F}.d\vec{r} = \int_A^C \left(-mg\hat{j}\right).\left(dx\,\hat{i}\right) = 0$$

$$\int_C^D \vec{F}.d\vec{r} = \int_C^D \left(-mg\hat{j}\right).\left(dy\,\hat{j}\right)$$

$$= -mg\int_0^h dy = -mgh$$

$$\int_D^B \vec{F}.d\vec{r} = \int_D^B \left(-mg\hat{j}\right).\left(-dx\,\hat{i}\right) = 0$$

Therefore, the total work done by the force along the path 2 is

$$W_{\text{path 2}} = \int_A^B \vec{F}.d\vec{r} = -mgh$$

- Note that the work done by the conservative force is independent of the path.

- Consider an object of mass 2 kg moved by an external force 20 N in a surface having coefficient of kinetic friction 0.9 to a distance 10 m. What is the work done by the external force and kinetic friction ? Comment on the result. (Assume g = 10 ms^{-2})

m = 2 kg, d = 10 m, F_{ext}= 20 N, μ_k= 0.9. When an object is in motion on the horizontal surface, it experiences two forces.

1. External force, F_{ext}=20 N
2. Kinetic friction

$$f_k = \mu_k mg = 0.9 \times (2) \times 10 = 18\text{N}.$$

The work done by the external force W_{ext}=$F\ d$ ×20×10 ×200J

- The work done by the force of kinetic friction W_k=f_k= (−18)×10 = −180 J. Here the negative sign implies that the force of kinetic friction is opposite to the direction of displacement.

  The total work done on the object W_{total}=W_{ext}+W_k= 200 J −180 J = 20 J.

- Since the friction is a non-conservative force, out of 200 J given by the external force, the 180 J is lost and it can not be recovered.

Law of conservation of energy

- When an object is thrown upwards its kinetic energy goes on decreasing and consequently its potential energy keeps increasing (neglecting air resistance). When it reaches the highest point its energy is completely potential. Similarly, when the object falls back from a height its kinetic energy increases whereas its potential energy decreases. When it touches the ground its energy is completely kinetic. At the intermediate points the energy is both kinetic and potential When the body reaches the ground the kinetic energy is completely dissipated into some other form of energy like sound, heat, light and deformation of the body etc.

- In this example the energy transformation takes place at every point. The sum of kinetic energy and potential energy i.e., the total mechanical energy always remains constant, implying that the total energy is conserved. This is stated as the law of conservation of energy.

- The law of conservation of energystates that energy can neither be created nor destroyed. It may be transformed from one form to another but the total energy of an isolated system remains constant.

- illustrates that, if an object starts from rest at height h, the total energy is purely potential energy (U=mgh) and the kinetic energy (KE) is zero at h. When the object falls at some distance y, the potential energy and the kinetic energy are not zero whereas, the total energy remains same as measured at height h. When the object is about to touch the ground, the potential energy is zero and total energy is purely kinetic.

An object of mass 1 kg is falling from the height h = 10 m. Calculate

1. The total energy of an object at h =10 m
2. Potential energy of the object when it is at h = 4 m
3. Kinetic energy of the object when it is at h = 4 m
4. What will be the speed of the object when it hits the ground? (Assume g =10 ms^{-2})

- The gravitational force is a conservative force. So the total energy remains constant throughout the motion. At h =10 m, the total energy E is entirely potential energy.

$$E = U = mgh = 1 \times 10 \times 10 = 100 \text{ J}$$

The potential energy of the object at h =4 m is

$$U = mgh = 1 \times 10 \times 4 = 40 \text{ J}$$

- Since the total energy is constant throughout the motion, the kinetic energy at h=4m must be $KE = E - U = 100 - 40 = 60$J

- Alternatively, the kinetic energy could also be found from velocity of the object at 4 m. At the height 4 m, the object has fallen through a height of 6 m.

The velocity after falling 6 m is calculated from the equation of motion,

$$v = \sqrt{2gh} = \sqrt{2 \times 10 \times 6} = \sqrt{120} \text{ m s}^{-1};$$

$$v^2 = 120$$

The kinetic energy is $KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 1 \times 120 = 60\,J$

- When the object is just about to hit the ground, the total energy is completely kinetic and the potential energy, U=0.

$$E = KE = \frac{1}{2}mv^2 = 100\ J$$

$$v = \sqrt{\frac{2}{m}KE} = \sqrt{\frac{2}{1} \times 100} = \sqrt{200} \approx 14.12\ m\ s^{-1}$$

- A body of mass 100 kg is lifted to a height 10 m from the ground in two different ways as shown in the figure. What is the work done by the gravity in both the cases? Why is it easier to take the object through a ramp?

m = 100 kg, h = 10 m
Along path (1):

- The minimum force $F1$ required to move the object to the height of 10 m shouldbe equal to the gravitational force, $F_1 = mg$ =100×10 =1000 N

The distance moved along path (1) is, h=10 m

$$W = F\ h = 1000 \times 10 = 10{,}000\ J$$

Along path (2):

- In the case of the ramp, the minimum force F_2 that we apply on the object to take it up is not equal to mg, it is rather equal to $mg \sin\theta$. (mg $\sin\theta < mg$) .

Here, angle θ = 30^o

Therefore, F_2 = mg sinθ = 100 × 10 × sin30o = 100 × 10 × 0.5 =500N

Hence, (mg $\sin\theta$ <mg).

$$l = \frac{h}{\sin 30} = \frac{10}{0.5} = 20\,m$$

The work done on the object along path (2) is, $W = F_2 l = 500 \times 20 = 10{,}000$ J

- Since the gravitational force is a conservative force, the work done by gravity on the object is independent of the path taken.

  In both the paths the work done by the gravitational force is 10,000 J

Along path (1): more force needs to be applied against gravity to cover lesser distance .

Along path (2): lesser force needs to be applied against the gravity to cover more distance.

- As the force needs to be applied along the ramp is less, it is easier to move the object along the ramp.
- An object of mass m is projected from the ground with initial speed v_0. Find the speed at height h.
- Since the gravitational force is conservative; the total energy is conserved throughout the motion.

	Initial	Final
Kinetic energy	$\frac{1}{2}mv_0^2$	$\frac{1}{2}mv^2$
Potential energy	0	mgh
Total energy	$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_0^2$	$\frac{1}{2}mv^2 + mgh$

- Final values of potential energy, kinetic energy and total energy are measured a the height h.
- By law of conservation of energy, the initial and final total energies are the same.

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgh$$

$$v_0^2 = v^2 + 2gh$$

$$v = \sqrt{v_0^2 - 2gh}$$

- Note that in section similar result is obtained using kinematic equation based on calculus method. However, calculation through energy conservation method is much easier than calculus method.

- An object of mass 2 kg attached to a spring is moved to a distance x=10 m from its equilibrium position. The spring constant k=1 N m^{-1} and assume that the surface is frictionless.

1. When the mass crosses the equilibrium position, what is the speed of the mass?
2. What is the force that acts on the object when the mass crosses the equilibrium position and extremum position x = ±10 m.

- Since the spring force is a conservative force, the total energy is constant. At x=10 m, the total energy is purely potential.

$$E = U = \frac{1}{2}kx^2 = \frac{1}{2}\times(1)\times(10)^2 = 50\ J$$

When the mass crosses the equilibrium position (x=0) , the potential energy

$$U = \frac{1}{2}\times 1\times(0) = 0\ J$$

The entire energy is purely kinetic energy at this position.

$$E = KE = \frac{1}{2}mv^2 = 50\,J$$

The speed

$$v = \sqrt{\frac{2KE}{m}} = \sqrt{\frac{2\times 50}{2}} = \sqrt{50}\,ms^{-1} \approx 7.07\,ms^{-1}$$

- Since the restoring spring force is F = - kx, when the object crosses the equilibrium position, it experiences no force. Note that at equilibrium position, the object moves very fast. When the object is at x = +10 m (elongation), the force F = − k x

- F = − (1) (10) = − 10 N. Here the negative sign implies that the force is towards equilibrium i.e., towards negative x-axis and when the object is at x = − 10m (compression), it experiences a forces F = − (1) (− 10) = +10 N. Here the positive sign implies that the force points towards positive x-axis.

- The object comes to momentary rest at x = ±10m even though it experiences a maximum force at both these points.

Motion in a vertical circle

- Imagine that a body of mass (m) attached to one end of a massless and inextensible string executes circular motion in a vertical plane with the other end of the string fixed. The length of the string becomes the radius ($\vec{r}$) of the circular path

- Let us discuss the motion of the body by taking the free body diagram (FBD) at a position where the position vector ($\vec{r}$) makes an angle θ with the vertically downward direction and the instantaneous velocity.

There are two forces acting on the mass.

1. Gravitational force which acts downward
2. Tension along the string.

- Applying Newton's second law on the mass, In the tangential direction,

$$mg\sin\theta = ma_t$$

$$mg\sin\theta = -m\left(\frac{dv}{dt}\right)$$

where, $a_t = -\frac{dv}{dt}$ is tangential retardation

In the radial direction,

$$T - mg\cos\theta = m\,a_r$$

$$T - mg\cos\theta = \frac{mv^2}{r}$$

where, $a_r = \frac{v^2}{r}$ is the centripetal acceleration.

- The circle can be divided into four sections A, B, C, D for better understanding of the motion. The four important facts to be understood from the two equations are as follows:

1. The mass is having tangential acceleration (g sin θ) for all values of θ (except θ = 0o), it is clear that this vertical cirular motion is not a uniform circular motion.
2. From the equations (4.28) and (4.29) it is understood that as the magnitude of velocity is not a constant in the course of motion, the tension in the string is also not constant

The equation (4.29), $T = mg\cos\theta + \frac{mv^2}{r}$ highlights that in sections A and D of the circle, $\left(\text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2};\ \cos\theta \text{ is positive}\right)$, the term mg cosθ is always greater than zero. Hence the tension cannot vanish even when the velocity vanishes.

The equation (4.29), $\frac{mv^2}{r} = T - mg\cos\theta$; further highlights that in sections B and C of the circle, $\left(\text{for } \frac{\pi}{2} < \theta < \frac{3\pi}{2};\ \cos\theta \text{ is negative}\right)$, the second term is always greater than zero. Hence velocity cannot vanish, even when the tension vanishes.

- These points are to be kept in mind while solving problems related to motion in vertical circle.
- To start with let us consider only two positions, say the lowest point 1 and the highest point 2 as shown in Figure 4.15 for further analysis. Let the velocity of the body at the lowest point 1 be $\vec{v}_1$, at the highest point 2 be $\vec{v}_2$ and $\vec{v}$ at any other point. The direction of velocity is tangential to the circular path at all points. Let $\vec{T}_1$ be the tension in the string at the lowest point and $\vec{T}_2$ be the tension at the highest point and $\vec{T}$ be the tension at any other point. Tension at each point acts towards the center. The tensions and velocities at these two points can be found by applying the law of conservation of energy.

$$T_1 - mg = \frac{mv_1^2}{r}$$

$$T_1 = \frac{mv_1^2}{r} + mg$$

- At the highest point 2, both the gravitational force $m\vec{g}$ on the body and the tension $\vec{T}_2$ act downwards, i.e. towards the center again.

$$T_2 + mg = \frac{mv_2^2}{r}$$

$$T_2 = \frac{mv_2^2}{r} - mg$$

$$T_1 - T_2 = \frac{mv_1^2}{r} + mg - \left(\frac{mv_2^2}{r} - mg\right)$$

$$= \frac{mv_1^2}{r} + mg - \frac{mv_2^2}{r} + mg$$

$$T_1 - T_2 = \frac{m}{r}\left[v_1^2 - v_2^2\right] + 2mg \qquad (3)$$

- The term $[v_1^2 - v_2^2]$ can be found easily by applying law of conservation of energy.

- Total Energy at point 1(E_1) is same as the total energy at a point 2 (E_2)

$$E_1 = E_2$$

Potential Energy at point 1, U_1=0 (by taking reference as point 1)

Kinetic Energy at point 1,

$$KE_1 = \frac{1}{2}mv_1^2$$

Total Energy at point 1,

$$E_1 = U_1 + KE_1 = 0 + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_1^2$$

Similarly, Potential Energy at point 2, U_2=mg (2r)

Kinetic Energy at point 2,

$$KE_2 = \frac{1}{2}mv_2^2$$

Total Energy at point 2,

$$E_2 = U_2 + KE_2 = \quad 2\text{ mg r} + \frac{1}{2}mv_2^2$$

From the law of conservation of energy given in equation

$$\frac{1}{2}mv_1^2 = 2mgr + \frac{1}{2}mv_2^2$$

After rearranging,

$$\frac{1}{2}m\left(v_1^2 - v_2^2\right) = 2mgr$$

$$v_1^2 - v_2^2 = 4gr$$

Substituting equation

$$T_1 - T_2 = \frac{m}{r}[4gr] + 2mg$$

Therefore, the difference in tension is

$$T_1 - T_2 = 6\text{ mg}$$

- The body must have a minimum speed at point 2 otherwise, the string will slack before reaching point 2 and the body will not loop the circle. To find this minimum speed let us take the tension $T_2 = 0$ in equation

$$0 = \frac{mv_2^2}{r} - mg$$

$$\frac{mv_2^2}{r} = mg$$

$$v_2^{\,2} = rg$$

$$v_2 = \sqrt{gr}$$

- The body must have a speed at point 2, $v_2 \geq \sqrt{gr}$ to stay in the circular path.

- To have this minimum speed $\left(v_2 = \sqrt{gr}\right)$ at point 2, the body must have minimum speed also at point 1.

- By making use of equation (4.36) we can find the minimum speed at point 1.

$$v_1^2 - v_2^2 = 4gr$$

Substituting equation

$$v_1^2 - gr = 4gr$$

$$v_1^2 = 5gr$$

$$v_1 = \sqrt{5gr}$$

The body must have a speed at point 1, $v_1 \geq \sqrt{5gr}$ to stay in the circular path.

It is clear that the minimum speed at the lowest point 1 should be $\sqrt{5}$ times more than the minimum speed at the highest point 2, so that the body loops without leaving the circle.

- Water in a bucket tied with rope is whirled around in a vertical circle of radius 0.5 m. Calculate the minimum velocity at the lowest point so that the water does not spill from it in the course of motion. ($g = 10\ ms^{-2}$)

Radius of circle r = 0.5 m

The required speed at the highest point

$v_2 = \sqrt{gr} = \sqrt{10 \times 0.5} = \sqrt{5}\ ms^{-1}$. The speed at lowest point $v_1 = \sqrt{5gr} = \sqrt{5} \times \sqrt{gr} = \sqrt{5} \times \sqrt{5} = 5\ ms^{-1}$

POWER
Definition of Power

- Power is a measure of how fast or slow a work is done. Power is defined as the rate of work done or energy delivered.

$$\text{Power } (P) = \frac{\text{work done}(W)}{\text{time taken}(t)}$$

$$P = \frac{W}{t}$$

Average power

The average power (P_{av}) is defined as the ratio of the total work done to the total time taken.

$$P_{av} = \frac{\text{total work done}}{\text{total time taken}}$$

Instantaneous power

- The instantaneous power (P_{inst}) is defined as the power delivered at an instant (as time interval approaches zero),

$$P_{inst} = \frac{dW}{dt}$$

Unit of Power

- Power is a scalar quantity. Its dimension is[ML^2T^{-3}]. The SI unit of power is watt (W),named after the inventor of the steam engineJames Watt. One watt is defined as the powerwhen one joule of work is done in one second,(1 W = 1 J s^{-1}).

