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UNIT – 6 GRAVITATION

INTRODUCTION

We are amazed looking at the glittering sky; we wonder how the Sun rises in the East and sets in the West, why there are comets or why stars twinkle. The sky has been an object of curiosity for human beings from time immemorial. We have always wondered about the motion of stars, the Moon, and the planets. From Aristotle to Stephen Hawking, great minds have tried to understand the movement of celestial objects in space and what causes their motion.

The ‘Theory of Gravitation’ was developed by Newton in the late 17th century to explain the motion of celestial objects and terrestrial objects and answer most of the queries raised. In spite of the study of gravitation and its effect on celestial objects, spanning last three centuries, “gravitation” is still one of the active areas of research in physics today. In 2017, the Nobel Prize in Physics was given for the detection of ‘Gravitational waves’ which was theoretically predicted by Albert Einstein in the year 1915. Understanding planetary motion, the formation of stars and galaxies, and recently massive objects like black holes and their life cycle have remained the focus of study for the past few centuries in physics.

Geocentric Model of Solar System

In the second century, Claudius Ptolemy, a famous Greco-Roman astronomer, developed a theory to explain the motion of celestial objects like the Sun, the Moon, Mars, Jupiter etc. This theory was called the geocentric model. According to the geocentric model, the Earth is at the center of the universe and all celestial objects including the Sun, the Moon, and other planets orbit the Earth. Ptolemy's model closely matched with the observations of the sky with our naked eye. But later, astronomers found that even though Ptolemy's model successfully explained the motion of the Sun and the Moon up to a certain level, the motion of Mars and Jupiter could not be explained effectively.

Heliocentric Model of Nicholas Copernicus

In the 15th century, a Polish astronomer, Nicholas Copernicus (1473-1543) proposed a new model called the 'Heliocentric model' in which the Sun was considered to be at the center of the solar system and all planets including the Earth orbited the Sun in circular orbits. This model successfully explained the motion of all celestial objects.

Around the same time, Galileo, a famous Italian physicist discovered that all objects close to Earth were accelerated towards the Earth at the same rate. Meanwhile, a noble man called Tycho Brahe (1546-1601) spent his entire lifetime in recording the observations of the stellar and planetary positions with his naked eye. The data that he compiled were analyzed later by his assistant Johannes Kepler (1571–1630) and eventually the analysis led to the deduction of the laws of the planetary motion. These laws are termed as 'Kepler's laws of planetary motion'.

Kepler's Laws of Planetary Motion

Law of orbits:

Each planet moves around the Sun in an elliptical orbit with the Sun at one of the foci.

The closest point of approach of the planet to the Sun 'P' is called perihelion and the farthest point 'A' is called aphelion (Figure 6.1). The semi-major axis is 'a' and semi-minor axis is 'b'. In fact, both Copernicus and Ptolemy considered planetary orbits to be circular, but Kepler discovered that the actual orbits of the planets are elliptical.

Law of area:

The radial vector (line joining the Sun to a planet) sweeps equal areas in equal intervals of time.

In Figure 6.2, the white shaded portion is the area DA swept in a small interval of time Dt, by a planet around the Sun. Since the Sun is not at the center of the ellipse, the planets travel faster when they are nearer to the Sun and slower when they are farther from it, to cover equal area in equal intervals of time. Kepler discovered the law of area by carefully noting the variation in the speed of planets.

Law of period:

The square of the time period of revolution of a planet around the Sun in its elliptical orbit is directly proportional to the cube of the semi-major axis of the ellipse. It can be written as:

$$T^2 \propto a^3$$

$$\frac{T^2}{a^3} = constant$$

where, T is the time period of revolution for a planet and a is the semi-major axis. Physically this law implies that as the distance of the planet from the Sun increases, the time period also increases but not at the same rate.

In Table 6.1, the time period of revolution of planets around the Sun along with their semi-major axes are given. From column four, we can realize that $\frac{T^2}{a^3}$ is nearly a constant endorsing Kepler's third law.

Planet	a ($10^{10} m$)	T (years)	$\frac{T^2}{a^3}$
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84	2.98
Neptune	450	165	2.99

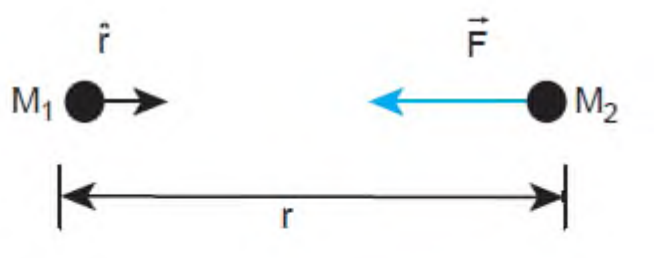
Universal Law of Gravitation

Even though Kepler's laws were able to explain the planetary motion, they failed to explain the forces responsible for it. It was Isaac Newton who analyzed Kepler's laws, Galileo's observations and deduced the law of gravitation.

Newton's law of gravitation states that a particle of mass M_1 attracts any other particle of mass M_2 in the universe with an attractive force. The strength of this force of attraction was found to be directly proportional to the product of their masses and is inversely proportional to the square of the distance between them. In mathematical form, it can be written as:

$$\vec{F} = -\frac{GM_1M_2}{r^2}\hat{r}$$

where $\hat{r}$ is the unit vector from M_1 towards M_2 as shown in Figure 6.3, and G is the Gravitational constant that has the value of 6.626×10^{-11}. Nm2kg^{-2}, and r is the distance between the two masses M_1 and M_2. In Figure 6.3, the vector $\vec{F}$ denotes the gravitational force experienced by M_2 due to M_1. Here the negative sign indicates that the gravitational force is always attractive in nature and the direction of the force is along the line joining the two masses.



In cartesian coordinates, the square of the distance is expressed as $r^2=(x^2+y^2+z^2)$ This is dealt in unit 2.

EXAMPLE

Consider two point masses m_1 and m_2 which are separated by a distance of 10 meter as shown in the following figure. Calculate the force of attraction between them and draw the directions of forces on each of them. Take m_1= 1 kg and m_2 = 2 kg

Solution

The force of attraction is given by

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

From the figure, r =10 m.

First, we can calculate the magnitude of the force

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{100}$$
$$= 13.34\times10^{-13}N.$$

It is to be noted that this force is very small. This is the reason we do not feel the gravitational force of attraction between each other. The small value of G plays a very crucial role in deciding the strength of the force.

The force of attraction ($\vec{F}_{21}$) experienced by the mass m_2 due to m_1 is in the negative 'y' direction ie., $\hat{r} = -\hat{j}$. According to Newton's third law, the mass m_2 also exerts equal and opposite force on m_1. So the force of attraction ($\vec{F}_{12}$) experienced by m1 due to m2 is in the direction of positive 'y' axis ie., $\hat{r} = \hat{j}$.

$$\vec{F}_{21} = -13.34\times10^{-13}\hat{j}$$

$$\vec{F}_{12} = 13.34\times10^{-13}\hat{j}$$

The direction of the force is shown in the figure,
Gravitational force of attraction between m_1 and m_2
$\vec{F}_{12} = -\vec{F}_{21}$ which confirms Newton's third law.

Important features of gravitational force:

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As the distance between two masses increases, the strength of the force tends to decrease because of inverse dependence on r^2. Physically it implies that the planet Uranus experiences less gravitational force from the Sun than the Earth since Uranus is at larger distance from the Sun compared to the Earth.

The gravitational forces between two particles always constitute an action-reaction pair. It implies that the gravitational force exerted by the Sun on the Earth is always towards the Sun. The reaction-force is exerted by the Earth on the Sun. The direction of this reaction force is towards Earth.

The torque experienced by the Earth due to the gravitational force of the Sun is given by

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \left(-\frac{GM_S M_E}{r^2} \hat{r} \right) = 0$$

$$\text{Since } \vec{r} = r\,\hat{r},\ \left(\hat{r} \times \hat{r}\right) = 0$$

- So $\vec{\tau} = \frac{d\vec{L}}{dt} = 0$. It implies that angular momentum $\vec{L}$ is a constant vector. The angular momentum of the Earth about the Sun is constant throughout the motion. It is true for all the planets. In fact, this constancy of angular momentum leads to the Kepler's second law.
- The expression $\vec{F} = -\frac{GM_1M_2}{r^2}\hat{r}$ has one inherent assumption that both M_1 and M_2 are treated as point masses. When it is said that Earth orbits around the Sun due to Sun's gravitational force, we assumed Earth and Sun to be point masses. This assumption is a good approximation because the distance between the two bodies is very much larger than their diameters. For some irregular and extended objects separated by a small distance, we cannot directly use the equation (6.3). Instead, we have to invoke separate mathematical treatment which will be brought forth in higher classes.
- However, this assumption about point masses holds even for small distance for one special case. To calculate force of attraction between a hollow sphere of mass M with uniform density and point mass m kept outside the hollow sphere, we can replace the hollow sphere of mass M as equivalent to a point mass M located at the center of the hollow sphere. The force of attraction between the hollow sphere of mass M and point mass m can be calculated by treating thehollow sphere also as another point mass. Essentially the entire mass of the hollow sphere appears to be concentrated at the center of the hollow sphere.
- There is also another interesting result. Consider a hollow sphere of mass M. If we place another object of mass 'm' inside this hollow sphere as in Figure 6.5(b), the force experienced by this mass 'm' will be zero. This calculation will be dealt with in higher classes.
- The triumph of the law of gravitation is that it concludes that the mango that is falling down and the Moon orbiting the Earth are due to the same gravitational force.