  The higher units are kilowatt(kW), megawatt(MW), and Gigawatt(GW).

$$1kW = 1000\ W = 10^3 \text{ watt}$$
$$1MW = 10^6 \text{ watt}$$
$$1GW = 10^9 \text{ watt}$$

- For motors, engines and some automobiles an old unit of power still commercially in use which is called as the horse-power (hp). We have a conversion for horse-power (hp) into watt (W) which is,

$$1 \text{ hp} = 746 \text{ W}$$

- All electrical goods come with a definite power rating in watt printed on them. A 100 watt bulb consumes 100 joule of electrical energy in one second. The energy measured in joule in terms of power in watt and time in second is written as, 1 J =1 W s. When electrical appliances are put in use for long hours, they consume a large amount of energy. Measuring the electrical energy in a small unit watt.second (W s) leads to handling large numerical values. Hence, electrical energy is measured in the unit called kilowatt hour (kWh).

$$1 \text{ electrical unit} = 1 \text{ kWh} = 1 \times (10^3 \text{ W}) \times (3600 \text{ s})$$

$$1 \text{ electrical unit} = 3600 \times 10^3 \text{ W s}$$

$$1 \text{ electrical unit} = 3.6 \times 10^6 \text{ J}$$

$$1 \text{ kWh} = 3.6 \times 10^6 \text{ J}$$

- Electricity bills are generated in units of kWh for electrical energy consumption. 1 unit of electrical energy is 1 kWh. (Note: kWh is unit of energy and not of power.)

- Calculate the energy consumed in electrical units when a 75 W fan is used for 8 hours daily for one month (30 days).

Power, P = 75 W

- Time of usage, t = 8 hour × 30 days = 240 hours
- Electrical energy consumed is the product of power and time of usage.

  Electrical energy = power × time of usage = P × t

$$= 75 \text{ watt} \times 240 \text{ hour}$$

$$= 18000 \text{ watt hour}$$

$$= 18 \text{ kilowatt hour} = 18 \text{ kWh}$$

$$1 \text{ electrical unit} = 1 \text{ kWh}$$

$$\text{Electrical energy} = 18 \text{ unit}$$

Incandescent lamps glow for 1000 hours. CFL lamps glow for 6000 hours. But LED lamps glow for 50000 hrs (almost 25 years at 5.5 hour per day).

Relation between power and velocity

The work done by a force $\vec{F}$ for a displacement $d\vec{r}$ is

$$W = \int \vec{F} \cdot d\vec{r}$$

Left hand side of the equation (4.40) can be written as

$$W = \int dW = \int \frac{dW}{dt} dt$$

Since, velocity is $\vec{v} = \frac{d\vec{r}}{dt}; d\vec{r} = \vec{v}dt.$. Right hand side of the equation (4.40) can be written as

$$\int \vec{F} \cdot d\vec{r} = \int \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt = \int \left(\vec{F} \cdot \vec{v} \right) dt \quad \left[\vec{v} = \frac{d\vec{r}}{dt} \right]$$

$$\int \frac{dW}{dt} dt = \int \left(\vec{F} \cdot \vec{v} \right) dt$$

$$\int \left(\frac{dW}{dt} - \vec{F} \cdot \vec{v} \right) dt = 0$$

- This relation is true for any arbitrary value of dt. This implies that the term within the bracket must be equal to zero, i.e.,

$$\frac{dW}{dt} - \vec{F} \cdot \vec{v} = 0$$

Or

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

- A vehicle of mass 1250 kg is driven with an acceleration 0.2 ms^{-2} along a straight level road against an external resistive force 500 N. Calculate the power delivered by the vehicle's engine if the velocity of the vehicle is 30 ms^{-1}.

- The vehicle's engine has to do work against resistive force and make vechile to move with an acceleration. Therefore, power delivered by the vehicle engine is

$$P = (\text{resistive force} + \text{mass} \times \text{acceleration})\ (\text{velocity})$$

$$P = \vec{F}_{-tot}\vec{v} = (F_{resistive} + F)\vec{v}$$

$$P = \vec{F}_{tot} \cdot \vec{v} = (F_{resistive} + ma)\vec{v}$$

$$= (500\ N + (1250\ kg) \times (0.2\ ms^{-2}))\ (30\ ms^{-1}) = 22.5\ kW$$

COLLISIONS

- Collision is a common phenomenon that happens around us every now and then. For example, carom, billiards, marbles, etc.,. Collisions can happen between two bodies with or without physical contacts.

- Linear momentum is conserved in all collision processes. When two bodies collide, the mutual impulsive forces acting between them during the collision time (Δt) produces a change in their respective momenta. That is, the first body exerts a force $\vec{F}_{12}$ on the second body. From Newton's third law, the second body exerts a force $\vec{F}_{21}$ on the first body. This causes a change in momentum $\Delta\vec{p}_1$ and $\Delta\vec{p}_2$ of the first body and second body respectively. Now, the relations could be written as,

$$\Delta\vec{p}_1 = \vec{F}_{12}\Delta t$$

$$\Delta\vec{p}_2 = \vec{F}_{21}\Delta t$$

Adding equation

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = \vec{F}_{12}\Delta t + \vec{F}_{21}\Delta t = \left(\vec{F}_{12} + \vec{F}_{21}\right)\Delta t$$

According to Newton's third law, $\vec{F}_{12} = -\vec{F}_{21}$

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = 0$$

$$\Delta\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

Dividing both sides by Δt and taking limit Δ t→0 , we get

$$\lim_{\Delta t \to 0} \frac{\Delta\left(\vec{p}_1 + \vec{p}_2\right)}{\Delta t} = \frac{d\left(\vec{p}_1 + \vec{p}_2\right)}{dt} = 0$$

- The above expression implies that the total linear momentum is a conserved quantity. Note: The momentum is a vector quantity. Hence, vector addition has to be followed to find the total momentum of the individual bodies in collision.

Types of collections

- In any collision process, the total linearmomentum and total energy are alwaysconserved whereas the total kinetic energyneed not be conserved always. Some partof the initial kinetic energy is transformedto other forms of energy. This is because,the impact of collisions and deformationoccurring due to collisions may in general,produce heat, sound, light etc. By takingthese effects into account, we classify thetypes of collisions as follows:

1. Elastic collision
2. Inelastic collision

Elastic collision

- In a collision, the total initial kinetic energy of the bodies (before collision) is equal to the total final kinetic energy of the bodies (after collision) then, it is called as elastic collision. i.e.,

- Total kinetic energy before collision = Total kinetic energy after collision

Inelastic collision

- In a collision, the total initial kinetic energy of the bodies (before collision) is not equal to the total final kinetic energy of the bodies (after collision) then, it is called as inelastic collision. i.e.,

- Total kinetic energy before collision ≠ Total kinetic energy after collision

$$\begin{pmatrix}\text{Total kinetic energy}\\ \text{after collision}\end{pmatrix} - \begin{pmatrix}\text{Total kinetic energy}\\ \text{before collision}\end{pmatrix} = \begin{pmatrix}\text{loss in energy}\\ \text{during collision}\end{pmatrix} = \Delta Q$$

- Even though kinetic energy is not conserved but the total energy is conserved. This is because the total energy contains the kinetic energy term and also a term ΔQ, which includes all the losses that take place during collision. Note that loss in kinetic energy during collision is transformed to another form of energy like sound, thermal, etc. Further, if the two colliding bodies stick together after collision such collisions are known as completely inelastic collision or perfectly inelastic collision. Such a collision is found very often. For example when a clay putty is thrown on a moving vehicle, the clay putty (or Bubblegum) sticks to the moving vehicle and they move together with the same velocity.

Elastic collisions in one dimension

- Consider two elastic bodies of masses m_1 and m_2 moving in a straight line (along positive x direction) on a frictionless horizontal surface.

Mass	Initial velocity	Final velocity
Mass m_1	u_1	v_1
Mass m_2	u_2	v_2

- In order to have collision, we assume that the mass m_1 moves faster than mass m_2 i.e., $u_1>u_2$. For elastic collision, the total linear momentum and kinetic energies of the two bodies before and after collision must remain the same.

S.No.	Elastic Collision	Inelastic Collision
1.	Total momentum is conserved	Total momentum is conserved
2.	Total kinetic energy is conserved	Total kinetic energy is not conserved
3.	Forces involved are conservative forces	Forces involved are non-conservative forces
4.	Mechanical energy is not dissipated.	Mechanical energy is dissipated into heat, light, sound etc.

	Momentum of mass m_1	Momentum of mass m_2	Total linear momentum
Before collision	$p_{i1} = m_1u_1$	$p_{i2} = m_2u_2$	$p_i = p_{i1} + p_{i2}$ $p_i = m_1u_1 + m_2u_2$
After collision	$p_{f1} = m_1v_1$	$p_{f2} = m_2v_2$	$p_f = p_{f1} + p_{f2}$ $p_f = m_1v_1 + m_2v_2$

From the law of conservation of linear momentum,

- Total momentum before collision (p_i) = Total momentum after collision (p_f)

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$\text{Or } \; m_1(u_1 - v_1) = m_2(v_2 - u_2)$$

	Kinetic energy of mass m_1	Kinetic energy of mass m_2	Total kinetic energy
Before collision	$KE_{i1} = \frac{1}{2}m_1u_1^2$	$KE_{i2} = \frac{1}{2}m_2u_2^2$	$KE_i = KE_{i1} + KE_{i2}$ $KE_i = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$
After collision	$KE_{f1} = \frac{1}{2}m_1v_1^2$	$KE_{f2} = \frac{1}{2}m_2v_2^2$	$KE_i = KE_{i1} + KE_{i2}$ $KE_f = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

Total kinetic energy before collision KE_i = Total kinetic energy after collision KE_f

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

After simplifying and rearranging the terms,

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$$

Using the formula $a^2 - b^2 = (a+b)(a-b)$ we can rewrite the above equation as

$$m_1(u_1 + v_1)(u_1 - v_1) = m_2(v_2 + u_2)(v_2 - u_2)$$

$$\frac{m_1(u_1 + v_1)(u_1 - v_1)}{m_1(u_1 - v_1)} = \frac{m_2(v_2 + u_2)(v_2 - u_2)}{m_2(v_2 - u_2)}$$

$$u_1 + v_1 = v_2 + u_2 \quad \text{Rearranging,} \quad (4.50)$$

$$u_1 - u_2 = v_2 - v_1$$

Equation (4.50) can be rewritten as

$$u_1 - u_2 = -(v_1 - v_2)$$

- This means that for any elastic head on collision, the relative speed of the two elastic bodies after the collision has the same magnitude as before collision but in opposite direction. Further note that this result is independent of mass.

Rewriting the above equation for v_1 and v_2,

$$v_1 = v_2 + u_2 - u_1$$

Or

$$v_2 = u_1 + v_1 - u_2$$

To find the final velocities v_1 and v_2:

- Substituting equation (4.52) in equation (4.47) gives the velocity of m_1 as

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - u_2 - u_2)$$

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - 2u_2)$$

$$m_1u_1 - m_1v_1 = m_2u_1 + m_2v_1 - 2m_2u_2$$

$$m_1u_1 - m_2u_1 + 2m_2u_2 = m_1v_1 + m_2v_1$$

$$(m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$$

$$\text{or } v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

- Similarly, by substituting (4.51) in equation (4.47) or substituting equation (4.53) in equation (4.52), we get the final velocity of m_2 as

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

When bodies has the same mass i.e., $m_1 = m_2$,

$$\Rightarrow v_1 = (0)\ u_1 + \left(\frac{2m_2}{2m_2}\right)u_2$$

$$v_1 = u_2$$

$$\Rightarrow v_2 = \left(\frac{2m_1}{2m_1}\right)u_1 + (0)u_2$$

$$v_2 = u_1$$

- The equations (4.55) and (4.56) show that in one dimensional elastic collision, when two bodies of equal mass collide after the collision their velocities are exchanged.

- When bodies have the same mass i.e., $m_1 = m_2$ and second body (usually called target) is at rest ($u_2 = 0$),

- By substituting $m_1 = m_2$ and $u_2 = 0$ in equations (4.53) and equations (4.54) we get,body moves with the initial velocity of the first body.

$$\Rightarrow v_1 = 0$$

$$\Rightarrow v_2 = u_1$$

The first body is very much lighter than the second body

$$\left(m_1 \ll m_2, \frac{m_1}{m_2} \ll 1\right) \text{ th n th ratio } \frac{m_1}{m_2} \approx 0$$

Dividing numerator and denominator of equation (4.53) by m2, we get

$$v_1 = \left(\frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1}\right) u_1 + \left(\frac{2}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_1 = \left(\frac{0-1}{0+1}\right) u_1$$

$$v_1 = -u_1$$

- Dividing numerator and denominator of equation (4.54) by m2, we get

$$v_2 = \left(\frac{2\frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right) u_1 + \left(\frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_2 = (0) u_1 + \left(\frac{1 - \frac{m_1}{m_2}}{\frac{m_1}{m_2} + 1}\right)(0)$$

$$v_2 = 0$$

- The equation (4.59) implies that the first body which is lighter returns back (rebounds) in the opposite direction with the same initial velocity as it has a negative sign. The equation (4.60) implies that the second body which is heavier in mass continues to remain at rest even after collision. For example, if a ball is thrown at a fixed wall, the ball will bounce back from the wall with the same velocity with which it was thrown but in opposite direction.

- The second body is very much lighter than the first body

$$\left(m_2 \ll m_1, \frac{m_2}{m_1} \ll 1\right) \text{ then the ratio } \frac{m_2}{m_1} \approx 0$$

$$v_1 = \left(\frac{1-\frac{m_2}{m_1}}{1+\frac{m_2}{m_1}}\right)u_1 + \left(\frac{2\frac{m_2}{m_1}}{1+\frac{m_2}{m_1}}\right)(0)$$

$$v_1 = \left(\frac{1-0}{1+0}\right)u_1 + \left(\frac{0}{1+0}\right)(0)$$

$$v_1 = u_1$$

- Dividing numerator and denominator of equation (4.58) by m1, we get

$$v_2 = \left(\frac{2}{1+\frac{m_2}{m_1}}\right)u_1 + \left(\frac{\frac{m_2}{m_1}-1}{1+\frac{m_2}{m_1}}\right)(0)$$

$$v_2 = \left(\frac{2}{1+0}\right)u_1$$

$$v_2 = 2u_1$$

- The equation (4.61) implies that the first body which is heavier continues to move with the same initial velocity. The equation (4.62) suggests that the second body which is lighter will move with twice the initial velocity of the first body. It means that the lighter body is thrown away from the point of collision.

- A lighter particle moving with a speed of 10 m s^{-1} collides with an object of double its mass moving in the same direction with half its speed. Assume that the collision is a one dimensional elastic collision. What will be the speed of both particles after the collision?

- Let the mass of the first body be m which moves with an initial velocity, u1 = 10 m s-1. Therefore, the mass of second body is 2m and its initial velocity is

$$u_2 = \frac{1}{2}u_1 = \frac{1}{2}\left(10\ \mathrm{m\ s^{-1}}\right),$$

- Then, the final velocities of the bodies can be calculated from the equation (4.53) and equation (4.54)

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = \left(\frac{m - 2m}{m + 2m}\right)10 + \left(\frac{2 \times 2m}{m + 2m}\right)5$$

$$v_1 = -\left(\frac{1}{3}\right)10 + \left(\frac{4}{3}\right)5 = \frac{-10 + 20}{3} = \frac{10}{3}$$

$$v_1 = 3.33\ \mathrm{ms^{-1}}$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

$$v_2 = \left(\frac{2m}{m + 2m}\right)10 + \left(\frac{2m - m}{m + 2m}\right)5$$

$$v_2 = \left(\frac{2}{3}\right)10 + \left(\frac{1}{3}\right)5 = \frac{20 + 5}{3} = \frac{25}{3}$$

$$v_2 = 8.33\ \mathrm{ms^{-1}}$$

As the two speeds v_1 and v_2 are positive, they move in the same direction with the velocities, 3.33 m s^{-1} and 8.33 m s^{-1} respectively.