Newton's inverse square Law:

Newton considered the orbits of the planets as circular. For circular orbit of radius r, the centripetal acceleration towards the center is

$$a=-\frac{v^2}{r}$$

Here v is the velocity and r, the distance of the planet from the center of the orbit

The velocity in terms of known quantities r and T, is

$$v=\frac{2\pi r}{T}$$

Here T is the time period of revolution of the planet. Substituting this value of v in equation (6.4) we get,

$$a=-\frac{\left(\frac{2\pi r}{T}\right)^2}{r}=-\frac{4\pi^2 r}{T^2}$$

Substituting the value of ‘a’ from (6.6) in Newton’s second law, F=ma, where ‘m’ is the mass of the planet.

$$F=-\frac{4\pi^2 mr}{T^2}$$

From Kepler’s third law,

$$\frac{r^3}{T^2}=k(constant)$$

$$\frac{r}{T^2}=\frac{k}{r^2}$$

By substituting equation 6.9 in the force expression, we can arrive at the law of gravitation.

$$F = -\frac{4\pi^2 mk}{r^2}$$

Here negative sign implies that the force is attractive and it acts towards the center. In equation (6.10), mass of the planet ‘m’ comes explicitly. But Newton strongly felt that according to his third law, if Earth is attracted by the Sun, then the Sun must also be attracted by the Earth with the same magnitude of force. So he felt that the Sun’s mass (M) should also occur explicitly in the expression for force (6.10). From this insight, he equated the constant $4p^2k$ to GM which turned out to be the law of gravitation

$$F = -\frac{GMm}{r^2}$$

Again the negative sign in the above equation implies that the gravitational force is attractive.

In the above discussion we assumed that the orbit of the planet to be circular which is not true as the orbit of the planet around the Sun is elliptical. But this circular orbit assumption is justifiable because planet’s orbit is very close to being circular and there is only a very small deviation from the circular shape.

EXAMPLE

Moon and an apple are accelerated by the same gravitational force due to Earth. Compare the acceleration of the two.

The gravitational force experienced by the apple due to Earth

$$F = -\frac{GM_E M_A}{R^2}$$

Here M_A– Mass of the apple, M_E– Mass of the Earth and R – Radius of the Earth.

Equating the above equation with Newton’s second law,

$$M_A a_A = -\frac{GM_E M_A}{R^2}.$$

Simplifying the above equation we get,

$$a_A = -\frac{GM_E}{R^2}$$

Here a_A is the acceleration of apple that is equal to 'g'. Similarly the force experienced by Moon due to Earth is given by

$$F = -\frac{GM_E M_m}{R_m^2}.$$

Here R_m- distance of the Moon from the Earth, M_m – Mass of the Moon

The acceleration experienced by the Moon is given by

$$a_m = -\frac{GM_E}{R_m^2}.$$

The ratio between the apple's acceleration to Moon's acceleration is given by

$$\frac{a_A}{a_m} = \frac{R_m^2}{R^2}.$$

From the Hipparchrus measurement, the distance to the Moon is 60 times that of Earth radius. $R_m = 60R$.

$$a_A / a_m = \frac{(60R)^2}{R^2} = 3600.$$

The apple's acceleration is 3600 times the acceleration of the Moon.

The same result was obtained by Newton using his gravitational formula. The apple's acceleration is measured easily and it is 9.8 m s−2. Moon orbits the Earth once in 27.3 days and by using the centripetal acceleration formula, (Refer unit 3).

$$\frac{a_A}{a_m} = \frac{9.8}{0.00272} = 3600$$

which is exactly what he got through his law of gravitation.

Gravitational Constant

In the law of gravitation, the value of gravitational constant G plays a very important role. The value of G explains why the gravitational force between the Earth and the Sun is so great while the same force between two small objects (for example between two human beings) is negligible.

The force experienced by a mass ‘m’ which is on the surface of the Earth (Figure 6.7) is given by

$$F = -\frac{GM_E m}{R_E^2}$$

M_E-mass of the Earth, m - mass of the object, R_E- radius of the Earth.

Equating Newton’s second law, F =mg, to equation (6.11) we get,

$$mg = -\frac{GM_E m}{R_E^2}$$

$$g = -\frac{GM_E}{R_E^2}$$

Now the force experienced by some other object of mass M at a distance r from the center of the Earth is given by,

$$F = -\frac{GM_E M}{r^2}$$

Using the value of g in equation (6.12), the force F will be,

$$F = -gM\frac{R_E^2}{r^2}$$

From this it is clear that the force can be calculated simply by knowing the value of g. It is to be noted that in the above calculation G is not required.

In the year 1798, Henry Cavendish experimentally determined the value of gravitational constant 'G' by using a torsion balance. He calculated the value of 'G' to be equal to $6.75 \times 10^{-11} Nm2kg^{-2}$. Using modern techniques a more accurate value of G could be measured. Th e currently accepted value of G is $6.67259 \times 10^{-11} Nm2kg^{-2}$.

GRAVITATIONAL FIELD AND GRAVITATIONAL POTENTIAL

Gravitational field

Force is basically due to the interaction between two particles. Depending upon the type of interaction we can have two kinds of forces: Contact forces and Non-contact forces (Figure 6.8).

Contact forces are the forces applied where one object is in physical contact with the other. The movement of the object is caused by the physical force exerted through the contact between the object and the agent which exerts force.

Consider the case of Earth orbiting around the Sun. Though the Sun and theEarth are not physically in contact with each other, there exists an interaction between them. This is because of the fact that the Earth experiences the gravitational force of the Sun. This gravitational force is a non-contact force.

It sounds mysterious that the Sun attracts the Earth despite being very far from it and without touching it. For contact forces like push or pull, we can calculate the strength of the force since we can feel or see. But how do we calculate the strength of non-contact force at different distances? To understand and calculate the strength of non-contact forces, the concept of 'field' is introduced.

The gravitational force on a particle of mass 'm_2' due to a particle of mass 'm_1' is

$$\vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

where $\hat{r}$ is a unit vector that points from m_1 to m_2 along the line joining the masses m_1 and m_2.

The gravitational field intensity $\vec{E}_1$ (here after called as gravitational field) at a point which is at a distance r from m_1 is defined as the gravitational force experienced by unit mass placed at that point. It given by the ratio $\frac{\vec{F}_{21}}{m_2}$ (where m_2 is the mass of the object on which $\vec{F}_{21}$ acts)

Using $\vec{E}_1 = \frac{\vec{F}_{21}}{m_2}$ in equation (6.14) we get,

$$\vec{E}_1 = -\frac{Gm_1}{r^2}\hat{r}$$

$\vec{E}_1$ is a vector quantity that points towards the mass m1 and is independent of mass m2, Here m2 is taken to be of unit magnitude . The unit is $\hat{r}$ along the line between m_1 and the point in question. The field $\vec{E}_1$ is due to the mass m_1. In general, the gravitational field intensity due to a mass M at a distance r is given by

$$\vec{E} = -\frac{GM}{r^2}\hat{r}$$

Now in the region of this gravitational field, a mass 'm' is placed at a point P (Figure 6.9). Mass 'm' interacts with the field $\vec{E}_1$ and experiences an attractive force due to M as shown in Figure 6.9. The gravitational force experienced by 'm' due to 'M' is given by

$$\vec{F}_m = m\vec{E}$$

Now we can equate this with Newton's second law $\vec{F} = m\vec{a}$

$$m\vec{a} = m\vec{E}$$

$$\vec{a} = \vec{E}$$

In other words, equation (6.18) implies that the gravitational field at a point is equivalent to the acceleration experienced by a particle at that point. However, it is to be noted that $\vec{a}$ and $\vec{E}$ are separate physical quantities that have the same magnitude and direction. The gravitational field $\vec{E}$ is the property of the source and acceleration $\vec{a}$ is the effect experienced by the test mass (unit mass) which is placed in the gravitational field $\vec{E}$. The noncontact interaction between two masses can now be explained using the concept of "Gravitational field".

Points to be noted:

The strength of the gravitational field decreases as we move away from the mass M as depicted in the Figure 6.10. The magnitude of $\vec{E}$ decreases as the distance r increases.

Figure 6.10 shows that the strength of the gravitational field at points P, Q, and R is given by $|\vec{E}_P| < |\vec{E}_Q| < |\vec{E}_R|$. It can be understood by comparing the length of the vectors at points P, Q, and R.

The "field" concept was introduced as a mathematical tool to calculate gravitational interaction. Later it was found that field is a real physical quantity and it carries energy and momentum in space. The concept of field is inevitable in understanding the behavior of charges.

The unit of gravitational field is Newton per kilogram (N / kg) or m s^{-2}.

Superposition principle for Gravitational field

Consider 'n' particles of masses $\vec{m}_1, \vec{m}_2, \vec{m}_3, \ldots\ldots$ distributed in space at positions - $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots\ldots$etc, with respect to point P. The total gravitational field at a point P due to all the masses is given by the vector sum of the gravitational field due to the individual masses (Figure 6.11). This principle is known as superposition of gravitational fields.

$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 + \ldots \vec{E}_n$$

$$= -\frac{Gm_1}{r_1^2}\hat{r}_1 - \frac{Gm_2}{r_2^2}\hat{r}_2 - \ldots - \frac{Gm_n}{r_n^2}\hat{r}_n$$

$$= -\sum_{i=1}^{n} \frac{Gm_i}{r_i^2}\hat{r}_i .$$

Instead of discrete masses, if we have continuous distribution of a total mass M, then the gravitational field at a point P is calculated using the method of integration.

EXAMPLE

Two particles of masses m_1 and m_2 are placed along the x and y axes respectively at a distance 'a' from the origin. Calculate the gravitational field at a point P shown in figure below.

Solution

Gravitational field due to m_1 at a point P is given by

$$\vec{E}_1 = -\frac{Gm_1}{a^2}\hat{j}$$

Gravitational field due to m_2 at the point p is given by,

$$\vec{E}_2 = -\frac{Gm_2}{a^2}\hat{i}$$

$$\vec{E}_{total} = -\frac{Gm_1}{a^2}\hat{j} - \frac{Gm_2}{a^2}\hat{i}$$

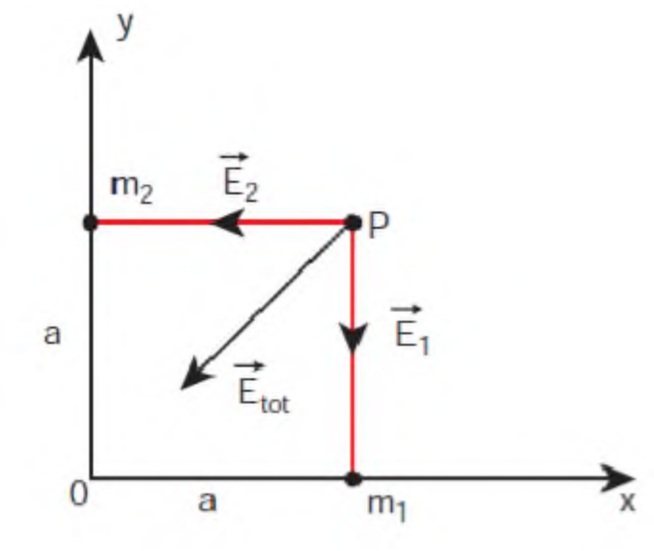
$$= -\frac{G}{a^2}\left(m_1\hat{j} + m_2\hat{i}\right)$$

The direction of the total gravitational field is determined by the relative value of m_1 and m_2

When $m_1 = m_2 = m$

$$\vec{E}_{total} = -\frac{Gm}{a^2}\left(\hat{i} + \hat{j}\right)$$

($\hat{i} + \hat{j} = \hat{j} + \hat{i}$ as vectors obeys commutation law).



$\vec{E}_{total}$ points towards the origin of the co-ordinate system and the magnitude of $\vec{E}_{total}$ is $\frac{Gm}{a^2}$.

EXAMPLE

Qualitatively indicate the gravitational field of Sun on Mercury, Earth, and Jupiter shown in figure.

Since the gravitational field decreases as distance increases, Jupiter experiences a weak gravitational field due to the Sun. Since Mercury is the nearest to the Sun, it experiences the strongest gravitational field.

Gravitational Potential Energy

The concept of potential energy and its physical meaning were dealt in unit 4. The gravitational force is a conservative force and hence we can define a gravitational potential energy associated with this conservative force field.

Two masses m_1 and m_2 are initially separated by a distance r′. Assuming m_1 to be fixed in its position, work must be done on m_2 to move the distance from r′ to r as shown in Figure 6.12(a).

To move the mass m_2 through an infinitesimal displacement $d\vec{r}$ from $\vec{r}$ to $\vec{r}+d\vec{r}$ (shown in the Figure 6.12(b)), work has to be done externally. This infinitesimal work is given by

$$dW = \vec{F}_{ext}.d\vec{r}$$

The work is done against the gravitational force, therefore,

$$\left|\vec{F}_{ext}\right| = \left|\vec{F}_G\right| = \frac{Gm_1m_2}{r^2}$$

Substituting Equation (6.22) in 6.21, we get

$$dW = \frac{Gm_1m_2}{r^2}\hat{r}.d\vec{r}$$

Also we know,

$$d\vec{r} = dr\,\hat{r} \qquad (6.24)$$

$$\Rightarrow dW = \frac{Gm_1m_2}{r^2}\hat{r}.\left(dr\hat{r}\right) \qquad (6.25)$$

$$\hat{r}.\hat{r} = 1\left(\textit{since both are unit vectors}\right)$$

$$\therefore \quad dW = \frac{Gm_1m_2}{r^2}dr \qquad (6.26)$$

Thus the total work done for displacing the particle from r’ to r is

$$W = \int_{r'}^{r} dW = \int_{r'}^{r} \frac{Gm_1m_2}{r^2}dr$$

$$W = -\left(\frac{Gm_1m_2}{r}\right)_{r'}^{r}$$

$$W = -\frac{Gm_1m_2}{r} + \frac{Gm_1m_2}{r'}$$

$$W = U\left(r\right) - U\left(r'\right)$$

$$\text{where } U(r) = \frac{-Gm_1m_2}{r}$$

This work done W gives the gravitational potential energy difference of the system of masses m_1 and m_2 when the separation between them are r and r′ respectively.

Case 1: If r<r'

Since gravitational force is attractive, m_2 is attracted by m_1.Then m_2 can move from r to r′ without any external work (Figure 6.13). Here work is done by the system spending its internal energy and hence the work done is said to be negative.

Case 2: If r>r'

Work has to be done against gravity to move the object from r to r′. Therefore work is done on the body by external force and hence work done is positive.

It is to be noted that only potential energy difference has physical significance. Now gravitational potential energy can be discussed by choosing one point as the reference point

Let us choose r'=¥ . Then the second term in the equation (6.28) becomes zero.

$$W = -\frac{Gm_1m_2}{r} + 0$$

Now we can define gravitational potential energy of a system of two masses m_1 and m_2 separated by a distance r as the amount of work done to bring the mass m_2 from infinity to a distance r assuming m_1 to be fixed in its position and is written as $U(r) = -\frac{Gm_1m_2}{r}$. It is to be noted that the gravitational potential energy of the system consisting of two masses m_1 and m_2 separated by a distance r, is the gravitational potential energy difference of the system when the masses areseparated by an infinite distance and by distance r. $U(r) = U(r) - U(¥)$. Here we choose $U(¥)$=0 as the reference point. The gravitational potential energy $U(r)$ is always negative because when two masses come together slowly from infinity, work is done by the system.

The unit of gravitational potential energy $U(r)$ is Joule and it is a scalar quantity. The gravitational potential energy depends upon the two masses and the distance between them.

Gravitational potential energy near the surface of the Earth

It is already discussed in chapter 4 that when an object of mass m is raised to a height h, the potential energy stored in the object is mgh (Figure 6.14). This can be derived using the general expression for gravitational potential energy

Consider the Earth and mass system, with r, the distance between the mass m and the Earth's centre. Then the gravitational potential energy,

$$U = -\frac{GM_e m}{r}$$

Here r = R_e+h, where R_e is the radius of the Earth. h is the height above the Earth's surface

$$U = -G\frac{M_e m}{(R_e + h)}$$

If h << Re, equation (6.31) can be modified as

$$U = -G\frac{M_e m}{R_e(1 + h/R_e)}$$

$$U = -G\frac{M_e m}{R_e}(1 + h/R_e)^{-1}$$

By using Binomial expansion and neglecting the higher order terms, we get

$$U = -G\frac{M_e m}{R_e}\left(1 - \frac{h}{R_e}\right).$$

We know that, for a mass m on the Earth's surface,

$$G\frac{M_e m}{R_e} = mgR_e$$

Substituting equation (6.34) in (6.33) we get,

$$U = -mgR_e + mgh$$

It is clear that the first term in the above expression is independent of the height h. For example, if the object is taken from height h_1 to h_2,then the potential energy at h_1 is

$$U(h_1) = -mgR_e + mgh_1$$

and the potential energy at h_2 is

$$U(h_2) = -mgR_e + mgh_2$$

The potential energy difference between h1 and h_2 is

$$U(h_2) - U(h_1) = mg(h_1 - h_2).$$

The term mgR_e in equations (6.36) and (6.37) plays no role in the result. Hence in the equation (6.35) the first term can be omitted or taken to zero. Thus it can be stated that The gravitational potential energy stored in the particle of mass m at a height h from the surface of the Earth is U =mgh. On the surface of the Earth, U = 0, since h is zero.

It is to be noted that mgh is the work done on the particle when we take the mass m from the surface of the Earth to a height h. This work done is stored as a gravitational potential energy in the mass m. Even though mgh is gravitational potential energy of the system (Earth and mass m), we can take mgh as the gravitational potential energy of the mass m since Earth is stationary when the mass moves to height h.

Gravitational potential V(r)

It is explained in the previous sections that the gravitational field $\vec{E}$ depends only on the source mass which creates the field. It is a vector quantity. We can also define a scalar quantity called "gravitational potential" which depends only on the source mass.

The gravitational potential at a distance r due to a mass is defined as the amount of work required to bring unit mass from infinity to the distance r and it is denoted as V(r). In other words, the gravitational potential at distance r is equivalent to gravitational potential energy per unit mass at the same distance r. It is a scalar quantity and its unit is J kg^{-1}

We can determine gravitational potential from gravitational potential energy. Consider two masses m_1 and m_2 separated by a distance r which has gravitational potential energy U (r)- (Figure 6.15). The gravitational potential due to mass m_1 at a point P which is at a distance r from m_1 is obtained by making m_2 equal to unity (m_2 = 1kg). Thus the gravitational potential V r - due to mass m_1 at a distance r is

$$V(r) = -\frac{Gm_1}{r}$$

Gravitational field and gravitational force are vector quantities whereas the gravitational potential and gravitational potential energy are scalar quantities. The motion of particles can be easily analyzed using scalar quantities than vector quantities. Consider the example of a falling apple:

Figure 6.16 shows an apple which falls on Earth due to Earth's gravitational force. This can be explained using the concept of gravitational potential V (r) - as follows.

The gravitational potential V (r) - at a point of height h from the surface of the Earth is given by,

$$V(r = R + h) = -\frac{GM_e}{(R + h)}$$

The gravitational potential V (r) - on the surface of Earth is given by,

$$V(r = R) = -\frac{GM_e}{R}$$

Thus we see that

$$V(r = R) < V(r = R + h).$$

It is already discussed in the previous section that the gravitational potential energy near the surface of the Earth at height h is mgh. The gravitational potential at this point is simply V(h)= U(h) / m = gh. In fact, the gravitational potential on the surface of the Earth is zero since h is zero. So the apple falls from a region of a higher gravitational potential to a region of lower gravitational potential. In general, the mass will move from a region of higher gravitational potential to a region of lower gravitational potential.