Perfect inelastic collision

- In a perfectly inelastic or completely inelastic collision, the objects stick together permanently after collision such that they move with common velocity. Let the two bodies with masses m_1 and m_2 move with initial velocities u_1 and u_2 respectively before collision. After perfect inelastic collision both the objects move together with a common velocity v

Since, the linear momentum is conserved during collisions,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$$

	Velocity		Linear momentum	
	Initial	Final	Initial	Final
Mass m_1	u_1	v	$m_1 u_1$	$m_1 v$
Mass m_2	u_2	v	$m_2 u_2$	$m_2 v$
	Total		$m_1 u_1 + m_2 u_2$	$(m_1 + m_2) v$

The common velocity can be computed by

$$v = \frac{m_1 u_1 + m_2 u_2}{(m_1 + m_2)}$$

- A bullet of mass 50 g is fired from below into a suspended object of mass 450 g. The object rises through a height of 1.8 m with bullet remaining inside the object. Find the speed of the bullet. Take g = 10 ms^{-2}.

$$m_1 = 50 \text{ g} = 0.05 \text{ kg}; \quad m_2 = 450 \text{ g} = 0.45 \text{ kg}$$

- The speed of the bullet is u1. The second body is at rest u_2= 0 . Let the common velocity of the bullet and the object after the bullet is embedded into the object is v.

$$v = \frac{m_1 u_1 + m_2 u_2}{(m_1 + m_2)}$$

$$v = \frac{0.05 u_1 + (0.45 \times 0)}{(0.05 + 0.45)} = \frac{0.05}{0.50} u_1$$

- The combined velocity is the initial velocity for the vertical upward motion of the combined bullet and the object. From second equation of motion,

$$v = \sqrt{2gh}$$

$$v = \sqrt{2 \times 10 \times 1.8} = \sqrt{36}$$

$$v = 6\,ms^{-1}$$

Substituting this in the above equation, the value of u_1 is

$$6 = \frac{0.05}{0.50}u_1 \quad \text{or} \quad u_1 = \frac{0.50}{0.05} \times 6 = 10 \times 6$$

$$u_1 = 60\,ms^{-1}$$

Loss of kinetic energy in perfect inelastic collision

- In perfectly inelastic collision, the loss in kinetic energy during collision is transformed to another form of energy like sound, thermal, heat, light etc. Let KE_i be the total kinetic energy before collision and KE_f be the total kinetic energy after collision.

Total kinetic energy before collision,

$$KE_i = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$$

$$KE_f = \frac{1}{2}(m_1 + m_2)v^2$$

Then the loss of kinetic energy is Loss of KE, $\Delta Q = KE_i - KE_f$

$$= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(m_1 + m_2)v^2$$

Substituting equation (4.63) in equation (4.66), and on simplifying (expand v by using the algebra

$$(a+b)^2 = a^2 + b^2 + 2ab,$$

$$\text{Loss of KE, } \Delta Q = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 - u_2)^2$$

Coefficient of restitution (e)

- Suppose we drop a rubber ball and a plasticball on the same floor. The rubber ballwill bounce back higher than the plasticball. This is because the loss of kineticenergy for an elastic ball is much lesserthan the loss of kinetic energy for a plasticball. The amount of kinetic energy afterthe collision of two bodies, in general,can be measured through a dimensionlessnumber called the coefficient of restitution(COR).

- It is defined as the ratio of velocity ofseparation (relative velocity) after collisionto the velocity of approach (relative velocity)before collision, i.e.,

$$e = \frac{\text{velocity of separation (after collision)}}{\text{velocity of approach (before collision)}}$$

$$= \frac{(v_2 - v_1)}{(u_1 - u_2)}$$

- In an elastic collision, we have obtained the velocity of separation is equal to the velocity of approach i.e.,

$$(u_1 - u_2) = (v_2 - v_1) \rightarrow \frac{(v_2 - v_1)}{(u_1 - u_2)} = 1 = e$$

- This implies that, coefficient of restitution for an elastic collision, e=1. Physically, it means that there is no loss of kinetic energy after the collision. So, the body bounces back with the same kinetic energy which is usually called as perfect elastic.

- In any real collision problems, there will be some losses in kinetic energy due to collision, which means e is not always equal to unity. If the ball is perfectly plastic, it will never bounce back and therefore their separation of velocity is zero after the collision. Hence, the value of coefficient of restitution, e=0.

- In general, the coefficient of restitution for a material lies between

$$0 < e < 1.$$

- Show that the ratio of velocities of equal masses in an inelastic collision when one of the masses is stationary is

$$\frac{v_1}{v_2} = \frac{1-e}{1+e}.$$

$$e = \frac{\text{velocity of separation (after collision)}}{\text{velocity of approach (before collision)}}$$

$$= \frac{(v_2 - v_1)}{(u_1 - u_2)} = \frac{(v_2 - v_1)}{(u_1 - 0)} = \frac{(v_2 - v_1)}{u_1}$$

$$\Rightarrow v_2 - v_1 = e u_1$$

From the law of conservation of linear momentum,

$$m u_1 = m v_1 + m v_2 \Rightarrow u_1 = v_1 + v_2 \ (2)$$

Using the equation (2) for u1 in (1), we get

$$v_2 - v_1 = e(v_1 + v_2)$$

On simplification, we get

$$\frac{v_1}{v_2} = \frac{1-e}{1+e}$$

UNIT- 5 MOTION OF SYSTEM OF PARTICLES AND RIGID BODIES

INTRODUCTION

Most of the objects that we come across in our day to day life consist of large number of particles. In the previous Units, we studied the motion of bodies without considering their size and shape. So far we have treated even the bulk bodies as only point objects. In this section, we will give importance to the size and shape of the bodies. These bodies are actually made up of a large number of particles. When such a body moves, we consider it as the motion of collection of particles as a whole. We define the concept of center of mass to deal with such a system of particles.

The forces acting on these bulk bodies are classified into internal and external forces. Internal forces are the forces acting among the particles within a system that constitute the body. External forces are the forces acting on the particles of a system from outside. In this unit, we deal with such system of particles which make different rigid bodies. A rigid body is the one which maintains its definite and fixed shape even when an external force acts on it. This means that, the interatomic distances do not change in a rigid body when an external force is applied. However, in real life situation, we have bodies which are not ideally rigid, because the shape and size of the body change when forces act on them. For the rigid bodies we study here, we assume that such deformations are negligible. The deformations produced on non-rigid bodies are studied separately in Unit 7 under elasticity of solids.

CENTER OF MASS

When a rigid body moves, all particles that constitute the body need not take the same path. Depending on the type of motion, different particles of the body may take different paths. For example, when a wheel rolls on a surface, the path of the center point of the wheel and the paths of other points of the wheel are different. In this Unit, we study about the translation, rotation and the combination of these motions of rigid bodies in detail.

Center of Mass of a Rigid Body

When a bulk object (say a bat) is thrown at an angle in air as shown in Figure 5.1; do all the points of the body take a parabolic path? Actually, only one point takes the parabolic path and all the other points take different paths.

The one point that takes the parabolic path is a very special point called center of mass (CM) of the body. Its motion is like the motion of a single point that is thrown. The center of mass of a body is defined as a point where the entire mass of the body appears to be concentrated. Therefore, this point can represent the entire body.

For bodies of regular shape and uniform mass distribution, the center of mass is at the geometric center of the body. As examples, for a circle and sphere, the center of mass is at their centers; for square and rectangle, at the point their diagonals meet; for cube and cuboid, it is at the point where their body diagonals meet. For other bodies, the center of mass has to be determined using some methods. The center of mass could be well within the body and in some cases outside the body as well.

Center of Mass for Distributed Point Masses

A point mass is a hypothetical point particle which has nonzero mass and no size or shape. To find the center of mass for a collection of n point masses, say, m1, m2, m3 . . . mn we have to first choose an origin and an appropriate coordinate system as shown in Figure 5.2. Let, x1, x2, x3 . . . xn be the X-coordinates of the positions of these point masses in the X direction from the origin.

The equation for the x coordinate of the center of mass is,

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i}$$

where, $\sum m_i$ is the total mass M of all theparticles, $\left(\sum m_i = m\right)$.

$$x_{CM} = \frac{\sum m_i x_i}{M}$$

Similarly, we can also find y and z coordinates of the center of mass for these distributed point masses as indicated in Figure (5.2).

$$y_{CM} = \frac{\sum m_i y_i}{M}$$

$$z_{CM} = \frac{\sum m_i z_i}{M}$$

Hence, the position of center of mass of these point masses in a Cartesian coordinate system is (x_{CM}, y_{CM}, z_{CM}). In general, the position of center of mass can be written in a vector form as,

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M}$$

where, $\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$ is the position vector of the center of mass and $\vec{r} = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ is the position vector of the distributed point mass; where, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along X, Y and Z-axes respectively.

Center of Mass of Two Point Masses

With the equations for center of mass, let us find the center of mass of two point masses m_1 and m_2, which are at positions x_1 and x_2 respectively on the X-axis. For this case, we can express the position of center of mass in the following three ways based on the choice of the coordinate system.

When the masses are on positive X-axis:

The origin is taken arbitrarily so that the masses m_1 and m_2 are at positions x_1 and x_2 on the positive X-axis as shown in Figure 5.3(a). The center of mass will also be on the positive X-axis at x_{CM} as given by the equation,

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

When the origin coincides with any one of the masses:

The calculation could be minimised if the origin of the coordinate system is made to coincide with any one of the masses as shown in Figure 5.3(b). When the origin coincides with the point mass m_1, its position x_1 is zero, (i.e. $x_1 = 0$). Then,

$$x_{CM} = \frac{m_1(0) + m_2x_2}{m_1 + m_2}$$

The equation further simplifies as,

$$x_{CM} = \frac{m_2x_2}{m_1 + m_2}$$

When the origin coincides with the center of mass itself:

If the origin of the coordinate system is made to coincide with the center of mass, then, $x_{CM} = 0$ and the mass m_1 is found to be on the negative X-axis as shown in Figure 5.3(c). Hence, its position x_1 is negative, (i.e. $-x_1$).

$$0 = \frac{m_1(-x_1) + m_2 x_2}{m_1 + m_2}$$

$$0 = m_1(-x_1) + m_2 x_2$$

$$m_1 x_1 = m_2 x_2$$

The equation given above is known as principle of moments.

EXAMPLE

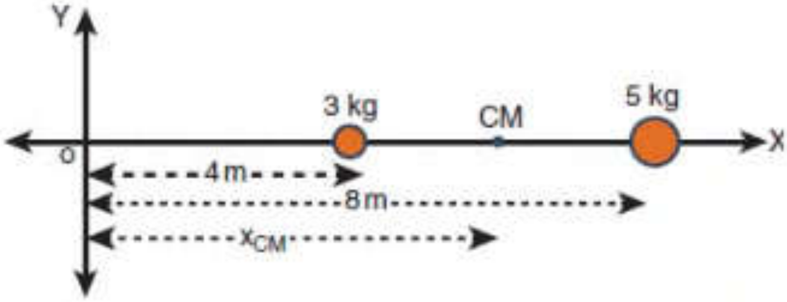
Two point masses 3 kg and 5 kg are at 4 m and 8 m from the origin on X-axis. Locate the position of center of mass of the two point masses (i) from the origin and (ii) from 3 kg mass.

Solution

Let us take, $m_1 = 3$ kg and $m_2 = 5$ kg

To find center of mass from the origin:

The point masses are at positions, $x_1 = 4$ m, $x_2 = 8$ m from the origin along X axis.



The center of mass x_{CM} can be obtained using equation

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$x_{CM} = \frac{(3\times4)+(5\times8)}{3+5}$$

$$x_{CM} = \frac{12+40}{8} = \frac{52}{8} = 6.5\,\text{m}$$

The center of mass is located 6.5 m from the origin on X-axis.

To find the center of mass from 3 kg mass:

The origin is shifted to 3 kg mass along X-axis. The position of 3 kg point mass is zero ($x_1 = 0$) and the position of 5 kg point mass is 4 m from the shifted origin ($x_2 = 4$ m).

$$x_{CM} = \frac{(3\times0)+(5\times4)}{3+5}$$

$$x_{CM} = \frac{0+20}{8} = \frac{20}{8} = 2.5\,\text{m}$$

The center of mass is located 2.5 m from 3 kg point mass, (and 1.5 m from the 5 kg point mass) on X-axis.

When we compare case (i) with case (ii), the x_{CM} = 2.5 m from 3 kg mass could also be obtained by subtracting 4 m (the position of 3 kg mass) from 6.5 m, where the center of mass was located in case (i)

EXAMPLE

From a uniform disc of radius R, a small disc of radius $\frac{R}{2}$ is cut and removed as shown in the diagram. Find the center of mass of the remaining portion of the disc.

Solution

Let us consider the mass of the uncut full disc be M. Its center of mass would be at the geometric center of the disc on which the origin coincides.

Let the mass of the small disc cut and removed be m and its center of mass is at a position $\frac{R}{2}$ to the right of the origin as shown in the figure.

Hence, the remaining portion of the disc should have its center of mass to the left of the origin; say, at a distance x. We can write from the principle of moments,

$$(M-m)x = (m)\frac{R}{2}$$
$$x = \left(\frac{m}{(M-m)}\right)\frac{R}{2}$$

If σ is the surface mass density (i.e. mass per unit surface area), $\sigma = \frac{M}{\pi R^2}$ then, the mass m of small disc is,

m = surfacemass density × surfacearea

$$m = \sigma \times \pi\left(\frac{R}{2}\right)^2$$
$$m = \left(\frac{M}{\pi R^2}\right)\pi\left(\frac{R}{2}\right)^2 = \frac{M}{\pi R^2}\pi\frac{R^2}{4} = \frac{M}{4}$$

substituting m in the expression for x

$$x = \frac{\frac{M}{4}}{\left(M-\frac{M}{4}\right)} \times \frac{R}{2} = \frac{\frac{M}{4}}{\left(\frac{3M}{4}\right)} \times \frac{R}{2}$$
$$x = \frac{R}{6}$$

The center of mass of the remaining portion is at a distance $\frac{R}{6}$ to the left from the center of the disc.

EXAMPLE

The position vectors of two point masses 10 kg and 5 kg are $\left(-3\hat{i}+2\hat{j}+4\hat{k}\right)$ m and $\left(3\hat{i}+6\hat{j}+5\hat{k}\right)$ m respectively. Locate the position of center of mass.

Solution

$$m_1 = 10\,kg$$

$$m_2 = 5\,kg$$

$$\vec{r}_1 = (-3\hat{i}+2\hat{j}+4\hat{k})\,m$$

$$\vec{r}_2 = (3\hat{i}+6\hat{j}+5\hat{k})\,m$$

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$\therefore \vec{r} = \frac{10(-3\hat{i}+2\hat{j}+4\hat{k})+5(3\hat{i}+6\hat{j}+5\hat{k})}{10+5}$$

$$= \frac{-30\hat{i}+20\hat{j}+40\hat{k}+15\hat{i}+30\hat{j}+25\hat{k}}{15}$$

$$= \frac{-15\hat{i}+50\hat{j}+65\hat{k}}{15}$$

$$\vec{r} = \left(-\hat{i}+\frac{10}{3}\hat{j}+\frac{13}{3}\hat{k}\right)\ m$$

The center of mass is located at position $\vec{r}$

Center of mass for uniform distribution of mass

If the mass is uniformly distributed in a bulk object, then a small mass (Δm) of the body can be treated as a point mass and the summations can be done to obtain the expressions for the coordinates of center of mass.

$$x_{CM} = \frac{\sum(\Delta m_i)x_i}{\sum \Delta m_i}$$

$$y_{CM} = \frac{\sum(\Delta m_i)y_i}{\sum \Delta m_i}$$

$$z_{CM} = \frac{\sum(\Delta m_i)z_i}{\sum \Delta m_i}$$

On the other hand, if the small mass taken is infinitesimally* small (dm) then, the summations can be replaced by integrations as given below.

$$x_{cm} = \frac{\int x\,dm}{\int dm}$$

$$y_{cm} = \frac{\int y\,dm}{\int dm}$$

$$z_{cm} = \frac{\int z\,dm}{\int dm}$$

EXAMPLE

Locate the center of mass of a uniform rod of mass M and length ℓ.

Solution

Consider a uniform rod of mass M and length ℓ whose one end coincides with the origin as shown in Figure. The rod is kept along the x axis. To find the center of mass

of this rod, we choose an infinitesimally small mass dm of elemental length dx at a distance x from the origin.

λ is the linear mass density (i.e. mass per unit length) of the rod $\lambda = \frac{M}{\ell}$

The mass of small element (dm) is, $dm = \frac{M}{\ell} dx$

Now, we can write the center of mass equation for this mass distribution as,

$$x_{CM} = \frac{\int x dm}{\int dm}$$

$$x_{CM} = \frac{\int_0^{\ell} x\left(\frac{M}{\ell} dx\right)}{M} = \frac{1}{\ell}\int_0^{\ell} x dx$$

$$= \frac{1}{\ell}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{\ell}\left(\frac{\ell^2}{2}\right)$$

$$x_{CM} = \frac{\ell}{2}$$

As the position $\frac{\ell}{2}$ is the geometric center of the rod, it is concluded that the center of mass of the uniform rod is located at its geometric center itself.

Motion of Center of Mass

When a rigid body moves, its center of mass will also move along with the body. For kinematic quantities like velocity (v_{CM}) and acceleration (a_{CM}) of the center of mass, we can differentiate the expression for position of center of mass with respect to time once and twice respectively. For simplicity, let us take the motion along X direction only.

$$\vec{v}_{CM} = \frac{d\vec{x}_{CM}}{dt} = \frac{\sum m_i \left(\frac{d\vec{x}_i}{dt}\right)}{\sum m_i}$$

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i}$$

$$\vec{a}_{CM} = \frac{d}{dt}\left(\frac{d\vec{x}_{CM}}{dt}\right) = \left(\frac{d\vec{v}_{CM}}{dt}\right) = \frac{\sum m_i \left(\frac{d\vec{v}_i}{dt}\right)}{\sum m_i}$$

$$\vec{a}_{CM} = \frac{\sum m_i \vec{a}_i}{\sum m_i}$$

In the absence of external force, i.e. $\vec{F}_{ext} = 0$ the individual rigid bodies of a system can move or shift only due to the internal forces. This will not affect the position of the center of mass. This means that the center of mass will be in a state of rest or uniform motion. Hence, $\vec{v}_{CM}$ will be zero when center of mass is at rest and constant when center of mass has uniform motion $\left(\vec{v}_{CM} = 0 \text{ or } \vec{v}_{CM} = \text{constant}\right)$. There will be no acceleration of center of mass, $\left(\vec{a}_{CM} = 0\right)$.

From equation

$$0 = \frac{\sum m_i \vec{v}_i}{\sum m_i} \quad \text{(or)} \quad \text{constant,}$$

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i}; \quad \vec{a}_{CM} = 0$$

Here, the individual particles may still move with their respective velocities and accelerations due to internal forces. In the presence of external force, (i.e. $\vec{F}_{ext} \neq 0$), the center of mass of the system will accelerate as given by the following equation.

$$\vec{F}_{ext} = \left(\sum m_i\right)\vec{a}_{CM}; \quad \vec{F}_{ext} = M\vec{a}_{CM}; \quad \vec{a}_{CM} = \frac{\vec{F}_{ext}}{M}$$

EXAMPLE

A man of mass 50 kg is standing at one end of a boat of mass 300 kg floating on still water. He walks towards the other end of the boat with a constant velocity of 2 m s-1 with respect to a stationary observer on land. What will be the velocity of the boat, (a) with respect to the stationary observer on land? (b) with respect to the man walking in the boat?

[Given: There is friction between the man and the boat and no friction between the boat and water.]

Solution

Mass of the man (m_1) is, m_1= 50 kg
Mass of the boat (m_2) is, m_2 = 300 kg

With respect to a stationary observer:

The man moves with a velocity, v_1 = 2 m s^{-1} and the boat moves with a velocity v_2 (which is to be found)

To determine the velocity of the boat with respect to a stationary observer on land:

As there is no external force acting on the system, the man and boat move due to the friction, which is an internal force in the boat-man system. Hence, the velocity of the center of mass is zero (v_{CM} = 0).

$$0 = \frac{\sum m_i v_i}{\sum m_i} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$0 = m_1 v_1 + m_2 v_2$$

$$-m_2 v_2 = m_1 v_1$$

$$v_2 = -\frac{m_1}{m_2} v_1$$

$$v_2 = -\frac{50}{300} \times 2 = -\frac{100}{300}$$

$$v_2 = -0.33\, m\, s^{-1}$$

The negative sign in the answer implies that the boat moves in a direction opposite to that of the walking man on the boat to a stationary observer on land.

To determine the velocity of the boat with respect to the walking man:

We can find the relative velocity as,

$$v_{21} = v_2 - v_1$$

where, v_{21} is the relative velocity of the boat with respect to the walking man.

$$v_{21} = (-0.33) - (2)$$

$$v_{21} = -2.33\, ms^{-1}$$

The negative sign in the answer implies that the boat appears to move in the opposite direction to the man walking in the boat.

Center of mass in explosions:

Many a times rigid bodies are broken in to fragments. If an explosion is caused by the internal forces in a body which is at rest or in motion, the state of the center of mass is not affected. It continues to be in the same state of rest or motion. But, the kinematic quantities of the fragments get affected. If the explosion is caused by an external agency, then the kinematic quantities of the center of mass as well as the fragments get affected.

EXAMPLE

A projectile of mass 5 kg, in its course of motion explodes on its own into two fragments. One fragment of mass 3 kg falls at three fourth of the range R of the projectile. Where will the other fragment fall?

Solution

It is an explosion of its own without any external influence. After the explosion, the center of mass of the projectile will continue to complete the parabolic path even though the fragments are not following the same parabolic path. After the fragments have fallen on the ground, the center of mass rests at a distance R (the range) from the point of projection as shown in the diagram.

If the origin is fixed to the final position of the center of mass, the principle of moments holds good.

$$m_1x_1 = m_2x_2$$

where, m_1 = 3 kg, m_2 = 2 kg, $x_1 = \frac{1}{4}R$. The value of x_2 = d

$$3 \times \frac{1}{4}R = 2 \times d;$$

$$d = \frac{3}{8}R$$

The distance between the point of launching and the position of 2 kg mass is R+d.

$$R + d = R + \frac{3}{8}R = \frac{11}{8}R = 1.375R$$

The other fragment falls at a distance of 1.375R from the point of launching. (Here R is the range of the projectile.)

TORQUE AND ANGULAR MOMENTUM

When a net force acts on a body, it produces linear motion in the direction of the applied force. If the body is fixed to a point or an axis, such a force rotates the body depending on the point of application of the force on the body. This ability of the force to produce rotational motion in a body is called torque or moment of force. Examples for such motion are plenty in day to day life. To mention a few; the opening and closing of a door about the hinges and turning of a nut using a wrench.

The extent of the rotation depends on the magnitude of the force, its direction and the distance between the fixed point and the point of application. When torque produces rotational motion in a body, its angular momentum changes with respect to time. In this Section we will learn about the torque and its effect on rigid bodies.

Definition of Torque

Torque is defined as the moment of the external applied force about a point or axis of rotation. The expression for torque is,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where, $\vec{r}$ is the position vector of the point where the force $\vec{F}$ is acting on the body as shown in Figure 5.4.

Here, the product of $\vec{r}$ and $\vec{F}$ is called the vector product or cross product. The vector product of two vectors results in another vector that is perpendicular to both the vectors (refer Section 2.5.2). Hence, torque ($\vec{\tau}$) is a vector quantity.

Tor que has a magnitude (rFsinθ) and direction perpendicular to $\vec{r}$ and $\vec{F}$. Its unit is N m.

$$\vec{\tau} = (rF\sin\theta)\hat{n}$$

Here, θ is the angle between $\vec{r}$ and $\vec{F}$, and ˆn is the unit vector in the direction of $\vec{\tau}$. Torque ($\vec{\tau}$) is sometimes called as a pseudo vector as it needs the other two vectors $\vec{r}$ and $\vec{F}$ for its existence.

The direction of torque is found using right hand rule. This rule says that if fingers of right hand are kept along the position vector with palm facing the direction of the force and when the fingers are curled the thumb points to the direction of the torque. This is shown in Figure 5.5.

The direction of torque helps us to find the type of rotation caused by the torque. For example, if the direction of torque is out of the paper, then the rotation produced by the torque is anticlockwise. On the other hand, if the direction of the torque is into the paper, then the rotation is clockwise as shown in Figure

In many cases, the direction and magnitude of the torque are found separately. For direction, we use the vector rule or right hand rule. For magnitude, we use scalar form as,

$$\tau = r\ F\ \sin\theta$$

The expression for the magnitude of torque can be written in two different ways by associating sin θ either with r or F in the following manner.

$$\tau = r(F \sin\theta) = r \times (F\perp)$$

$$\tau = (r \sin\theta)F = (r\perp) \times F$$

Here, (F sinθ) is the component of $\vec{F}$ perpendicular to $\vec{r}$. Similarly, (r sinθ) is the component of $\vec{r}$ perpendicular to $\vec{F}$.

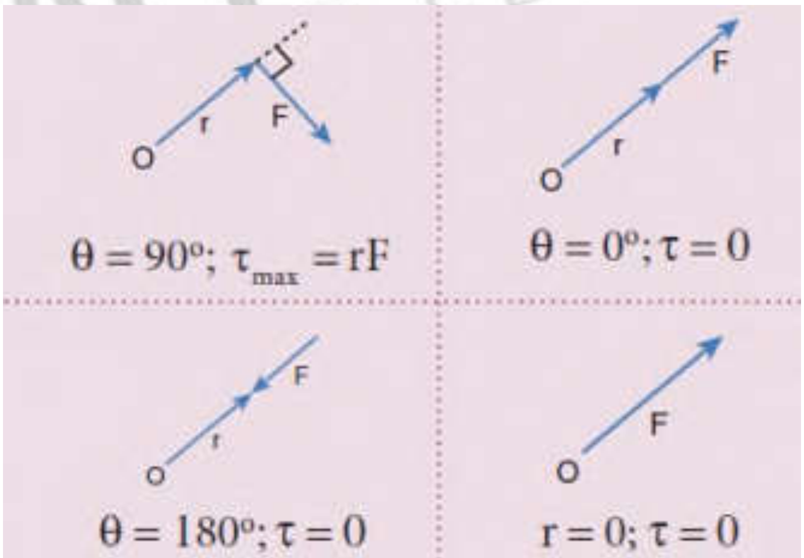
Based on the angle θ between $\vec{r}$ and $\vec{F}$ the torque takes different values.

The torque is maximum when, $\vec{r}$ and $\vec{F}$ are perpendicular to each other. That is when $\theta = 90^o$ and $\sin 90^o = 1$, Hence, $\tau_{max} = rF$.

The torque is zero when $\vec{r}$ and $\vec{F}$ are parallel or antiparallel. If parallel, then $\theta = 0^o$ and $\sin 0^o = 0$. If antiparallel, then $\theta = 180^o$ and $\sin 180^o = 0$. Hence, $\tau = 0$.

The torque is zero if the force acts at the reference point. i.e. as $\vec{r} = 0$, $\tau = 0$.

The Value of τ for different cases.



EXAMPLE

If the force applied is perpendicular to the handle of the spanner as shown in the diagram, find the (i) torque exerted by the force about the center of the nut, (ii) direction of torque and (iii) type of rotation caused by the torque about the nut.

Solution

Arm length of the spanner, r = 15 cm = 15×10^{-2}m

Force, F = 2.5 N
Angle between r and F, $\theta = 90^o$

Torque, $\tau = rF \sin\theta$

$$\tau = 15 \times 10^{-2} \times 2.5 \times \sin(90^\circ)$$
$$[\text{here, } \sin 90^\circ = 1]$$
$$\tau = 37.5 \times 10^{-2} \text{ N m}$$

As per the right hand rule, the direction of torque is out of the page.

The type of rotation caused by the torque is anticlockwise.

EXAMPLE

A force of $\left(4\hat{i} - 3\hat{j} + 5\hat{k}\right)$ N is applied at a point whose position vector is $\left(7\hat{i} + 4\hat{j} - 2\hat{k}\right)$ m. Find the torque of force about the origin.

Solution

$$\vec{r} = 7\hat{i} + 4\hat{j} - 2\hat{k}$$
$$\vec{F} = 4\hat{i} - 3\hat{j} + 5\hat{k}$$

Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 4 & -2 \\ 4 & -3 & 5 \end{vmatrix}$$
$$\vec{\tau} = \hat{i}(20-6) - \hat{j}(35+8) + \hat{k}(-21-16)$$
$$\vec{\tau} = \left(14\hat{i} - 43\hat{j} - 37\hat{k}\right) \text{ N m}$$

EXAMPLE

A crane has an arm length of 20 m inclined at 30^o with the vertical. It carries a container of mass of 2 ton suspended from the top end of the arm. Find the torque produced by the gravitational force on the container about the point where the arm is fixed to the crane. [Given: 1 ton = 1000 kg; neglect the weight of the arm. g = 10 m s^{-2}]

Solution

The force F at the point of suspension is due to the weight of the hanging mass.

$$F = mg = 2 \times 1000 \times 10 = 20000 \text{ N};$$
$$\text{The arm length, } r = 20 \text{ m}$$

We can solve this problem by three different methods.

Method – I

The angle (θ) between the arm length (r) and the force (F) is, θ = 150^o
The torque (τ) about the fixed point of the arm is,

$$\tau = rF \sin\theta$$
$$\tau = 20 \times 20000 \times \sin(150^\circ)$$
$$= 400000 \times \sin(90^\circ + 60^\circ)$$
$$[\text{here, } \sin(90^\circ + \theta) = \cos\theta]$$
$$= 400000 \times \cos(60^\circ)$$
$$= 400000 \times \frac{1}{2} \quad \left[\cos 60^\circ = \frac{1}{2}\right]$$
$$= 200000 \text{ N m}$$
$$\tau = 2 \times 10^5 \text{ N m}$$

Method – II

Let us take the force and perpendicular distance from the point where the arm is fixed to the crane.

$$\tau = (r \perp) F$$

$$\tau = r \cos\phi \, mg$$

$$\tau = 20 \times \cos 60^\circ \times 20000$$

$$= 20 \times \frac{1}{2} \times 20000$$

$$= 200000 \, Nm$$

$$\tau = 2 \times 10^5 \, Nm$$

Method – III

Let us take the distance from the fixed point and perpendicular force.

$$\tau = r(F \perp)$$

$$\tau = r \, mg \, \cos\phi$$

$$\tau = 20 \times 20000 \times \cos 60^\circ$$

$$= 20 \times 20000 \times \frac{1}{2}$$

$$= 200000 \, Nm$$

$$\tau = 2 \times 10^5 \, Nm$$

All the three methods, give the same answer.