EXAMPLE

Water falls from the top of a hill to the ground. Why?

This is because the top of the hill is a point of higher gravitational potential than the surface of the Earth i.e. V_{hill} V_{ground}

The motion of particles can be analysed more easily using scalars like U(r) or V(r) than vector quantities like $\vec{F}$ or $\vec{E}$. Inmodern theories of physics, the concept of potential plays a vital role.

EXAMPLE

Consider four masses m_1, m_2, m_3, and m_4 arranged on the circumference of a circle as shown in figure below

Calculate
(a) The gravitational potential energy of the system of 4 masses shown in figure.
(b) The gravitational potential at the point O due to all the 4 masses

Solution

The gravitational potential energy U(r) can be calculated by finding the sum of gravitational potential energy of each pair of particles.

$$U = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_1m_4}{r_{14}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_2m_4}{r_{24}} - \frac{Gm_3m_4}{r_{34}}$$

Here r_{12}, r_{13} ... are distance between pair of particles

$$r_{14}^{\ 2} = R^2 + R^2 = 2R^2$$

$$r_{14} = \sqrt{2}R = r_{12} = r_{23} = r_{34}$$

$$r_{13} = r_{24} = 2R$$

$$U = -\frac{Gm_1m_2}{\sqrt{2}R} - \frac{Gm_1m_3}{2R} - \frac{Gm_1m_4}{\sqrt{2}R} - \frac{Gm_2m_3}{\sqrt{2}R} - \frac{Gm_2m_4}{2R} - \frac{Gm_3m_4}{\sqrt{2}R}$$

$$U = -\frac{G}{R}\left[\frac{m_1m_2}{\sqrt{2}} + \frac{m_1m_3}{2} + \frac{m_1m_4}{\sqrt{2}} + \frac{m_2m_3}{\sqrt{2}} + \frac{m_2m_4}{2} + \frac{m_3m_4}{\sqrt{2}}\right]$$

If all the masses are equal, then $m_1 = m_2 = m_3 = m_4 = M$

$$U = -\frac{GM^2}{R}\left[\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}}\right]$$

$$U = -\frac{GM^2}{R}\left[1 + \frac{4}{\sqrt{2}}\right]$$

$$U = -\frac{GM^2}{R}\left[1 + 2\sqrt{2}\right]$$

The gravitational potential V(r) at a point O is equal to the sum of the gravitational potentials due to individual mass. Since potential is a scalar, the net potential at point O is the algebraic sum of potentials due to each mass.

$$V_O(r) = -\frac{Gm_1}{R} - \frac{Gm_2}{R} - \frac{Gm_3}{R} - \frac{Gm_4}{R}$$

$$\text{If } m_1 = m_2 = m_3 = m_4 = M$$

$$V_O(r) = -\frac{4GM}{R}$$

ACCELERATION DUE TO GRAVITY OF THE EARTH

When objects fall on the Earth, the acceleration of the object is towards the Earth. From Newton's second law, an object is accelerated only under the action of a force. In the case of Earth, this force is the gravitational pull of Earth. This force produces a constant acceleration near the Earth's surface in all bodies, irrespective of their masses. The gravitational force exerted by Earth on the mass m near the surface of the Earth is given by

$$\vec{F} = -\frac{GmM_e}{R_e^{\,2}}\hat{r}$$

Now equating Gravitational force to Newton's second law,

$$m\vec{a} = -\frac{GmM_e}{R_e^{\,2}}\hat{r}$$

hence, acceleration is,

$$\vec{a} = -\frac{GM_e}{R_e^{\,2}}\hat{r}$$

The acceleration experienced by the object near the surface of the Earth due to its gravity is called acceleration due to gravity. It is denoted by the symbol g. The magnitude of acceleration due to gravity is

$$|g| = \frac{GM_e}{R_e^{\,2}}.$$

It is to be noted that the acceleration experienced by any object is independent of its mass. The value of g depends only on the mass and radius of the Earth. Infact, Galileo arrived at the same conclusion 400 years ago that all objects fall towards the Earth with the same acceleration through various quantitative experiments. The acceleration due to gravity g is found to be 9.8 m s^{-2} on the surface of the Earth near the equator.

Variation of g with altitude, depth and latitude

Consider an object of mass m at a height h from the surface of the Earth. Acceleration experienced by the object due to Earth is

$$g' = \frac{GM}{(R_e + h)^2}$$

$$g' = \frac{GM}{R_e^2\left(1 + \frac{h}{R_e}\right)^2}$$

$$g' = \frac{GM}{R_e^2}\left(1 + \frac{h}{R_e}\right)^{-2}$$

If $h << R_e$

We can use Binomial expansion. Taking the terms upto first order

$$g' = \frac{GM}{R_e^2}\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

We find that $g' < g$. This means that as altitude h increases the acceleration due to gravity g decreases

EXAMPLE

Calculate the value of g in the following two cases:
(a) If a mango of mass ½ kg falls from a tree from a height of 15 meters, what is the acceleration due to gravity when it begins to fall?

Solution

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = 9.8\left(1 - \frac{2\times 15}{6400\times 10^3}\right)$$

$$g' = 9.8\left(1 - 0.469\times 10^{-5}\right)$$

But $1 - 0.00000469 \cong 1$

Therefore g'=g

(b) Consider a satellite orbiting the Earth in a circular orbit of radius 1600 km above the surface of the Earth. What is the acceleration experienced by the satellite due to Earth's gravitational force?

Solution

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = g\left(1 - \frac{2\times 1600\times 10^3}{6400\times 10^3}\right)$$

$$g' = g\left(1 - \frac{2}{4}\right)$$

$$g' = g\left(1 - \frac{1}{2}\right) = g/2$$

The above two examples show that the acceleration due to gravity is a constant near the surface of the Earth.

Variation of g with depth:

Consider a particle of mass m which is in a deep mine on the Earth. (Example: coal mines in Neyveli). Assume the depth of the mine as d. To calculate g' at a depth d, consider the following points.

The part of the Earth which is above the radius ($R_e - d$) do not contribute to the acceleration. The result is proved earlier and is given as

$$g' = \frac{GM'}{(R_e - d)^2}$$

Here M' is the mass of the Earth of radius ($R_e - d$)
Assuming the density of Earth ρ to be constant

$$\rho = \frac{M}{V}$$

where M is the mass of the Earth and V its volume, Thus,

$$\rho = \frac{M'}{V'}$$

$$\frac{M'}{V'} = \frac{M}{V} \text{ and } M' = \frac{M}{V}V'$$

$$M' = \left(\frac{M}{\frac{4}{3}\pi R_e^{\ 3}}\right)\left(\frac{4}{3}\pi (R_e - d)^3\right)$$

$$M' = \frac{M}{R_e^{\ 3}}(R_e - d)^3$$

$$g' = G\frac{M}{R_e^{\ 3}}(R_e - d)^3 . \frac{1}{(R_e - d)^2}$$

$$g' = GM\frac{R_e\left(1 - \frac{d}{R_e}\right)}{R_e^{\ 3}}$$

$$g' = GM\frac{\left(1-\frac{d}{R_e}\right)}{R_e^{\ 2}}$$

Thus

$$g' = g\left(1-\frac{d}{R_e}\right)$$

Here also $g' < g$. As depth increases, g′ decreases. It is very interesting to know that acceleration due to gravity is maximum on the surface of the Earth but decreases when we go either upward or downward.

Variation of g with latitude:

Whenever we analyze the motion of objects in rotating frames [explained in chapter 3] we must take into account the centrifugal force. Even though we treat the Earth as an inertial frame, it is not exactly correct because the Earth spins about its own axis. So when an object is on the surface of the Earth, it experiences a centrifugal force that depends on the latitude of the object on Earth. If the Earth were not spinning, the force on the object would have been mg. However, the object experiences an additional centrifugal force due to spinning of the Earth.

This centrifugal force is given by $m\omega^2 R'$.

$$R' = R\cos\lambda$$

where λ is the latitude. The component of centrifugal acceleration experienced by the object in the direction opposite to g is

$$a_{PQ} = \omega^2 R'\cos\lambda = \omega^2 R\cos^2\lambda$$
$$since\ R' = R\cos\lambda$$

Therefore,

$$g' = g - \omega^2 R\cos^2\lambda$$

From the expression (6.52), we can infer that at equator, $\lambda=0$; $g'=g-w^2R$. The acceleration due to gravity is minimum. At poles $\lambda= 90$; $g'=g$, it is maximum. At the equator, g'is minimum.

EXAMPLE

Find out the value of g′ in your school laboratory?

Solution

Calculate the latitude of the city or village where the school is located. The information is available in Google search. For example, the latitude of Chennai is approximately 13 degree.

$$g' = g - \omega^2 R \cos^2 \lambda$$

Here $\omega^2R = (2x3.14/86400)^2 \times (6400x10^3) = 3.4x10^{-2}$ m s^{-2}.

It is to be noted that the value of λ should be in radian and not in degree. 13 degree is equivalent to 0.2268 rad.

$$g' = 9.8 - \left(3.4 \times 10^{-2}\right) \times \left(\cos 0.2268\right)^2$$

$$g' = 9.7677 \text{ m s}^{-2}$$

ESCAPE SPEED AND ORBITAL SPEED

Hydrogen and helium are the most abundant elements in the universe but Earth's atmosphere consists mainly of nitrogen and oxygen. The following discussion brings forth the reason why hydrogen and helium are not found in abundance on the Earth's atmosphere. When an object is thrown up with some initial speed it will reach a certain height after which it will fall back to Earth. If the same object is thrown again with a higher speed, it reaches a greater height than the previous one and falls back to Earth. This leads to the question of what should be the speed of an object thrown vertically up such that it escapes the Earth's gravity and would never come back.