Torque about an Axis

In the earlier sections, we have dealt with the torque about a point. In this section we will deal with the torque about an axis. Let us consider a rigid body capable of rotating about an axis AB as shown in Figure 5.8. Let the force F act at a point P on the rigid body. The force F may not be on the plane ABP. We can take the origin O at any random point on the axis AB.

The torque of the force $\vec{F}$ about O is, $\vec{\tau} = \vec{r} \times \vec{F}$. The component of the torque along the axis is the torque of $\vec{F}$ about the axis. To find it, we should first find the vector $\vec{\tau} = \vec{r} \times \vec{F}$ and then find the angle φ between τ and AB. (Remember here, $\vec{F}$ is not on the plane ABP). The torque about AB is the parallel component of the torque along AB, which is $\left|\vec{r}\times\vec{F}\right|$ cos φ. And the torque perpendicular to the axis AB is $\left|\vec{r}\times\vec{F}\right|$sin φ.

The torque about the axis will rotate the object about it and the torque perpendicular to the axis will turn the axis of rotation. When both exist simultaneously on a rigid body, the body will have a precession. One can witness the precessional motion in a spinning top when it is about to come to rest as shown in Figure 5.9.

Study of precession is beyond the scope of the higher secondary physics course. Hence, it is assumed that there are constraints to cancel the effect of the perpendicular components of the torques, so that the fixed position of the axis is maintained. Therefore, perpendicular components of the torque need not be taken into account.

Hereafter, for the calculation of torques on rigid bodies we will:

1. Consider only those forces that lie on planes perpendicular to the axis (and do not intersect the axis).
2. Consider position vectors which are perpendicular to the axis

EXAMPLE

Three mutually perpendicular beams AB, OC, GH are fixed to form a structure which is fixed to the ground firmly as shown in the Figure. One string is tied to the point C and its free end D is pulled with a force F. Find the magnitude and direction of the torque produced by the force,

Solution

1. Torque about point D is zero. (as F passes through D).
   Torque about point C is zero. (as F passes through C).
   Torque about point O is $\left(\overrightarrow{OC}\right)\times\vec{F}$ and direction is along GH.
   Torque about point B is $\left(\overrightarrow{BD}\right)\times\vec{F}$ and direction is along GH.

   (The ⊥ of $\overrightarrow{BD}$ with respect to $\vec{F}$ is $\overrightarrow{OC}$).

2. Torque about axis CD is zero (as F is parallel to CD).
   Torque about axis OC is zero (as F intersects OC).
   Torque about axis AB is zero (as F is parallel to AB).
   Torque about axis GH is $\left(\overrightarrow{OC}\right)\times\vec{F}$ and direction is along GH.

The torque of a force about an axis is independent of the choice of the origin as long as it is chosen on that axis itself. This can be shown as below.

Let O be the origin on the axis AB, which is the rotational axis of a rigid body. F is the force acting at the point P. Now, choose another point O′ anywhere on the axis as shown in Figure 5.10

The torque of F about O′ is,

$$\overrightarrow{O'P} \times \vec{F} = \left(\overrightarrow{O'O} + \overrightarrow{OP}\right) \times \vec{F}$$
$$= \left(\overrightarrow{O'O} \times \vec{F}\right) + \left(\overrightarrow{OP} \times \vec{F}\right)$$

As $\overrightarrow{O'O} \times \vec{F}$ is perpendicular to $\overrightarrow{O'O}$, this term will not have a component along AB. Thus, the component of $\overrightarrow{O'P} \times \vec{F}$ is equal to that of $\overrightarrow{OP} \times \vec{F}$.

Torque and Angular Acceleration

Let us consider a rigid body rotating about a fixed axis. A point mass m in the body will execute a circular motion about a fixed axis as shown in Figure 5.11. A tangential force $\vec{F}$ acting on the point mass produces the necessary torque for this rotation. This force $\vec{F}$ is perpendicular to the position vector $\vec{r}$ of the point mass

The torque produced by the force on the point mass m about the axis can be written as,

$$\tau = r F \sin 90 = r F \quad [\because \sin 90 = 1]$$
$$\tau = r m a \quad [\because (F = ma)]$$
$$\tau = r m r \alpha = m r^2 \alpha \quad [\because (a = r\alpha)]$$

$$\tau = \left(m r^2\right) \alpha$$

Hence, the torque of the force acting on the point mass produces an angular acceleration (α) in the point mass about the axis of rotation.

In vector notation,

$$\vec{\tau} = \left(m r^2\right) \vec{\alpha}$$

The directions of τ and α are along the axis of rotation. If the direction of τ is in the direction of α, it produces angular acceleration. On the other hand if, τ is opposite to α, angular deceleration or retardation is produced on the point mass.

The term mr2 in equations 5.14 and 5.15 is called moment of inertia (I) of the point mass. A rigid body is made up of many such point masses. Hence, the moment of inertia of a rigid body is the sum of moments of inertia of all such individual point masses that constitute the body $\left(I = \sum m_i r_i^2\right)$. Hence, torque for the rigid body can be written as,

$$\vec{\tau} = \left(\sum m_i r_i^2\right)\vec{\alpha}$$
$$\vec{\tau} = I\vec{\alpha}$$

We will learn more about the moment of inertia and its significance for bodies with different shapes in section 5.4.

Angular Momentum

The angular momentum in rotational motion is equivalent to linear momentum in translational motion. The angular momentum of a point mass is defined as the moment of its linear momentum. In other words, the angular momentum L of a point mass having a linear momentum p at a position r with respect to a point or axis is mathematically written as,

$$\vec{L} = \vec{r} \times \vec{p}$$

The magnitude of angular momentum could be written as,

$$L = r\,p\sin\theta$$

where, θ is the angle between $\vec{r}$ and $\vec{p}$. $\vec{L}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$. As we have written in the case of torque, here also we can associate sin θ with either $\vec{r}$ or $\vec{p}$.

$$L = r(p\sin\theta) = r(p\perp)$$
$$L = (r\sin\theta)p = (r\perp)p$$

where, $p\perp$ is the component of linear momentum p perpendicular to r, and $r\perp$ is the component of position r perpendicular to p.

The angular momentum is zero (L = 0), if the linear momentum is zero (p = 0) or if the particle is at the origin ($\vec{r}$ = 0) or if $\vec{r}$ and $\vec{p}$ are parallel or antiparallel to each other (θ= 00 or 1800).

There is a misconception that the angular momentum is a quantity that is associated only with rotational motion. It is not true. The angular momentum is also associated with bodies in the linear motion. Let us understand the same with the following example.

EXAMPLE

A particle of mass (m) is moving with constant velocity (v). Show that its angular momentum about any point remains constant throughout the motion.

Solution

Let the particle of mass m move with constant velocity $\vec{v}$. As it is moving with constant velocity, its path is a straight line. Its momentum ($\vec{p}$ =m$\vec{v}$) is also directed along the same path. Let us fix an origin (O) at a perpendicular distance (d) from the path. At a particular instant, we can connect the particle which is at positon Q with a position vector $\left(\vec{r} = \overrightarrow{QQ}\right)$).

Take, the angle between the $\vec{r}$ and $\vec{p}$ as θ. The magnitude of angular momentum of that particle at that instant is,

$$L = OQ\,p\sin\theta = OQ\,mv\sin\theta = mv\left(OQ\sin\theta\right)$$

The term (OQ sinθ) is the perpendicular distance (d) between the origin and line along which the mass is moving. Hence, the angular momentum of the particle about the origin is,

$$L = mvd$$

The above expression for angular momentum L, does not have the angle θ. As the momentum (p = mv) and the perpendicular distance (d) are constants, the angular momentum of the particle is also constant. Hence, the angular momentum is associated with bodies with linear motion also. If the straight path of the particle passes through the origin, then the angular momentum is zero, which is also a constant

Angular Momentum and Angular Velocity

Let us consider a rigid body rotating about a fixed axis. A point mass m in the body will execute a circular motion about the fixed axis as shown in Figure

The point mass m is at a distance r from the axis of rotation. Its linear momentum at any instant is tangential to the circular path. Then the angular momentum $\vec{L}$ is perpendicular to $\vec{r}$ and $\vec{p}$. Hence, it is directed along the axis of rotation. The angle θ between $\vec{r}$ and $\vec{p}$ in this case is 90o. The magnitude of the angular momentum L could be written as,

$$L = r\,mv \sin 90^{\circ} = r\,mv$$

where, v is the linear velocity. The relation between linear velocity v and angular velocity ω in a circular motion is, v = rω. Hence,

$$L = rmr\omega$$
$$L = \left(mr^2\right)\omega$$

The directions of L and ω are along the axis of rotation. The above expression can be written in the vector notation as,

$$\vec{L} = \left(mr^2\right)\vec{\omega}$$

As discussed earlier, the term mr2 in equations 5.22 and 5.23 is called moment of inertia (I) of the point mass. A rigid body is made up of many such point masses. Hence, the moment of inertia of a rigid body is the sum of moments of inertia of all such individual point masses that constitute the body $\left(I = \sum m_i r_i^2\right)$. Hence, the angular momentum of the rigid body can be written as,

$$\vec{L} = \left(\sum m_i r_i^2\right)\vec{\omega}$$
$$\vec{L} = I\vec{\omega}$$

The study about moment of inertia (I) is reserved for Section 5.4.

Torque Angular Momentum

We have the expression for magnitude ofangular momentum of a rigid body as,L = Iω. The expression for magnitude oftorque on a rigid body is, τ = Iα

We can further write the expression for torque as,

$$\tau = I\frac{d\omega}{dt} \quad \because \left(\alpha = \frac{d\omega}{dt}\right)$$

Where, ω is angular velocity and α is angular acceleration. We can also write equation 5.26 as,

$$\tau = \frac{d(I\omega)}{dt}$$

$$\tau = \frac{dL}{dt}$$

The above expression says that an external torque on a rigid body fixed to an axis produces rate of change of angular momentum in the body about that axis. This is the Newton's second law in rotational motion as it is in the form of $F = \frac{dp}{dt}$ which holds good for translational motion.

Conservation of angular momentum:

From the above expression we could conclude that in the absence of external torque, the angular momentum of the rigid body or system of particles is conserved.

$$\text{If } \tau = 0 \text{ then, } \frac{dL}{dt} = 0;\ L = \text{constant}$$

The above expression is known as law of conservation of angular momentum. We will learn about this law further insection 5.5.

EQUILIBRIUM OF RIGID BODIES

When a body is at rest without any motion on a table, we say that there is no force acting on the body. Actually it is wrong because, there is gravitational force acting on the body downward and also the normal force exerted by table on the body upward. These two forces cancel each other and thus there is no net force acting on the body. There is a lot of difference between the terms "no force" and "no net force" acting on a body. The same argument holds good for rotational conditions in terms of torque or moment of force.

A rigid body is said to be in mechanical equilibrium when both its linear momentum and angular momentum remain constant.

When the linear momentum remains constant, the net force acting on the body is zero.

$$\vec{F}_{net} = 0$$

In this condition, the body is said to be in translational equilibrium. This implies that the vector sum of different forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots$ acting in different directions on the body is zero.

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n = 0$$

If the forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots$ act in different directions on the body, we can resolve them into horizontal and vertical components and then take the resultant in the respective directions. In this case there will be horizontal as well as vertical equilibria possible.

Similarly, when the angular momentum remains constant, the net torque acting on the body is zero.

$$\vec{\tau}_{net} = 0$$

Under this condition, the body is said to be in rotational equilibrium. The vector sum of different torques $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3, \ldots$ producing different senses of rotation on the body is zero.

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \cdots + \vec{\tau}_n = 0$$

Thus, we can also conclude that a rigid body is in mechanical equilibrium when the net force and net torque acts on the body is zero.

$$\vec{F}_{net} = 0 \text{ and } \vec{\tau}_{net} = 0$$

As the forces and torques are vector quantities, the directions are to be taken with proper sign conventions.

Types of Equilibrium

Based on the above discussions, we come to aconclusion that different types of equilibriumare possible based on the different conditions.They are consolidated in Table 5.2.

EXAMPLE

Arun and Babu carry a wooden log of mass 28 kg and length 10 m which has almost uniform thickness. They hold it at 1 m and 2 m from the ends respectively. Who will bear more weight of the log? [$g = 10\ ms^{-2}$]

Solution

Let us consider the log is in mechanical equilibrium. Hence, the net force and net torque on the log must be zero. The gravitational force acts at the center of mass of the log downwards. It is cancelled by the normal reaction forces R_A and R_B applied upwards by Arun and Babu at points A and B respectively. These reaction forces are the weights borne by them.

The total weight, W = mg = 28 × 10 = 280 N, has to be borne by them together. The reaction forces are the weights borne by each of them separately. Let us show all the forces acting on the log by drawing a free body diagram of the log.

For translational equilibrium:

The net force acting on the log must be zero.

$$R_A + (-mg) + R_B = 0$$

Here, the forces R_A an $R_B$are taken positive as they act upward. The gravitational force acting downward is taken negative.

$$R_A + R_B = mg$$

For rotational equilibrium:

The net torque acting on the log must be zero. For ease of calculation, we can take the torque caused by all the forces about the point A on the log. The\ forces are perpendicular to the distances. Hence,

$$(0R_A) + (-4mg) + (7R_B) = 0.$$

Here, the reaction force R_A cannot produce any torque as the reaction forces pass through the point of reference A. The torque of force mg produces a clockwise turn about the point A which is taken negative and torque of force R_B causes anticlockwise turn about A which is taken positive.

$$7R_B = 4mg$$

$$R_B = \frac{4}{7}mg$$

$$R_B = \frac{4}{7} \times 28 \times 10 = 160\ N$$

By substituting for R_B we get,

$$R_A = mg - R_B$$

$$R_A = 28 \times 10 - 160 = 280 - 160 = 120\ N$$

As R_B is greater than R_A, it is concluded that Babu bears more weight than Arun. The one closer to center of mass of the log bears more weight.

Couple

Consider a thin uniform rod AB. Its center of mass is at its midpoint C. Let two forces which are equal in magnitude and opposite in direction be applied at the two ends A and B of the rod perpendicular to it. The two forces are separated by a distance of 2r as shown in Figure 5.13.

As the two equal forces are opposite in direction, they cancel each other and the net force acting on the rod is zero. Now the rod is in translational equilibrium. But, the rod is not in rotational equilibrium. Let us see how it is not in rotational equilibrium. The moment of the force applied at the end A taken with respect to the center point C, produces an anticlockwise rotation. Similarly, the moment of the force applied at the end B also produces an anticlockwise rotation. The moments of both the forces cause the same sense of rotation in the rod. Thus, the rod undergoes a rotational motion or turning even though the rod is in translational equilibrium.

A pair of forces which are equal in magnitude but opposite in direction and separated by a perpendicular distance so that their lines of action do not coincide that causes a turning effect is called a couple. We come across couple in many of our daily activities as shown in Figure 5.14.

Principle of Moments

Consider a light rod of negligible mass which is pivoted at a point along its length. Let two parallel forces F1 and F2 act at the two ends at distances d1 and d2 from the point of pivot and the normal reaction force N at the point of pivot as shown in Figure 5.15. If the rod has to remain stationary in horizontal position, it should be in translational and rotational equilibrium. Then, both the net force and net torque must be zero.