Consider an object of mass M on the surface of the Earth. When it is thrown up with an initial speed vi, the initial total energy of the object is

$$E_i = \frac{1}{2}Mv_i^2 - \frac{GMM_E}{R_E}$$

where, M_E is the mass of the Earth and R_E- the radius of the Earth. The term $-\frac{GMM_E}{R_E}$ is the potential energy of the mass M.

When the object reaches a height far away from Earth and hence treated as approaching infinity, the gravitational potential energy becomes zero $[U(¥)=0]$ and the kinetic energy becomes zero as well. Therefore the final total energy of the object becomes zero. This is for minimum energy and for minimum speed to escape. Otherwise Kinetic energy can be nonzero.

$$E_f = 0$$

According to the law of energy conservation,

$$E_i = E_f$$

Substituting (6.53) in (6.54) we get,

$$\frac{1}{2}Mv_i^2 - \frac{GMM_E}{R_E} = 0$$

$$\frac{1}{2}Mv_i^2 = \frac{GMM_E}{R_E}$$

Consider the escape speed, the minimum speed required by an object to escape Earth's gravitational field, hence replace v_i with v_e . i.e,

$$\frac{1}{2}Mv_e^2 = \frac{GMM_E}{R_E}$$

$$v_e^2 = \frac{GMM_E}{R_E}.\frac{2}{M}$$

$$v_e^2 = \frac{2GM_E}{R_E}$$

Using $g = \frac{GM_E}{R_e^2}$

$$v_e^2 = 2gR_E$$

$$v_e = \sqrt{2gR_E}$$

From equation (6.56) the escape speed depends on two factors: acceleration due to gravity and radius of the Earth. It is completely independent of the mass of the object. By substituting the values of g (9.8 m s^{-2}) and R_e = 6400km , the escape speed of the Earth is v_e = 11.2 kms^{-1} . The escape speed is independent of the direction in which the object is thrown. Irrespective of whether the object is thrown vertically up, radially outwards or tangentially it requires the same initial speed to escape Earth's gravity. It is shown in Figure 6.19

Lighter molecules such as hydrogen and helium have enough speed to escape from the Earth, unlike the heavier ones such as nitrogen and oxygen. (The average speed of hydrogen and helium atoms compaired with the escape speed of the Earth,is presented in the kinetic theory of gases, unit 9).

Satellites, orbital speed and time period

We are living in a modern world with sophisticated technological gadgets and are able to communicate to any place on Earth. This advancement was made possible because of our understanding of solar system. Communication mainly depends on the satellites that orbit the Earth (Figure 6.20). Satellites revolve around the Earth just like the planets revolve around the Sun. Kepler's laws are applicable to manmade satellites also.

For a satellite of mass M to move in a circular orbit, centripetal force must be acting on the satellite. This centripetal force is provided by the Earth's gravitational force.

$$\frac{Mv^2}{(R_E + h)} = \frac{GMM_E}{(R_e + h)^2}$$

$$v^2 = \frac{GM_E}{(R_E + h)}$$

$$v = \sqrt{\frac{GM_E}{(R_E + h)}}$$

As h increases, the speed of the satellite decreases.

Time period of the satellite:

The distance covered by the satellite during one rotation in its orbit is equal to $2p(R_E + h)$ and time taken for it is the time period, T. Then

$$Speed\ v = \frac{Distance\ travelled}{Time\ taken} = \frac{2\pi\left(R_E + h\right)}{T}$$

From equation (6.58)

$$\sqrt{\frac{GM_E}{\left(R_E + h\right)}} = \frac{2\pi\left(R_E + h\right)}{T}$$

$$T = \frac{2\pi}{\sqrt{GM_E}}\left(R_E + h\right)^{3/2}$$

Squaring both sides of the equation (6.60), we get

$$T^2 = \frac{4\pi^2}{GM_E}\left(R_E + h\right)^3$$

$$\frac{4\pi^2}{GM_E} = constant\ say\ c$$

$$T^2 = c\left(R_E + h\right)^3$$

Equation (6.61) implies that a satellite orbiting the Earth has the same relation between time and distance as that of Kepler's law of planetary motion. For a satellite orbiting near the surface of the Earth, h is negligible compared to the radius of the Earth R_E. Then,

$$T^2 = \frac{4\pi^2}{GM_E}R_E^{\ 3}$$

$$T^2 = \frac{4\pi^2}{GM_E / R_E^2} R_E$$

$$T^2 = \frac{4\pi^2}{g} R_E$$

$$since \; GM_E / R_E^2 = g$$

$$T = 2\pi\sqrt{\frac{R_E}{g}}$$

By substituting the values of R_E= 6.4 ×10^6m and g = 9.8 m s^{-2}, the orbital time period is obtained as T ≅ 85 minutes.

EXAMPLE

Moon is the natural satellite of Earth and it takes 27 days to go once around its orbit. Calculate the distance of the Moon from the surface of the Earth assuming the orbit of the Moon as circular.

Solution

We can use Kepler's third law,

$$T^2 = c\left(R_E + h\right)^3$$

$$T^{2/3} = c^{1/3}\left(R_E + h\right)$$

$$\left(\frac{T^2}{c}\right)^{1/3} = \left(R_E + h\right)$$

$$\left(\frac{T^2 GM_E}{4\pi^2}\right)^{\frac{1}{3}} = \left(R_E + h\right);$$

$$c = \frac{4\pi^2}{GM_E}$$

$$h = \left(\frac{T^2 GM_E}{4\pi^2}\right)^{1/3} - R_E$$

Here h is the distance of the Moon from the surface of the Earth. Here,

R_E – radius of the Earth = $6.4 \times 10^6\, m$

M_E – mass of the Earth = $6.02 \times 10^{24}\, kg$

G – Universal gravitational constant = $6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

By substituting these values, the distance to the Moon from the surface of the Earth is calculated to be 3.77×10^5 km.

Energy of an Orbiting Satellite

The total energy of a satellite orbiting the Earth at a distance h from the surface of Earth is calculated as follows; The total energy of the satellite is the sum of its kinetic energy and the gravitational potential energy. The potential energy of the satellite is,

$$U = -\frac{GM_s M_E}{(R_E + h)}$$

Here M_s - mass of the satellite, M_E - mass of the Earth, R_E - radius of the Earth.
The Kinetic energy of the satellite is

$$K.E = \frac{1}{2} M_s v^2$$

Here v is the orbital speed of the satellite and is equal to

$$v = \sqrt{\frac{GM_E}{(R_E + h)}}$$

Substituting the value of v in (6.64), the kinetic energy of the satellite becomes,

$$K.E = \frac{1}{2}\frac{GM_E M_s}{(R_E + h)}$$

Therefore the total energy of the satellite is

$$E = \frac{1}{2}\frac{GM_E M_s}{(R_E + h)} - \frac{GM_s M_E}{(R_E + h)}$$

$$E = -\frac{GM_s M_E}{2(R_E + h)}$$

The negative sign in the total energy implies that the satellite is bound to the Earth and it cannot escape from the Earth.

As h approaches ¥ , the total energy tends to zero. Its physical meaning is that the satellite is completely free from the influence of Earth’s gravity and is not bound to Earth at large distances.

EXAMPLE

Calculate the energy of the (i) Moon orbiting the Earth and (ii) Earth orbiting the Sun.

Solution

Assuming the orbit of the Moon to be circular, the energy of Moon is given by,

$$E_m = -\frac{GM_E M_m}{2R_m}$$

where M_E is the mass of Earth 6.02 ×10^{24} kg; M_m is the mass of Moon 7.35 ×10^{22} kg; and R_m is the distance between the Moon and the center of the Earth 3.84 ×10^5km

$$G = 6.67 \times 10^{-11} \text{N m}^2 \text{kg}^{-2}.$$

$$E_m = -\frac{6.67 \times 10^{-11} \times 6.02 \times 10^{24} \times 7.35 \times 10^{22}}{2 \times 3.84 \times 10^5 \times 10^3}$$

$$E_m = -38.42 \times 10^{-19} \times 10^{46}$$

$$E_m = -38.42 \times 10^{46} \; Joule$$

The negative energy implies that the Moon is bound to the Earth.

Same method can be used to prove that the energy of the Earth is also negative.

Geo-stationary and polar satellite

The satellites orbiting the Earth have different time periods corresponding to different orbital radii. Can we calculate the orbital radius of a satellite if its time period is 24 hours?

Kepler's third law is used to find the radius of the orbit.

$$T^2 = \frac{4\pi^2}{GM_E}(R_E + h)^3$$

$$(R_E + h)^3 = \frac{GM_E T^2}{4\pi^2}$$

$$R_E + h = \left(\frac{GM_E T^2}{4\pi^2}\right)^{1/3}$$

Substituting for the time period (24 hrs = 86400 seconds), mass, and radius of the Earth, h turns out to be 36,000 km. Such satellites are called "geo-stationary satellites", since they appear to be stationary when seen from Earth.

India uses the INSAT group of satellites that are basically geo-stationary satellites for the purpose of telecommunication. Another type of satellite which is placed at a distance of 500 to 800 km from the surface of the Earth orbits the Earth from north to south direction. This type of satellite that orbits Earth from North Pole to South Pole is called a polar satellite. The time period of a polar satellite is nearly 100 minutes and the satellite completes many revolutions in a day. A Polar satellite covers a small strip of area from pole to pole during one revolution. In the next revolution it covers a different strip of area since the Earth would have moved by a small angle. In this way polar satellites cover the entire surface area of the Earth.

Weightlessness Weight of an object

Objects on Earth experience the gravitational force of Earth. The gravitational force acting on an object of mass m is mg. This force always acts downwards towards the center of the Earth. When we stand on the floor, there are two forces acting on us. One is the gravitational force, acting downwards and the other is the normal force exerted by the floor upwards on us to keep us at rest. The weight of an object $\vec{W}$ is defined as the downward force whose magnitude W is equal to that of upward force that must be applied to the object to hold it at rest or at constant velocity relative to the earth. The direction of weight is in the direction of gravitational force. So the magnitude of weight of an object is denoted as, W=N=mg. Note that even though magnitude of weight is equal to mg, it is not same as gravitational force acting on the object.