For net force to be zero, $-F_1 + N - F_2 = 0$

$$N = F_1 + F_2$$

For net torque to be zero, $d_1F_1 - d_2F_2 = 0$

$$d_1F_1 = d_2F_2$$

The above equation represents the principle of moments. This forms the principle for beam balance used for weighing goods with the condition $d_1 = d_2$; $F_1 = F_2$. We can rewrite the equation 5.33 as,

$$\frac{F_1}{F_2} = \frac{d_2}{d_1}$$

If F_1 is the load and F_2 is our effort, we get advantage when, $d_1 < d_2$. This implies that $F_1 > F_2$. Hence, we could lift a large load with small effort. The ratio $\left(\frac{d_2}{d_1}\right)$ is called mechanical advantage of the simple lever. The pivoted point is called fulcrum.

Mechanical Advantage MA= $\frac{d_2}{d_1}$

There are many simple machines that work on the above mentioned principle.

Center of Gravity

Each rigid body is made up of several point masses. Such point masses experience gravitational force towards the center of Earth. As the size of Earth is very large compared to any practical rigid body we come across in daily life, these forces appear to be acting parallelly downwards as shown in Figure 5.16

The resultant of these parallel forces always acts through a point. This point is called center of gravity of the body (with respect to Earth). The center of gravity of a body is the point at which the entire weight of the body acts irrespective of the position and orientation of the body. The center of gravity and center of mass of a rigid body coincide when the gravitational field is uniform across the body. The concept of gravitational field is dealt in Unit 6.

We can also determine the center of gravity of a uniform lamina of even an irregular shape by pivoting it at various points by trial and error. The lamina remains horizontal when pivoted at the point where the net gravitational force acts, which is the center of gravity as shown in Figure 5.17. When a body is supported at the center of gravity, the sum of the torques acting on all the point masses of the rigid body becomes zero. Moreover the weight is compensated by the normal reaction force exerted by the pivot. The body is in static equilibrium and hence it remains horizontal.

There is also another way to determine the center of gravity of an irregular lamina. If we suspend the lamina from different points like P, Q, R as shown in Figure 5.18, the vertical lines PP', QQ', RR' all pass through the center of gravity. Here, reaction force acting at the point of suspension and the gravitational force acting at the center of gravity cancel each other and the torques caused by them also cancel each other.

Bending of Cyclist in Curves

Let us consider a cyclist negotiating a circular level road (not banked) of radius r with a speed v. The cycle and the cyclist are considered as one system with mass m. The center gravity of the system is C and it goes in a circle of radius r with center at O. Let us choose the line OC as X-axis and the vertical line through O as Z-axis as shown in Figure 5.19.

The system as a frame is rotating about Z-axis. The system is at rest in this rotating frame. To solve problems in rotating frame of reference, we have to apply a centrifugal force (pseudo force) on the system which will be $\frac{mv^2}{r}$ This force will act through the center of gravity. The forces acting on the system are, (i) gravitational force (mg), (ii) normal force (N), (iii) frictional force (f) and (iv) centrifugal force $\left(\frac{mv^2}{r}\right)$. As the system is in equilibrium in the rotational frame of reference, the net external force and net external torque must be zero. Let us consider all torques about the point A in Figure 5.20.

The torque due to the gravitational force about point A is (mgAB) which causes a clockwise turn that is taken as negative. The torque due to the centripetal force is $\left(\frac{mv^2}{r}BC\right)$ which causes an anticlockwise turn that is taken as positive.

$$-mg\,AB + \frac{mv^2}{r}BC = 0$$

$$mg\,AB = \frac{mv^2}{r}BC$$

From Δ ABC,
AB = ACsinθ and BC = ACcosθ

$$mg\,AC\,\sin\theta = \frac{mv^2}{r} AC\cos\theta$$

$$\tan\theta = \frac{v^2}{rg}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

While negotiating a circular level road of radius r at velocity v, a cyclist has to bend by an angle θ from vertical given by the above expression to stay in equilibrium (i.e. to avoid a fall).

EXAMPLE

A cyclist while negotiating a circular path with speed 20 m s^{-1} is found to bend an angle by 30o with vertical. What is the radius of the circular path? (given, g = 10 m s^{-2})

Solution

Speed of the cyclist, v = 20 m s^{-1}
Angle of bending with vertical, θ = 30^o

Equation for angle of bending, tanθ= $\frac{V^2}{rg}$

Rewriting the above equation for radius

$$r = \frac{v^2}{\tan\theta\, g}$$

Substituting,

$$r = \frac{(20)^2}{(\tan 30^\circ)\times 10} = \frac{20\times 20}{(\tan 30^\circ)\times 10}$$

$$= \frac{400}{\left(\frac{1}{\sqrt{3}}\right)\times 10}$$

$$r = \left(\sqrt{3}\right)\times 40 = 1.732\times 40$$

$$r = 69.28\,\text{m}$$

MOMENT OF INERTIA

In the expressions for torque and angular momentum for rigid bodies (which are considered as bulk objects), we have come across a term $\sum m_i r_i^2$ This quantity is called moment of inertia (I) of the bulk object. For point mass mi at a distance r_i from the fixed axis, the moment of inertia is given as $m_i r_i^2$

Moment of inertia for point mass,

$$I = m_i r_i^2$$

Moment of inertia for bulk object

$$I = \sum m_i r_i^2$$

In translational motion, mass is a measure of inertia; in the same way, for rotational motion, moment of inertia is a measure of rotational inertia. The unit of moment of inertia is, kg m^2. Its dimension is M L^2. In general, mass is an invariable quantity of matter (except for motion comparable to that of light). But, the moment of inertia of a body is not an invariable quantity. It depends not only on the mass of the body, but also on the way the mass is distributed around the axis of rotation.

To find the moment of inertia of a uniformly distributed mass; we have to consider an infinitesimally small mass (dm) as a point mass and take its position (r) with respect to an axis. The moment of inertia of this point mass can now be written as,

$$dI = (dm)r^2$$

We get the moment of inertia of the entire bulk object by integrating the above expression.

$$I = \int dI = \int (dm) r^2$$

$$I = \int r^2 dm$$

We can use the above expression for determining the moment of inertia of some of the common bulk objects of interest like rod, ring, disc, sphere etc.

Moment of Inertia of a Uniform Rod

Let us consider a uniform rod of mass (M) and length (ℓ) as shown in Figure 5.21. Let us find an expression for moment of inertia of this rod about an axis that passes through the center of mass and perpendicular to the rod. First an origin is to be fixed for the coordinate system so that it coincides with the center of mass, which is also the geometric center of the rod. The rod is now along the x axis. We take an infinitesimally small mass (dm) at a distance (x) from the origin. The moment of inertia (dI) of this mass (dm) about the axis is,

$$dI = (dm) x^2$$

As the mass is uniformly distributed, the mass per unit length (λ) of the rod is $\lambda = \frac{M}{\ell}$

The (dm) mass of the infinitesimally small length as, dm $= \lambda \, dx = \frac{M}{\ell} dx$

The moment of inertia (I) of the entire rod can be found by integrating dI,

$$I = \int dI = \int (dm) x^2 = \int \left(\frac{M}{\ell} dx \right) x^2$$

$$I = \frac{M}{\ell} \int x^2 dx$$

As the mass is distributed on either side of the origin, the limits for integration are taken from -ℓ /2 to ℓ /2.

$$I = \frac{M}{\ell}\int_{-\ell/2}^{\ell/2} x^2 dx = \frac{M}{\ell}\left[\frac{x^3}{3}\right]_{-\ell/2}^{\ell/2}$$

$$I = \frac{M}{\ell}\left[\frac{\ell^3}{24} - \left(-\frac{\ell^3}{24}\right)\right] = \frac{M}{\ell}\left[\frac{\ell^3}{24} + \frac{\ell^3}{24}\right]$$

$$I = \frac{M}{\ell}\left[2\left(\frac{\ell^3}{24}\right)\right]$$

$$I = \frac{1}{12}M\ell^2 \qquad (5.41)$$

EXAMPLE

Find the moment of inertia of a uniform rod about an axis which is perpendicular to the rod and touches any one end of the rod.

Solution

The concepts to form the integrand to find the moment of inertia could be borrowed from the earlier derivation. Now, the origin is fixed to the left end of the rod and the limits are to be taken from 0 to ℓ.

$$I = \frac{M}{\ell}\int_{0}^{\ell} x^2 dx = \frac{M}{\ell}\left[\frac{x^3}{3}\right]_{0}^{\ell} = \frac{M}{\ell}\left[\frac{\ell^3}{3}\right]$$

$$I = \frac{1}{3}M\ell^2$$

Moment of Inertia of a Uniform Ring

Let us consider a uniform ring of mass M and radius R. To find the moment of inertia of the ring about an axis passing through its center and perpendicular to the plane, let us take an infinitesimally small mass (dm) of length (dx) of the ring. This (dm) is located at a distance R, which is the radius of the ring from the axis.

The moment of inertia (dI) of this small mass (dm) is,

$$dI = (dm)R^2$$

The length of the ring is its circumference (2πR). As the mass is uniformly distributed, the mass per unit length (λ) is,

$$\lambda = \frac{\text{mass}}{\text{length}} = \frac{M}{2\pi R}$$

The mass (dm) of the infinitesimally small length is, $dm = \lambda\, dx = \frac{M}{2\pi R} dx$

Now, the moment of inertia (I) of the entire ring is,

$$I = \int dI = \int (dm)R^2 = \int \left(\frac{M}{2\pi R} dx \right) R^2$$

$$I = \frac{MR}{2\pi} \int dx$$

To cover the entire length of the ring, the limits of integration are taken from 0 to 2πR.

$$I = \frac{MR}{2\pi} \int_0^{2\pi R} dx$$

$$I = \frac{MR}{2\pi} [x]_0^{2\pi R} = \frac{MR}{2\pi} [2\pi R - 0]$$

$$I = MR^2 \qquad (5.42)$$

Moment of Inertia of a Uniform Disc

Consider a disc of mass M and radius R. This disc is made up of many infinitesimally small rings as shown in Figure 5.23. Consider one such ring of mass (dm) and thickness (dr) and radius (r). The moment of inertia (dI) of this small ring is,

$$dI = (dm)r^2$$

As the mass is uniformly distributed, the mass per unit area (σ) is, $\sigma = \frac{\text{mass}}{\text{area}} = \frac{M}{\pi R^2}$

The mass of the infinitesimally small ring is,

$$dm = \sigma 2\pi r dr = \frac{M}{\pi R^2} 2\pi r dr$$

where, the term (2π r dr) is the area of this elemental ring (2π r is the length and dr is the thickness). $dm = \frac{2M}{R^2} r dr$

$$dI = \frac{2M}{R^2} r^3 dr$$

The moment of inertia (I) of the entire disc is,

$$I = \int dI$$

$$I = \int_0^R \frac{2M}{R^2} r^3 dr = \frac{2M}{R^2} \int_0^R r^3 dr$$

$$I = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2}\left[\frac{R^4}{4} - 0\right]$$

$$I = \frac{1}{2} MR^2$$

Radius of Gyration

For bulk objects of regular shape with uniform mass distribution, the expression for moment of inertia about an axis involves their total mass and geometrical features like radius, length, breadth, which take care of the shape and the size of the objects. But, we need an expression for the moment of inertia which could take care of not only the mass, shape and size of objects, but also its orientation to the axis of rotation. Such an expression should be general so that it is applicable even for objects of irregular shape and non-uniform distribution of mass. The general expression for moment of inertia is given as,

$$I = MK^2$$

where, M is the total mass of the object and K is called the radius of gyration.

The radius of gyration of an object is the perpendicular distance from the axis of rotation to an equivalent point mass, which would have the same mass as well as the same moment of inertia of the object.

As the radius of gyration is distance, its unit is m. Its dimension is L.

A rotating rigid body with respect to any axis, is considered to be made up of point masses $m_1, m_2, m_3, \ldots m_n$ at perpendicular distances (or positions) $r_1, r_2, r_3 \ldots r_n$ respectively as shown in Figure 5.24.

The moment of inertia of that object can be written as,

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2$$

If we take all the n number of individual masses to be equal,

$$m = m_1 = m_2 = m_3 = \ldots = m_n$$

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \cdots + mr_n^2$$

$$= m\left(r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2\right)$$

$$= nm\left(\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}\right)$$

$$I = MK^2$$

where, nm is the total mass M of the body and K is the radius of gyration.

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2}{n}}$$

The expression for radius of gyration indicates that it is the root mean square (rms) distance of the particles of the body from the axis of rotation.

In fact, the moment of inertia of any object could be expressed in the form, $I = MK^2$

For example, let us take the moment of inertia of a uniform rod of mass M and length ℓ. Its moment of inertia with respect to a perpendicular axis passing through the center of mass is, $I = \frac{1}{12} M\ell^2$

In terms of radius of gyration, $I = MK^2$

$$\text{Hence, } MK^2 = \frac{1}{12} M\ell^2$$

$$K^2 = \frac{1}{12}\ell^2$$

$$K = \frac{1}{\sqrt{12}}\ell \text{ or } K = \frac{1}{2\sqrt{3}}\ell \text{ or } K = (0.289)\ell$$

EXAMPLE

Find the radius of gyration of a disc of mass M and radius R rotating about an axis passing through the center of mass and perpendicular to the plane of the disc.

Solution

The moment of inertia of a disc about an axis passing through the center of mass and perpendicular to the disc is, $I = \frac{1}{2} MR^2$

In terms of radius of gyration, $I = MK^2$

$$\text{Hence, } MK^2 = \frac{1}{2} MR^2; \quad K^2 = \frac{1}{2} R^2$$

$$K = \frac{1}{\sqrt{2}} R \text{ or } K = \frac{1}{1.414} R \text{ or } K = (0.707) R$$

From the case of a rod and also a disc, we can conclude that the radius of gyration of the rigid body is always a geometrical feature like length, breadth, radius or their combinations with a positive numerical value multiplied to it.

Obesity and associated ailments like back pain, joint pain etc. are due to the shift in center of mass of the body. Due to this shift in center of mass, unbalanced torque acting on the body leads to ailments. As the mass is spread away from center of the body the moment of inertia is more and turning will also be diffi cult.

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Parallel axis theorem:

Parallel axis theorem states that the moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its center of mass and the product of the mass of the body and the square of the perpendicular distance between the two axes.

If I_C is the moment of inertia of the body of mass M about an axis passing through the center of mass, then the moment of inertia I about a parallel axis at a distance d from it is given by the relation,

$$I = I_C + Md^2$$

Let us consider a rigid body as shown in Figure 5.25. Its moment of inertia about an axis AB passing through the center of mass is I_C. DE is another axis parallel to AB at a perpendicular distance d from AB. The moment of inertia of the body about DE is I. We attempt to get an expression for I in terms of I_C. For this, let us consider a point mass m on the body at position x from its center of mass.

The moment of inertia of the point mass about the axis DE is, $m(x + d)^2$.

The moment of inertia I of the whole body about DE is the summation of the above expression.

$$I = \sum m(x + d)^2$$

This equation could further be written as,

$$I = \sum m\left(x^2 + d^2 + 2xd\right)$$

$$I = \sum\left(mx^2 + md^2 + 2dmx\right)$$

$$I = \sum mx^2 + \sum md^2 + 2d\sum mx$$

Here, Σmx^2 is the moment of inertia of the body about the center of mass. Hence, $I_C = \sum mx^2$

The term, $\Sigma mx = 0$ because, x can take positive and negative values with respect to the axis AB. The summation (Σmx) will be zero.

$$\text{Thus, } I = I_C + \sum md^2 = I_C + \left(\sum m\right)d^2$$

Here, Σm is the entire mass M of the object ($\Sigma m = M$)

$$I = I_C + Md^2$$

Hence, the parallel axis theorem is proved.

Perpendicular axis theorem:

This perpendicular axis theorem holds good only for plane laminar objects. The theorem states that the moment of inertia of a plane laminar body about an axis perpendicular to its plane is equal to the sum of moments of inertia about two perpendicular axes lying in the plane of the body such that all the three axes are mutually perpendicular and have a common point.