Apparent weight in elevators

Everyone who used an elevator would have felt a jerk when the elevator takes off or stops. Why does it happen? Understanding the concept of weight is crucial for explaining this effect. Let us consider a man inside an elevator in the following scenarios. When a man is standing in the elevator, there are two forces acting on him.

1. Gravitational force which acts downward. If we take the vertical direction as positive y direction, the gravitational force acting on the man is $\vec{F}_G = -mg\hat{j}$
2. The normal force exerted by floor on the man which acts vertically upward, $\vec{N} = N\hat{j}$

Case (i) When the elevator is at rest

The acceleration of the man is zero. Therefore the net force acting on the man is zero. With respect to inertial frame (ground), applying Newton's second law on the man,

$$\vec{F}_G + \vec{N} = 0$$
$$-mg\hat{j} + N\hat{j} = 0$$

By comparing the components, we can write

$$N - mg = 0 \text{ (or) } N = mg$$

Since weight, W =N, the apparent weight of the man is equal to his actual weight.

Case (ii) When the elevator is moving uniformly in the upward or downward direction

In uniform motion (constant velocity), the net force acting on the man is still zero. Hence, in this case also the apparent weight of the man is equal to his actual weight. It is shown in Figure 6.23(a)

Case (iii) When the elevator is accelerating upwards

If an elevator is moving with upward acceleration $(\vec{a} = a\hat{j})$ with respect to inertial frame (ground), applying Newton's second law on the man,

$$\vec{F}_G + \vec{N} = m\vec{a}$$

Writing the above equation in terms of unit vector in the vertical direction,

$$-mg\hat{j} + N\hat{j} = ma\hat{j}$$

By comparing the components,

$$N = m(g + a)$$

Therefore, apparent weight of the man is greater than his actual weight. It is shown in Figure 6.23(b)

Case (iv) When the elevator is accelerating downwards

If the elevator is moving with downward acceleration $(\vec{a} = -a\hat{j})$ by applying Newton's second law on the man, we can write

$$\vec{F}_G + \vec{N} = m\vec{a}$$

Writing the above equation in terms of unit vector in the vertical direction

$$-mg\hat{j} + N\hat{j} = -ma\hat{j}$$

By comparing the components,

$$N = m(g - a)$$

Therefore, apparent weight W = N = m(g-a) of the man is lesser than his actual weight. It is shown in Figure 6.23(c)

Weightlessness of freely falling bodies

Freely falling objects experience only gravitational force. As they fall freely, they are not in contact with any surface (by neglecting air friction). The normal force acting on the object is zero. The downward acceleration is equal to the acceleration due to the gravity of the Earth. i.e (a = g). From equation (6.69) we get.

$$a = g \quad \therefore N = m(g - g) = 0.$$

This is called the state of weightlessness. When the lift falls (when the lift wire cuts) with downward acceleration a = g, the person inside the elevator is in the state of weightlessness or free fall. It is shown in Figure 6.23(d)

When the apple was falling from the tree it was weightless.As soon as it hit Newton's head, it gained weight! and Newton gained physics!

Weightlessness in satellites:

There is a wrong notion that the astronauts in satellites experience no gravitational force because they are far away from the Earth. Actually the Earth satellites that orbit very close to Earth experience only gravitational force. The astronauts inside the satellite also experience the same gravitational force. Because of this, they cannot exert any force on the floor of the satellite. Thus, the floor of the satellite also cannot exert any normal force on the astronaut. Therefore, the astronauts inside a satellite are in the state of weightlessness. Not only the astronauts, but all the objects in the satellite will be in the state of weightlessness which is similar to that of a free fall. It is shown in the Figure 6.24.

ELEMENTARY IDEAS OF ASTRONOMY

Astronomy is one of the oldest sciences in the history of mankind. In the olden days, astronomy was an inseparable part of physical science. It contributed a lot to the development of physics in the 16th century. In fact Kepler's laws and Newton's theory of gravitation were formulated and verified using astronomical observations and data accumulated over the centuries by famous astronomers like Hippachrus, Aristachrus, Ptolemy, Copernicus and Tycho Brahe. Without Tycho Brahe's astronomical observations, Kepler's laws would not have emerged. Without Kepler's laws, Newton's theory of gravitation would not have been formulated.

It was mentioned in the beginning of this chapter that Ptolemy's geocentric model was replaced by Copernicus' heliocentric model. It is important to analyze and explain the shortcoming of the geocentric model over heliocentric model.

Heliocentric system over geocentric system

When the motion of the planets are observed in the night sky by naked eyes over a period of a few months, it can be seen that the planets move eastwards and reverse their motion for a while and return to eastward motion again. This is called "retrograde motion" of planets.

Figure 6.25 shows the retrograde motion of the planet Mars. Careful observation for a period of a year clearly shows that Mars initially moves eastwards (February to June), then reverses its path and moves backwards (July, August, September). It changes its direction of motion once again and continues its forward motion (October onwards). In olden days, astronomers recorded the retrograde motion of all visible planets and tried to explain the motion. According to Aristotle, the other planets and the Sun move around the Earth in the circular orbits. If it was really a circular orbit it was not known how the planet could reverse its motion for a brief interval. To explain this retrograde motion, Ptolemy introduced the concept of "epicycle" in his geocentric model. According to this theory, while the planet orbited the Earth, it also underwent another circular motion termed as "epicycle". A combination of epicycle and circular motion around the Earth gave rise to retrograde motion of the planets with respect to Earth (Figure 6.26). Essentially Ptolemy retained the Earth centric idea of Aristotle and added the epicycle motion to it.

But Ptolemy's model became more and more complex as every planet was found to undergo retrograde motion. In the 15th century, the Polish astronomer Copernicus proposed the heliocentric model to explain this problem in a simpler manner. According to this model, the Sun is at the center of the solar system and all planets orbited the Sun. The retrograde motion of planets with respect to Earth is because of the relative motion of the planet with respect to Earth. The retrograde motion from the heliocentric point of view is shown in Figure 6.27.

Figure 6.27 shows that the Earth orbits around the Sun faster than Mars. Because of the relative motion between Mars and Earth, Mars appears to move backwards from July to October. In the same way the retrograde motion of all other planets was explained successfully by the Copernicus model. It was because of its simplicity, the heliocentric model slowly replaced the geocentric model. Historically, if any natural phenomenon has one or more explanations, the simplest one is usually accepted. Though this was not the only reason to disqualify the geocentric model, a detailed discussion on correctness of the Copernicus model over to Ptolemy's model can be found in astronomy books

Kepler's Third Law and The Astronomical Distance

When Kepler derived his three laws, he strongly relied on Tycho Brahe's astronomical observation. In his third law, he formulated the relation between the distance of a planet from the Sun to the time period of revolution of the planet. Astronomers cleverly used geometry and trigonometry to calculate the distance of a planet from the Sun in terms of the distance between Earth and Sun. Here we can see how the distance of Mercury and Venus from the Sun were measured. The Venus and Mercury, being inner planets with respect to Earth, the maximum angular distance they can subtend at a point on Earth with respect to the Sun is 46 degree for Venus and 22.5 degree for Mercury. It is shown in the Figure 6.28

Figure 6.29 shows that when Venus is at maximum elongation (i.e., 46 degree) with respect to Earth, Venus makes 90 degree to Sun. This allows us to find the distance between Venus and Sun. The distance between Earth and Sun is taken as one Astronomical unit (1 AU).

The trigonometric relation satisfied by this right angled triangle is shown in Figure 6.29.

$$\sin\theta = \frac{r}{R}$$

where R = 1 AU.

$$r = R\sin\theta = (1AU)(\sin 46°)$$

Here sin46- 0.77. Hence Venus is at a distance of 0.77 AU from Sun. Similarly, the distance between Mercury (θ is 22.5 degree) and Sun is calculated as 0.38 AU. To find the distance of exterior planets like Mars and Jupiter, a slightly different method is used. The distances of planets from the Sun is given in the table below.

Planet	semi major axis of the orbit(a)	Period T (years)	a^3/T^2
Mercury	0.389 AU	87.77	7.64
Venus	0.724 AU	224.70	7.52
Earth	1.000 AU	365.25	7.50
Mars	1.524 AU	686.98	7.50
Jupiter	5.200 AU	4332.62	7.49
Saturn	9.510 AU	10,759.20	7.40

It is to be noted that to verify the Kepler's law we need only high school level geometry and trigonometry.

Measurement of radius of the Earth

Around 225 B.C a Greek librarian "Eratosthenes" who lived at Alexandria measured the radius of the Earth with a small error when compared with results using modern measurements. The technique he used involves lower school geometry and brilliant insight. He observed that during noon time of summer solstice the Sun's rays cast no shadow in the city Syne which was located 500 miles away from Alexandria. At the same day and same time he found that in Alexandria the Sun's rays made 7.2 degree with local vertical as shown in the Figure 6.30. He realized that this difference of 7.2 degree was due to the curvature of the Earth.

$$\frac{1}{8}\, radian.\ \text{So}\ \theta = \frac{1}{8}\, rad;$$

If S is the length of the arc between the cities of Syne and Alexandria, and if R is radius of Earth, then

$$S = R\theta = 500\,miles,$$

so radius of the Earth

$$R = \frac{500}{\theta}\text{ miles}$$

$$R = 500\frac{miles}{\frac{1}{8}}$$

$$\text{R} = 4000 \text{ miles}$$

1 mile is equal to 1.609 km. So, he measured the radius of the Earth to be equal to R = 6436 km, which is amazingly close to the correct value of 6378 km.

The distance of the Moon from Earth was measured by a famous Greek astronomer Hipparchus in the 3rd century BC.