Let the X and Y-axes lie in the plane and Z-axis perpendicular to the plane of the laminar object. If the moments of inertia of the body about X and Y-axes are I_X and I_Y respectively and I_Z is the moment of inertia about Z-axis, then the perpendicular axis theorem could be expressed as,

$$I_Z = I_X + I_Y$$

To prove this theorem, let us consider a plane laminar object of negligible thickness on which lies the origin (O). The X and Y-axes lie on the plane and Z-axis is perpendicular to it as shown in Figure 5.26. The lamina is considered to be made up of a large number of particles of mass m. Let us choose one such particle at a point P which has coordinates (x, y) at a distance r from O.

The moment of inertia of the particle about Z-axis is, mr^2

The summation of the above expression gives the moment of inertia of the entire lamina about Z-axis as, $I_Z = \sum mr^2$

$$\text{Here, } r^2 = x^2 + y^2$$

$$\text{Then, } I_Z = \sum m\left(x^2 + y^2\right)$$

$$I_Z = \sum mx^2 + \sum my^2$$

In the above expression, the term $\sum mx^2$ is the moment of inertia of the body about the Y-axis and similarly the term $\sum my^2$ is the moment of inertia about X-axis. Thus,

$$I_X = \sum my^2 \quad \text{and} \quad I_Y = \sum mx^2$$

Substituting in the equation for I_z gives,

$$I_Z = I_X + I_Y$$

Thus, the perpendicular axis theorem is proved.

EXAMPLE

Find the moment of inertia of a disc of mass 3 kg and radius 50 cm about the following axes.

1. axis passing through the center and perpendicular to the plane of the disc,
2. axis touching the edge and perpendicular to the plane of the disc and
3. axis passing through the center and lying on the plane of the disc.

Solution

The mass, M = 3 kg, radius R = 50 cm = 50×10^{-2} m = 0.5 m

The moment of inertia (I) about an axis passing through the center and perpendicular to the plane of the disc is,

$$I = \frac{1}{2}MR^2$$

$$I = \frac{1}{2} \times 3 \times (0.5)^2 = 0.5 \times 3 \times 0.5 \times 0.5$$

$$I = 0.375\,\text{kg m}^2$$

The moment of inertia (I) about an axis touching the edge and perpendicular to the plane of the disc by parallel axis theorem is,

$$I = I_C + M d^2$$

$$\text{where, } I_C = \frac{1}{2}MR^2 \text{ and } d = R$$

$$I = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

$$I = \frac{3}{2} \times 3 \times (0.5)^2 = 1.5 \times 3 \times 0.5 \times 0.5$$

$$I = 1.125\,\text{kg m}^2$$

The moment of inertia (I) about an axis passing through the center and lying on the plane of the disc is,

$$I_Z = I_X + I_Y$$

$$\text{where, } I_X = I_Y = I \text{ and } I_Z = \frac{1}{2}MR^2$$

$$I_Z = 2I; \; I = \frac{1}{2} I_Z$$

$$I = \frac{1}{2} \times \frac{1}{2} MR^2 = \frac{1}{4} MR^2$$

$$I = \frac{1}{4} \times 3 \times (0.5)^2 = 0.25 \times 3 \times 0.5 \times 0.5$$

$$I = 0.1875 \, \text{kg m}^2$$

EXAMPLE

Find the moment of inertia about the geometric center of the given structure made up of one thin rod connecting two similar solid spheres as shown in Figure.

Solution

The structure is made up of three objects; one thin rod and two solid spheres.

The mass of the rod, M = 3 kg and the total length of the rod, ℓ = 80 cm = 0.8 m

The moment of inertia of the rod about its center of mass is,

$$I_{rod} = \frac{1}{12} M\ell^2 \quad I_{rod} = \frac{1}{12} \times 3 \times (0.8)^2 = \frac{1}{4} \times 0.64$$

$$I_{rod} = 0.16 \, \text{kg m}^2$$

The mass of the sphere, M = 5 kg and the radius of the sphere, R = 10 cm = 0.1m $I_C = \frac{2}{5} MR^2$

The moment of inertia of the sphere about geometric center of the structure is,

$$I_{sph} = I_C + Md^2$$

Where, d = 40 cm + 10 cm = 50 cm = 0.5 m

$$I_{sph} = \frac{2}{5}MR^2 + Md^2$$
$$I_{sph} = \frac{2}{5} \times 5 \times (0.1)^2 + 5 \times (0.5)^2$$
$$I_{sph} = (2 \times 0.01) + (5 \times 0.25) = 0.02 + 1.25$$
$$I_{sph} = 1.27 \text{ kg m}^2$$

As there are one rod and two similar solid spheres we can write the total moment of inertia (I) of the given geometric structure as,

$$I = I_{rod} + (2 \times I_{sph})$$

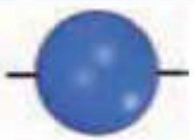
$$I = (0.16) + (2 \times 1.27) = 0.16 + 2.54$$
$$I = 2.7 \text{ kg m}^2$$

Moment of Inertia of Different Rigid Bodies

The moment of inertia of different objects about different axes is given in the Table 5.3.

Table 5.3 Moment of Inertia of Different Rigid Bodies

No.	Object	About an axis	Diagram	Moment of Inertia (I) kg m²	Radius of Gyration (K)	Ratio $\left(\frac{K^2}{R^2}\right)$
1.	**Thin Uniform Rod** Mass = M Length = ℓ	Passing through the center and perpendicular to the length		$\frac{1}{12}M\ell^2$	$\frac{1}{\sqrt{12}}$	--
		Touching one end and perpendicular to the length		$\frac{1}{3}M\ell^2$	$\frac{1}{\sqrt{3}}$	
2.	**Thin Uniform Rectangular Sheet** Mass = M; Length = ℓ; Breadth = b	Passing through the center and perpendicular to the plane of the sheet		$\frac{1}{12}M(\ell^2+b^2)$	$\sqrt{\frac{(\ell^2+b^2)}{12}}$	--
3.	Thin Uniform Ring Mass = M Radius = R	Passing through the center and perpendicular to the plane		MR^2	R	1
		Touching the edge perpendicular to the plane (perpendicular tangent)		$2MR^2$	$(\sqrt{2})R$	2
		Passing through the center lying on the plane (along diameter)		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Touching the edge parallel to the plane (parallel tangent)		$\frac{3}{2}MR^2$	$\left(\sqrt{\frac{3}{2}}\right)R$	$\frac{3}{2}$
4.	**Thin Uniform Disc** Mass = M Radius = R	Passing through the center and perpendicular to the plane		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Touching the edge perpendicular to the plane (perpendicular tangent to the plane)		$\frac{3}{2}MR^2$	$\left(\sqrt{\frac{3}{2}}\right)R$	$\frac{3}{2}$
		Passing through the center lying on the plane (along diameter)		$\frac{1}{4}MR^2$	$\left(\frac{1}{2}\right)R$	$\frac{1}{4}$
		Touching the edge parallel to the plane (parallel tangent to the plane)		$\frac{5}{4}MR^2$	$\left(\sqrt{\frac{5}{4}}\right)R$	$\frac{5}{4}$

5.	**Thin Uniform Hollow Cylinder** Mass = M Length = ℓ; Radius = R	Passing through the center and along the axis of the cylinder		MR^2	R	1
		Passing perpendicular to the length and passing through the center		$M\left(\frac{R^2}{2}+\frac{\ell^2}{12}\right)$	$\sqrt{\frac{R^2}{2}+\frac{\ell^2}{12}}$	- -
6.	Uniform Solid Cylinder Mass = M Length = ℓ; Radius = R	Passing through the center and along the axis of the cylinder		$\frac{1}{2}MR^2$	$\left(\frac{1}{\sqrt{2}}\right)R$	$\frac{1}{2}$
		Passing perpendicular to the length and passing through the center		$M\left(\frac{R^2}{4}+\frac{\ell^2}{12}\right)$	$\sqrt{\frac{R^2}{4}+\frac{\ell^2}{12}}$	- -
7.	Thin Hollow Sphere (Thin Spherical Shell) Mass = M Radius = R	Passing through the center (along diameter)		$\frac{2}{3}MR^2$	$\left(\sqrt{\frac{2}{3}}\right)R^2$	$\frac{2}{3}$
		Touching the edge (tangent)		$\frac{5}{3}MR^2$	$\left(\sqrt{\frac{5}{3}}\right)R$	$\frac{5}{3}$
8.	Uniform Solid Sphere Mass = M Radius = R	Passing through the center (along diameter)		$\frac{2}{5}MR^2$	$\left(\sqrt{\frac{2}{5}}\right)R^2$	$\frac{2}{5}$
		Touching the edge (tangent)		$\frac{7}{5}MR^2$	$\left(\sqrt{\frac{7}{5}}\right)R^2$	$\frac{7}{5}$

ROTATIONAL DYNAMICS

The relations among torque, angular acceleration, angular momentum, angular velocity and moment of inertia were seen in Section 5.2. In continuation to that, in this section, we will learn the relations among the other dynamical quantities like work, kinetic energy in rotational motion of rigid bodies. Finally a comparison between the translational and rotational quantities is made with a tabulation.

Effect of Torque on Rigid Bodies

A rigid body which has non zero externaltorque (τ) about the axis of rotation wouldhave an angular acceleration (α) aboutthat axis. The scalar relation between thetorque and angular acceleration is,

$$\tau = I\alpha$$

where, I is the moment of inertia of the rigid body. The torque in rotational motion is equivalent to the force in linear motion.

EXAMPLE

A disc of mass 500 g and radius 10 cm can freely rotate about a fixed axis as shown in figure. light and inextensible string is wound several turns around it and 100 g body is suspended at its free end. Find the acceleration of this mass. [Given: The string makes the disc to rotate and does not slip over it. g = 10 m s^{-2}.]

Solution

Let the mass of the disc be m1 and its radius R. The mass of the suspended body is m_2.

$$m_1 = 500\text{ g} = 500\times10^{-3}\text{ kg} = 0.5\text{ kg}$$
$$m_2 = 100\text{ g} = 100\times10^{-3}\text{ kg} = 0.1\text{ kg}$$
$$R = 10\text{ cm} = 10\times10^{-2}\text{ m} = 0.1\text{ m}$$

As the light inextensible string is wound around the disc several times it makes the disc rotate without slipping over it. The translational acceleration of m_2 and tangential acceleration of m_1 will be the same. Let us draw the free body diagram (FBD) of m_1 and m_2 separately.

Its gravitational force (m_1g) acts downward and normal force N exerted by the fixed support at the center acts upward. The tension T acts downward at the edge. The gravitational force (m_1g) and the normal force (N) cancel each other. m_1g= N

The tension T produces a torque (R T), which produces a rotational motion in the disc with angular acceleration, $\left(\alpha = \frac{a}{R^2}\right)$. Here, a is the linear acceleration of a point at the edge of the disc. If the moment of inertia of the disc is I and its radius of gyration is K, then

$$RT = I\alpha; \quad RT = \left(m_1K^2\right)\frac{a}{R}$$

$$T = \left(m_1K^2\right)\frac{a}{R^2}$$

FBD of the body:

Its gravitational force (m_2g) acts downward and the tension T acts upward. As ($T < m_2g$), there is a resultant force (m2a) acting on it downward.

Substituting for T from the equation for disc,

$$m_2g - \left(m_1K^2\right)\frac{a}{R^2} = m_2a$$

$$m_2g = \left(m_1K^2\right)\frac{a}{R^2} + m_2a$$

$$m_2g = \left[\left(m_1\frac{K^2}{R^2}\right) + m_2\right]a$$

$$a = \frac{m_2}{\left[\left(m_1\frac{K^2}{R^2}\right) + m_2\right]}g$$

The expression $\left(\frac{K^2}{R^2}\right)$ for a disc rotatingabout an axis passing through the centerand perpendicular to the plane is $\frac{K^2}{R^2} = \frac{1}{2}$. Now the expression for acceleration further simplifies as,

$$a = \frac{m_2}{\left[\left(\frac{m_1}{2}\right) + m_2\right]} g \; ; \quad a = \frac{2m_2}{[m_1 + 2m_2]} g$$

substituting the values,

$$a = \frac{2 \times 0.1}{[0.5 + 0.2]} \times 10 = \frac{0.2}{0.7} \times 10$$

$$a = 2.857\, \text{m s}^{-2}$$

Conservation of Angular Momentum

When no external torque acts on the body, the net angular momentum of a rotating rigid body remains constant. This is known as law of conservation of angular momentum.

$$\tau = \frac{dL}{dt}$$

$$\text{If } \tau = 0 \text{ then, } L = \text{constant}$$

As the angular momentum is $L = I\omega$, the conservation of angular momentum could further be written for initial and final situations as,

$$I_i \omega_i = I_f \omega_f \text{ (or) } I\omega = \text{constant}$$

The above equations say that if I increases ω will decrease and vice-versa to keep the angular momentum constant.

There are several situations where the principle of conservation of angular momentum is applicable. One striking example is an ice dancer as shown in Figure 5.27. The dancer spins slowly when the hands are stretched out and spins faster when the hands are brought close to the body. Stretching of hands away from body increases moment of inertia, thus the angular velocity decreases resulting in slower spin. When the hands are brought close to the body, the moment of inertia decreases, and thus the angular velocity increases resulting in faster spin.

A diver while in air as in Figure 5.28 curls the body close to decrease the moment of inertia, which in turn helps to increase the number of somersaults in air.

EXAMPLE

A jester in a circus is standing with his arms extended on a turn table rotating with angular velocity ω. He brings his arms closer to his body so that his moment of inertia is reduced to one third of the original value. Find his new angular velocity. [Given: There is no external torque on the turn table in the given situation.]

Solution

Let the moment of inertia of the jester with his arms extended be I. As there is no external torque acting on the jester and the turn table, his total angular momentum is conserved. We can write the equation,

$$I_i\omega_i = I_f\omega_f$$

$$I\omega = \frac{1}{3}I\omega_f \quad \because \left(I_f = \frac{1}{3}I\right)$$

$$\omega_f = 3\omega$$

The above result tells that the final angular velocity is three times that of initial angular velocity.

Work done by Torque

Let us consider a rigid body rotatingabout a fixed axis. Figure 5.29 shows apoint P on the body rotating about an axisperpendicular to the plane of the page. Atangential force F is applied on the body.

It produces a small displacement ds on the body. The work done (dw) by the force is,

$$dw = Fds$$

As the distance ds, the angle of rotation dθ and radius r are related by the expression,

$$ds = r\, d\theta$$

The expression for work done now becomes,

$$dw = F\, ds;\ dw = F\, r\, d\theta$$

The term (Fr) is the torque τ produced by the force on the body.

$$dw = \tau d\theta$$

This expression gives the work done by the external torque τ, which acts on the body rotating about a fixed axis through an angle dθ.

The corresponding expression for work done in translational motion is,

$$dw = Fds$$

Kinetic Energy in Rotation

Let us consider a rigid body rotating with angular velocity ω about an axis as shown in Figure 5.30. Every particle of the body will have the same angular velocity ω and different tangential velocities v based on its positions from the axis of rotation.

Let us choose a particle of mass m_i situated at distance r_i from the axis of rotation. It has a tangential velocity v_i given by the relation, $v_i = r_i \omega$. The kinetic energy KE_i of the particle is,

$$KE_i = \frac{1}{2} m_i v_i^2$$

Writing the expression with the angular velocity,

$$KE_i = \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2}(m_i r_i^2)\omega^2$$

For the kinetic energy of the whole body, which is made up of large number of such particles, the equation is written with summation as,

$$KE = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

where, the term $\sum m_i r_i^2$ is the moment of inertia I of the whole body. $I = \sum m_i r_i^2$

Hence, the expression for KE of the rigid body in rotational motion is,

$$KE = \frac{1}{2} I \omega^2$$

This is analogous to the expression for kinetic energy in translational motion.

$$KE = \frac{1}{2}Mv^2$$

Relation between rotational kinetic energy and angular momentum

Let a rigid body of moment of inertia I rotate with angular velocity ω.

The angular momentum of a rigid body is, L = Iω

The rotational kinetic energy of the rigid body is

$$KE = \frac{1}{2}I\omega^2$$

By multiplying the numerator and denominator of the above equation with I, we get a relation between L and KE as,

$$KE = \frac{1}{2}\frac{I^2\omega^2}{I} = \frac{1}{2}\frac{(I\omega)^2}{I}$$

$$KE = \frac{L^2}{2I}$$

EXAMPLE

Find the rotational kinetic energy of a ring of mass 9 kg and radius 3 m rotating with 240 rpm about an axis passing through its center and perpendicular to its plane. (rpm is a unit of speed of rotation which means revolutions per minute)

Solution

The rotational kinetic energy is, $KE = \frac{1}{2}I\omega^2$ The moment of inertia of the ring is, $I=MR^2$

$$I = 9 \times 3^2 = 9 \times 9 = 81\,kg\,m^2$$

The angular speed of the ring is,

$$\omega = 240\,\text{rpm} = \frac{240 \times 2\pi}{60}\,\text{rad s}^{-1}$$

$$KE = \frac{1}{2} \times 81 \times \left(\frac{240 \times 2\pi}{60}\right)^2 = \frac{1}{2} \times 81 \times (8\pi)^2$$

$$KE = \frac{1}{2} \times 81 \times 64 \times (\pi)^2 = 2592 \times (\pi)^2$$

$$KE \approx 25920\,\text{J} \quad \because (\pi)^2 \approx 10$$

$$KE = 25.920\ \text{kJ}$$

Power Delivered by Torque

Power delivered is the work done per unit time. If we differentiate the expression for work done with respect to time, we get the instantaneous power (P).

$$P = \frac{dw}{dt} = \tau \frac{d\theta}{dt} \quad \because (dw = \tau d\theta)$$

$$P = \tau\omega \qquad (5.54)$$

The analogous expression for instantaneous power delivered in translational motion is,

$$P = \vec{F} \cdot \vec{v}$$

Comparison of Translational and Rotational Quantities

Many quantities in rotational motion have expressions similar to that of translational motion. The rotational terms are compared with the translational equivalents in Table 5.4.

S.No	Translational Motion	Rotational motion about a fixed axis
1	Displacement, x	Angular displacement, θ
2	Time, t	Time, t
3	Velocity, $v = \frac{dx}{dt}$	Angular velocity, $\omega = \frac{d\theta}{dt}$
4	Acceleration, $a = \frac{dv}{dt}$	Angular acceleration, $\alpha = \frac{d\omega}{dt}$
5	Mass, m	Moment of inertia, I
6	Force, F = ma	Torque, $\tau = I\alpha$
7	Linear momentum, p = mv	Angular momentum, $L = I\omega$
8	Impulse, $F\Delta t = \Delta p$	Impulse, $\tau\Delta t = \Delta L$
9	Work done, w = F s	Work done, $w = \tau\theta$
10	Kinetic energy, $KE = \frac{1}{2}mv^2$	Kinetic energy, $KE = \frac{1}{2}I\omega^2$
11	Power, P = F v	Power, $P = \tau\omega$

ROLLING MOTION

The rolling motion is the most commonly observed motion in daily life. The motion of wheel is an example of rolling motion. Round objects like ring, disc, sphere etc. are most suitable for rolling .

Let us study the rolling of a disc on a horizontal surface. Consider a point P on the edge of the disc. While rolling, the point undergoes translational motion along with its center of mass and rotational motion with respect to its center of mass.

Combination of Translation and Rotation

We will now see how these translational and rotational motions are related in rolling. If the radius of the rolling object is R, in one full rotation, the center of mass is displaced by $2\pi R$ (its circumference). One would agree that not only the center of mass, but all the points on the disc are displaced by the same $2\pi R$ after one full rotation. The only difference is that the center of mass takes a straight path; but, all the other points undergo a path which has a combination of the translational and rotational motion. Especially the point on the edge undergoes a path of a cycloid as shown in the Figure 5.31.

As the center of mass takes only a straight line path, its velocity v_{CM} is only translational velocity v_{TRANS} ($v_{CM} = v_{TRANS}$). All the other points have two velocities. One is the translational velocity v_{TRANS}, (which is also the velocity of center of mass) and the other is the rotational velocity v_{ROT} ($v_{ROT} = r\omega$). Here, r is the distance of the point from the center of mass and ω is the angular velocity. The rotational velocity v_{ROT} is perpendicular to the instantaneous position vector from the center of mass as shown in Figure 5.32(a). The resultant of these two velocities is v. This resultant velocity v is perpendicular to the position vector from the point of contact of the rolling object with the surface on which it is rolling as shown in Figure 5.32(b).

We shall now give importance to the point of contact. In pure rolling, the point of the rolling object which comes in contact with the surface is at momentary rest. This is the case with every point that is on the edge of the rolling object. As the rolling proceeds, all the points on the edge, one by one come in contact with the surface; remain at momentary rest at the time of contact and then take the path of the cycloid as already mentioned.

Hence, we can consider the pure rolling in two different ways.

1. The combination of translational motion and rotational motion about the center of mass.

(or)

2. The momentary rotational motion about the point of contact

As the point of contact is at momentary rest in pure rolling, its resultant velocity v is zero (v = 0). For example, in Figure 5.33, at the point of contact, v_{TRANS} is forward (to right) and v_{ROT} is backwards (to the left).

That implies that, v_{TRANS} and v_{ROT} are equal in magnitude and opposite in direction ($v = v_{TRANS} - v_{ROT} = 0$). Hence, we conclude that in pure rolling, for all the points on the edge, the magnitudes of v_{TRANS} and v_{ROT} are equal ($v_{TRANS} = v_{ROT}$). As $v_{TRANS} = v_{CM}$ and $v_{ROT} = R\omega$, in pure rolling we have,

$$v_{CM} = R\,\omega$$

We should remember the special feature of the equation 5.55. In rotational motion, as per the relation $v = r\omega$, the center point will not have any velocity as r is zero. But in rolling motion, it suggests that the center point has a velocity v_{CM} given by equation 5.55.

For the topmost point, the two velocities v_{TRANS} and v_{ROT} are equal in magnitude and in the same direction (to the right). Thus, the resultant velocity v is the sum of these two velocities, $v = v_{TRANS} + v_{ROT}$. In other form, $v = 2\,v_{CM}$ as shown in Figure 5.34.

Slipping and Sliding

When the round object moves, it always tends to roll on any surface which has a coefficient of friction any value greater than zero ($\mu > 0$). The friction that enabling the rolling motion is called rolling friction. In pure rolling, there is no relative motion of the point of contact with the surface. When the rolling object speeds up or slows down, it must accelerate or decelerate respectively. If this suddenly happens it makes the rolling object to slip or slide.

Sliding

Sliding is the case when $v_{CM} > R\omega$ (or $v_{TRANS} > v_{ROT}$). The translation is more than the rotation. This kind of motion happens when sudden break is applied in a moving vehicles, or when the vehicle enters into a slippery road. In this case, the point of contact has more of v_{TRANS} than v_{ROT}. Hence, it has a resultant velocity v in the forward direction as shown in Figure 5.35. The kinetic frictional force (f_k) opposes the relative motion. Hence, it acts in the opposite direction of the relative velocity. This frictional force reduces the translational velocity and increases the rotational velocity till they become equal and the object sets on pure rolling. Sliding is also referred as forward slipping.

Slipping

Slipping is the case when $v_{CM} < R\omega$ (or $v_{TRANS} < v_{ROT}$). The rotation is more than the translation. This kind of motion happens when we suddenly start the vehicle from rest or the vehicle is stuck in mud. In this case, the point of contact has more of v_{ROT} than v_{TRANS}. It has a resultant velocity v in the backward direction as shown in Figure 5.36. The kinetic frictional force (f_k) opposes the relative motion. Hence it acts in the opposite direction of the relative velocity. This frictional force reduces the rotational velocity and increases the translational velocity till they become equal and the object sets pure rolling. Slipping is sometimes empahasised as backward slipping.

EXAMPLE

A rolling wheel has velocity of its center of mass as 5 m s-1. If its radius is 1.5 m and angular velocity is 3 rad s^{-1}, then check whether it is in pure rolling or not.

Solution

Translational velocity (v_{TRANS}) or velocity of center of mass, $v_{CM} = 5$ m s^{-1}

The radius is, R = 1.5 m and the angular velocity is, $\omega = 3$ rad s^{-1}

Rotational velocity, $v_{ROT} = R\omega$

$$v_{ROT} = 1.5 \times 3$$

$$v_{ROT} = 4.5 \text{ m s}^{-1}$$

$v_{CM} > R\omega$ (or) $v_{TRANS} > R\omega$, It is not in pure rolling, but sliding.

Kinetic Energy in Pure Rolling

As pure is the combination of translational and rotational motion, we can write the total kinetic energy (KE) as the sum of kinetic energy due to translational motion (KE_{TRANS}) and kinetic energy due to rotational motion (KE_{ROT}).

$$KE = KE_{TRANS} + KE_{ROT}$$

If the mass of the rolling object is M, the velocity of center of mass is v_{CM}, its moment of inertia about center of mass is I_{CM} and angular velocity is ω, then

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

With center of mass as reference: The moment of inertia (I_{CM}) of a rolling object about the center of mass is,

$I_{CM}=MK^2$ and $v_{CM} = R\omega$. Here, K is radius of gyration.

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}\left(MK^2\right)\frac{v_{CM}^2}{R^2}$$

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

$$KE = \frac{1}{2}Mv_{CM}^2\left(1 + \frac{K^2}{R^2}\right)$$

With point of contact as reference:

We can also arrive at the same expression by taking the momentary rotation happening with respect to the point of contact (another approach to rolling). If we take the point of contact as O, then,

$$KE = \frac{1}{2}I_o\omega^2$$

Here, Io is the moment of inertia of the object about the point of contact. By parallel axis theorem, $I_o = I_{CM} + MR^2$. Further we can write, $I_o = MK^2 + MR^2$. With $v_{CM} = R\omega$ or

$$\omega = \frac{v_{CM}}{R}$$

$$KE=\frac{1}{2}\left(MK^2+MR^2\right)\frac{v_{CM}^2}{R^2}$$

$$KE=\frac{1}{2}Mv_{CM}^2\left(1+\frac{K^2}{R^2}\right)$$

As the two equations 5.59 and 5.60 are the same, it is once again confirmed that the pure rolling problems could be solved by considering the motion as any one of the following two cases.

1. The combination of translational motion and rotational motion about the center of mass.
2. The momentary rotational motion about the point of contact.

EXAMPLE

A solid sphere is undergoing pure rolling. What is the ratio of its translational kinetic energy to rotational kinetic energy?

Solution

The expression for total kinetic energy in pure rolling is,

$$KE=KE_{TRANS}+KE_{ROT}$$

For any object the total kinetic energy as per equation 5.58 and 5.59 is,

$$KE=\frac{1}{2}Mv_{CM}^2+\frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

$$KE=\frac{1}{2}Mv_{CM}^2\left(1+\frac{K^2}{R^2}\right)$$

Then,

$$\frac{1}{2}Mv_{CM}^2\left(1+\frac{K^2}{R^2}\right)=\frac{1}{2}Mv_{CM}^2+\frac{1}{2}Mv_{CM}^2\left(\frac{K^2}{R^2}\right)$$

The above equation suggests that in pure rolling the ratio of total kinetic energy, translational kinetic energy and rotational kinetic energy is given as,

$$KE : KE_{TRANS} : KE_{ROT} :: \left(1+\frac{K^2}{R^2}\right):1:\left(\frac{K^2}{R^2}\right)$$

$$\text{Now, } KE_{TRANS} : KE_{ROT} :: 1:\left(\frac{K^2}{R^2}\right)$$

$$\text{For a solid sphere, } \frac{K^2}{R^2}=\frac{2}{5}$$

$$\text{Then, } KE_{TRANS} : KE_{ROT} :: 1:\frac{2}{5} \quad \text{or}$$

$$KE_{TRANS} : KE_{ROT} :: 5:2$$

Rolling on Inclined Plane

Let us assume a round object of mass m and radius R is rolling down an inclined plane without slipping as shown in Figure 5.37. There are two forces acting on the object along the inclined plane. One is the component of gravitational force (mg sinθ) and the other is the static frictional force (f). The other component of gravitation force (mg cosθ) is cancelled by the normal force (N) exerted by the plane. As the motion is happening along the incline, we shall write the equation for motion from the free body diagram (FBD) of the object.

For translational motion, mg sinθ is the supporting force and f is the opposing force

$$mg\sin\theta - f = ma$$

For rotational motion, let us take the torque with respect to the center of the object. Then mg sinθ cannot cause torque as it passes through it but the frictional force f can set torque of Rf.

$$Rf = I\alpha$$

By using the relation, a = r α, and moment of inertia $I=mK^2$, we get,

$$Rf = mK^2 \frac{a}{R}; \quad f = ma\left(\frac{K^2}{R^2}\right)$$

Now equation (5.59) becomes,

$$mg\sin\theta - ma\left(\frac{K^2}{R^2}\right) = ma$$

$$mg\sin\theta = ma + ma\left(\frac{K^2}{R^2}\right)$$

$$a\left(1+\frac{K^2}{R^2}\right) = g\sin\theta$$

After rewriting it for acceleration, we get,

$$a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

We can also find the expression for final velocity of the rolling object by using third equation of motion for the inclined plane. $v^2 = u^2 + 2as$. If the body starts rolling from rest, u=0. When h is the vertical height of the incline, the length of the incline s is, $s = \frac{h}{\sin\theta}$

$$v^2 = 2\frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}\left(\frac{h}{\sin\theta}\right) = \frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}$$

By taking square root,

$$v = \sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}$$

The time taken for rolling down the incline could also be written from first equation of motion as, v = u + at. For the object which starts rolling from rest, u=0. Then,

$$t = \frac{v}{a}$$

$$t = \left(\sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}\right)\left(\frac{\left(1+\frac{K^2}{R^2}\right)}{g\sin\theta}\right)$$

$$t = \sqrt{\frac{2h\left(1+\frac{K^2}{R^2}\right)}{g\sin^2\theta}} \qquad (5.64)$$

The equation suggests that for a given incline, the object with the least value of radius of gyration K will reach the bottom of the incline first.

EXAMPLE

Four round objects namely a ring, a disc, a hollow sphere and a solid sphere with same radius R start to roll down an incline at the same time. Find out which object will reach the bottom first.

Solution

For all the four objects namely the ring, disc, hollow sphere and solid sphere, the radii of gyration K are R, $\sqrt{\frac{1}{2}}R, \sqrt{\frac{2}{3}}R, \sqrt{\frac{2}{5}}R,$ ref Table (5.3)). With numerical values the radius of gyration K are 1R, 0.707R, 0.816R, 0.632R respectively. The expression for time taken for rolling has the radius of gyration K in the numerator as per equation 5.63

$$t = \sqrt{\frac{2h\left(1 + \frac{K^2}{R^2}\right)}{g \sin^2\theta}}$$

The one with least value of radius of gyration K will take the shortest time to reach the bottom of the inclined plane. The order of objects reaching the bottom is first, solid sphere; second, disc; third, hollow sphere and last, ring.