Interesting Astronomical Facts
Lunar eclipse and measurement of shadow of Earth

On January 31, 2018 there was a total lunar eclipse which was observed from various places including Tamil Nadu. It is possible to measure the radius of shadow of the Earth at the point where the Moon crosses. Figure 6.31 illustrates this.

When the Moon is inside the umbra shadow, it appears red in color. As soon as the Moon exits from the umbra 1737shadow, it appears in crescent shape. Figure 6.32 is the photograph taken by digital camera during Moon's exit from the umbra shadow.

By finding the apparent radii of the Earth's umbra shadow and the Moon, the ratio of the these radii can be calculated. This is shown in Figures 6.33 and 6.34.

The apparent radius of Earth's umbra shadow = R_S = 13.2 cm

The apparent radius of the Moon = R_m= 5.15 cm

$$\text{The ratio } \frac{R_s}{R_m} \approx 2.56$$

The radius of the Earth's umbra shadow is $R_s = 2.56 \times R_m$

The radius of Moon $R_m = 1737$ km

The radius of the Earth's umbra shadow

is $R_s = 2.56 \times 1737 km \cong 4446 km$.

The correct radius is 4610 km.

The percentage of error in the calculation

$$= \frac{4610 - 4446}{4610} \times 100 = 3.5\%.$$

The error will reduce if the pictures taken using a high quality telescope are used. It is to be noted that this calculation is done using very simple mathematics.

Early astronomers proved that Earth is spherical in shape by looking at the shape of the shadow cast by Earth on the Moon during lunar eclipse

Why there are no lunar eclipse and solar eclipse every month?

If the orbits of the Moon and Earth lie on the same plane, during full Moon of every month, we can observe lunar eclipse. If this is so during new Moon we can observe solar eclipse. But Moon's orbit is tilted 5° with respect to Earth's orbit. Due to this 5° tilt, only during certain periods of the year, the Sun, Earth and Moon align in straight line leading to either lunar eclipse or solar eclipse depending on the alignment. This is shown in Figure 6.35

Why do we have seasons on Earth?

The common misconception is that 'Earth revolves around the Sun, so when the Earth is very far away, it is winter and when the Earth is nearer, it is summer'. Actually, the seasons in the Earth arise due to the rotation of Earth around the Sun with 23.5° tilt. This is shown in Figure 6.36

Due to this 23.5° tilt, when the northern part of Earth is farther to the Sun, the southern part is nearer to the Sun. So when it is summer in the northern hemisphere, the southern hemisphere experience winter.

Star's apparent motion and spinning of the Earth

The Earth's spinning motion can be proved by observing star's position over a night. Due to Earth's spinning motion, the stars in sky appear to move in circular motion about the pole star as shown in Figure 6.37

Recent developments of astronomy and gravitation

Till the 19th century astronomy mainly depended upon either observation with the naked eye or telescopic observation. After the discovery of the electromagnetic spectrum at the end of the 19th century, our understanding of the universe increased enormously. Because of this development in the late 19th century it was found that Newton's law of gravitation could not explain certain phenomena and showed some discrepancies. Albert Einstein formulated his 'General theory of relativity' which was one of the most successful theories of 20th century in the field of gravitation.

In the twentieth century both astronomy and gravitation merged together and have grown in manifold. The birth and death of stars were more clearly understood. Many Indian physicists made very important contributions to the field of astrophysics and gravitation.

Subramanian Chandrasekar formulated the theory of black holes and explained the life of stars. These studies brought him the Nobel prize in the year 1983. Another very notable Indian astrophysicist Meghnad Saha discovered the ionization formula which was useful in classifying stars. This formula is now known as "Saha ionization formula". In the field of gravitation Amal Kumar Raychaudhuri solved an equation now known as "Raychaudhuri equation" which was a very important contribution. Another notable Indian Astrophysicist Jayant V Narlikar made pioneering contribution in the field of astrophysics and has written interesting books on astronomy and astrophysics. IUCAA (Inter University Center for Astronomy and Astrophysics) is one of the important Indian research institutes where active research in astrophysics and gravitation are conducted. The institute was founded by Prof. J.V. Narlikar. Students are encouraged to read more about the recent developments in these fields.

UNIT 7: PROPERTIES OF MATTER

Microscopic understanding of various states of matter:

- In extreme environments, matter can exist in other states such as plasma, Bose-Einstein condensates. Additional states, such as quark-gluon plasmas are also believed to be possible. A major part of the atomic matter of the universe is hot plasma in the form of rarefied interstellar medium and dense stars.
- If a body regains its original shape and size after the removal of deforming force, it is said to be elastic and the property is called elasticity. The force which changes the size or shape of a body is called a deforming force.

Examples: Rubber, metals, steel ropes

- If a body does not regain its original shape and size after removal of the deforming force, it is said to be a plastic body and the property is called plasticity. Example: Glass

Stress:

When a body is subjected to such a deforming force, internal force is developed in it, called as restoring force. The SI unit of stress is $N\,m^{-2}$or pascal (Pa) and its dimension is $[M\,L^{-1}T^{-2}]$. Stress is a tensor.

$$stress, \sigma = \frac{Force}{Area} = \frac{F}{A}$$

<u>Types of stress:</u>

1. Tangential or shearing stress
2. Longitudinal or perpendicular stress:
   a) Compressive stress
   b) Tensile stress
3. Volume stress: pressure exerted on a body which is immersed in a liquid.

Strain:

- Strain measures the degree of deformation and is a dimensionless quantity.

$$\boldsymbol{strain, \varepsilon = \frac{change\ in\ size}{original\ size} = \frac{\Delta l}{l}}$$

$$\boldsymbol{longitudnal\ strain, \varepsilon_l = \frac{increase\ in\ length\ of\ the\ rod}{original\ length\ of\ the\ rod} = \frac{\Delta l}{l}}$$

Tensile strain: If the length is increased from its natural length
Compressive strain: If the length is decreased from its natural length

$$\boldsymbol{shearing\ strain, \varepsilon_s = \frac{x}{h} = \tan\theta \approx \theta = angle\ of\ shear}$$

$$\boldsymbol{volume\ strain, \varepsilon_V = \frac{\Delta V}{V}}$$

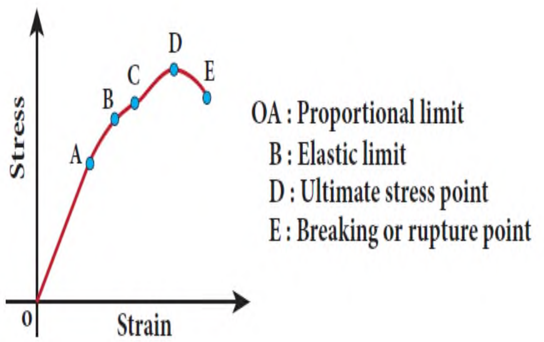
Elastic limit:

- The maximum stress within which the body regains its original size and shape after the removal of deforming force is called the elastic limit.

Hooke's law states that the stress is proportional to the strain in the elastic limit.

$$\sigma \propto \varepsilon$$

Stress- strain curve:



The point A is called limit of proportionality because above this point Hooke's law is not valid. The slope of the line OA gives the Young's modulus of the wire.
The point B is known as yield point (elastic limit).
If the wire is stretched beyond the point B (elastic limit), stress increases and the wire will not regain its original length after the removal of stretching force.

- The maximum stress (here D) beyond which the wire breaks is called *breaking stress* or *tensile strength.* The corresponding point D is known as *fracture point*. The region BCDE represents the plastic behaviour of the material of the wire.

Moduli of elasticity:

$$\boldsymbol{young\ modulus, Y = \frac{longitudnal\ stress}{longitudnal\ strain} = \frac{\sigma_l}{\varepsilon_l}}$$

The SI unit of young modulus is $N\ m^{-2}$or pascal (Pa)

$$\boldsymbol{bulk\ modulus, K = \frac{normal\ stress}{volume\ strain} = -\frac{\sigma_n}{\varepsilon_V} = -\frac{\Delta P}{\frac{\Delta V}{V}}}$$

The SI unit of bulk modulus is $N\ m^{-2}$or pascal (Pa).

$$\textbf{rigidity modulus or shear modulus}, \eta_R = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{\sigma_s}{\varepsilon_s} = \frac{\frac{F_t}{\Delta A}}{\theta}$$

The SI unit of rigidity modulus is $N\ m^{-2}$or pascal (Pa).

Compressibility:

- The reciprocal of the bulk modulus is called compressibility. It is defined as the fractional change in volume per unit increase in pressure. Since, gases have small value of bulk modulus than solids, their, values of compressibility is very high

$$\boldsymbol{compressibility, C} = \frac{1}{K} = -\frac{\varepsilon_V}{\sigma_n} = -\frac{\frac{\Delta V}{V}}{\Delta P}$$

In a cycle, the tyre should be less compressible for its easy rolling. In fact the rear tyre is less compressible than front tyre for a smooth ride.

Materials	Young modulus(Y) 10^{10} N m^2	Bulk modulus(k) 10^{10} N m^2	Shear modulus(n_g) 10^{10} N m^2
steel	20.0	15.8	8.0
Aluminium	7.0	7.0	2.5
Copper	12.0	12.0	4.0
Iron	19.0	8.0	5.0
glass	7.0	3.6	3.0

Material	Poisson's ratio
Rubber	0.4999
Gold	0.42-0.44
Copper	0.33
Stainless steel	0.30-0.31
Steel	0.21-0.26
Cast iron	0.21-0.26
Concrete	0.1-0.2
Glass	0.18-0.3
Foam	0.10-0.50
Cork	0.0

Poisson's ratio:

$$\boldsymbol{Poisson'sratio, \mu} = \frac{\boldsymbol{lateral\ strain}}{\boldsymbol{longitudnal\ strain}} = -\frac{\frac{d}{D}}{\frac{l}{L}} = -\frac{L}{l} \times \frac{d}{D}$$

- Negative sign indicates the elongation along longitudinal and contraction along lateral dimension. So, Poisson's ratio has no unit and no dimension (dimensionless number)

Elastic energy:

$$\boldsymbol{elastic\ potential\ energy, W = \frac{1}{2}Fl}$$

$$\boldsymbol{energy\ density, u = \frac{elastic\ potential\ energy}{volume} = \frac{1}{2}\frac{Fl}{AL} = \frac{1}{2}\frac{F}{A}\frac{l}{L} = \frac{1}{2}(stress \times strain)}$$

Applications of elasticity:

- To reduce the bending of a beam for a given load, one should use the material with a higher Young's modulus of elasticity (Y). The Young's modulus of steel is greater than aluminium or copper. Iron comes next to steel. This is the reason why steel is mostly preferred in the design of heavy duty machines and iron rods in the construction of buildings.Steel is more elastic than rubber. If an equal stress is applied to both steel and rubber, the steel produces less strain. The object which has higher young's modulus is more elastic.

Fluids:

$$Pressure, P = \frac{F}{A}$$

- Pressure is a scalar quantity. The SI unit of pressure is $N\,m^{-2}$or pascal (Pa) and its dimension is $[M\,L^{-1}T^{-2}]$. One 'atm' is defined as the pressure exerted by the atmosphere at sea level. i.e., 1 atm = 1.013 x 10^5Pa or $N\,m^{-2}$

$$density, \rho = \frac{m}{V}$$

- The dimensions and S.I unit are $\left[M\,L^{-3}\right]$ and $kg\,m^{-3}$. It is a positive scalar quantity.

$$relative\ density = \frac{density\ of\ substance}{density\ of\ water\ at\ 4°C}$$

It is a dimensionless positive scalar quantity.
The density of mercury is $13.6 \times 10^3 kg\,m^{-3}$. Its relative density is equal to 13.6.

Pressure due to fluid column at rest:

$$\boldsymbol{P = P_a + \rho g h}$$

Where P_a is the atmospheric pressure.

- v The liquid pressure is the same at all points at the same depth. This statement can be demonstrated by an experiment called ‘hydrostatic paradox’.

- v The decrease of atmospheric pressure with altitude has an unwelcome consequence in daily life. For example, it takes longer time to cook at higher altitudes. Nose bleeding is another common experience at higher altitude because of larger difference in atmospheric pressure and blood pressure.

- v Take a metal container with an opening. Connect a vacuum pump to the opening. Evacuate the air from inside the container.Due to the force of atmospheric pressure acting on its outer surface, the shape of the container crumbles.

- ü Take a glass tumbler. Fill it with water to the brim. Slide a cardboard on its rim so that no air remains in between the cardboard and the tumbler. Invert the tumbler gently. The water does not fall down. This is due to the fact that the weight of water in the tumbler is balanced by the upward thrust caused due to the atmospheric pressure acting on the lower surface of the cardboard that is exposed to air.

Pascal’s law and its application:

- · If the pressure in a liquid is changed at a particular point, the change is transmitted to the entire liquid without being diminished in magnitude.

Application of Pascal’s law- hydraulic lift

$$\boldsymbol{P} = \frac{\boldsymbol{F_1}}{\boldsymbol{A_1}} = \frac{\boldsymbol{F_2}}{\boldsymbol{A_2}}$$

$$\boldsymbol{F_2} = \frac{\boldsymbol{F_1}}{\boldsymbol{A_1}} \times \boldsymbol{A_2} = \frac{\boldsymbol{A_2}}{\boldsymbol{A_1}} \times \boldsymbol{F_1}$$

- · Therefore by changing the force on the smaller piston A, the force on the piston B has been increased by the factor $\frac{A_2}{A_1}$and this factor is called the mechanical advantage of the lift.

Buoyancy:

Archimedes principle:

- · When a body is partially or wholly immersed in a fluid, it experiences an upward thrust equal to the weight of the fluid displaced by it and its upthrust acts through the centre of gravity of the liquid displaced.

Upthrust or buoyant force = weight of liquid displaced.

Law of floatation:

- The law of floatation states that a body will float in a liquid if the weight of the liquid displaced by the immersed part of the body equals the weight of the body

- If an object floats, the volume of fluid displaced is equal to the volume of the object submerged and the percentage of the volume of the object submerged is equal to the relative density of an object with respect to the density of the fluid in which it floats.

- When the ballast tanks are filled with air, the overall density of the submarine becomes lesser than the surrounding water, and it surfaces (positive buoyancy). If the tanks are flooded with water replacing air, the overall density becomes greater than the surrounding water and submarine begins to sink (negative buoyancy). To keep the submarine at any depth, tanks are filled with air and water (neutral buoyancy).

Examples of floating bodies:

i) A person can swim in sea water more easily than in river water.
ii) Ice floats on water.
iii) The ship is made of steel but its interior is made hollow by giving it a concave shape.

Viscosity:

- An ideal liquid is incompressible (i.e., bulk modulus is infinity) and in which no shearing forces can be maintained (i.e., the coefficient of viscosity is zero). Viscosity is defined as 'the property of a fluid to oppose the relative motion between its layers'.

$$F = -\eta A \frac{dv}{dx}$$

- Where η is called the coefficient of viscosity of the liquid and the negative sign implies that the force is frictional and it opposes the relative motion. The kinetic energy of the substance is dissipated as heat energy. The dimensional formula for coefficient of viscosity is $\left[M\, L^{-1}T^{-1}\right]$

Streamline flow (or) steady flow (or) laminar flow :

- When a liquid flows such that each particle of the liquid passing through a point moves along the same path with the same velocity as its predecessor then the flow of liquid is said to be a *streamlined flow*. This is possible below critical velocity.

Turbulent flow:

- Velocity changes both in magnitude and direction from particle to particle and hence, the path taken by the particles in turbulent flow becomes erratic and whirlpool-like circles called eddy current or eddies when velocity of flow is greater than the critical velocity v_s.

Reynold’s number:

S.no	Reynold’s number	Flow
1	Rc< 1000	Streamline
2	1000<Rc<2000	Unsteady
3	Rc>2000	Turbulent

$$Reynold's number, R_C (or) K = \frac{\rho v D}{\eta}$$

- *Law of similarity* which states that when there are two geometrically similar flows, both are essentially equal to each other, as long as they embrace the same Reynold’s number.

Terminal velocity:

The forces acting on the sphere falling in a liquid are

- gravitational force of the sphere acting vertically downwards,
- upthrust U due to buoyancy and
- viscous drag acting upwards

- A stage is reached when the net downward force balances the upward force and hence the resultant force on the sphere becomes zero. The maximum constant velocity acquired by a body while falling freely through a viscous medium is called the terminal velocity v_T

$$terminal\ velocity, v_T = \frac{2}{9} \times \frac{r^2(\rho - \sigma)g}{\eta}$$

1) If σ is greater than ρ, then the term (ρ - σ) becomes negative leading to a negative terminal velocity. That is why air bubbles rise up through water or any fluid. This is also the reason for the clouds in the sky to move in the upward direction.

2) The terminal speed of a sphere is directly proportional to the square of the radius of the sphere. Hence, larger raindrops fall with greater speed as compared to the smaller raindrops.

3) If the density of the material of the sphere is less than the density of the medium, then the sphere shall attain terminal velocity in the upward direction. That is why gas bubbles rise up in soda water.

Stoke's law:

$$\boldsymbol{viscous\ force, F = 6\pi\eta r v}$$

· Since the raindrops are smaller in size and their terminal velocities are small, remain suspended in air in the form of clouds. As they grow up in size, their terminal velocities increase and they start falling in the form of rain.

This law explains the following:

a) Floatation of clouds
b) Larger raindrops hurt us more than the smaller ones
c) A man coming down with the help of a parachute acquires constant terminal velocity.

Poiseuille's equation:

· He derived an expression for the volume of the liquid flowing per second through the capillary tube. The conditions to be retained are,

- The flow of liquid through the tube is streamlined.
- The tube is horizontal so that gravity does not influence the flow
- The layer in contact with the wall of the tube is at rest
- The pressure is uniform over any cross section of the tube

$$volume\ of\ the\ liquid\ flowing\ per\ second, v = \frac{\pi r^4 P}{8\eta l}$$

Applications of viscosity:

(1) The oil used as a lubricant for heavy machinery parts should have a high viscous coefficient. To select a suitable lubricant, we should know its viscosity and how it varies with temperature

(2) The highly viscous liquid is used to damp the motion of some instruments and is used as brake oil in hydraulic brakes.
(3) Blood circulation through arteries and veins depends upon the viscosity of fluids.

(4) Millikan conducted the oil drop experiment to determine the charge of an electron. He used the knowledge of viscosity to determine the charge.

Surface tension:

- The force between the like molecules which holds the liquid together is called '*cohesive force*'. When the liquid is in contact with a solid, the molecules of the solid and liquid will experience an attractive force which is called '*adhesive force*'.

- Laplace and Gauss developed the theory of surface and motion of a liquid under various situations.

- These molecular forces are effective only when the distance between the molecules is very small about 10^{-9}m (i.e., 10 Å). The distance through which the influence of these molecular forces can be felt in all directions constitute a range and is called *sphere of influence*.

- When any molecule is brought towards the surface from the interior of the liquid, work is done against the cohesive force among the molecules of the surface. This work is stored as potential energy in molecules. So the molecules on the surface will have greater potential energy than that of molecules in the interior of the liquid. But for a system to be under stable equilibrium, its potential energy (or surface energy) must be a minimum. Therefore, in order to maintain stable equilibrium, a liquid always tends to have a minimum number of molecules. In other words, the liquid tends to occupy a minimum surface area. This behaviour of the liquid gives rise to surface tension.

Examples of surface tension:

1) Water bugs and water striders walk on the surface of water. The water molecules are pulled inwards and the surface of water acts like a springy or stretched membrane.

2) The hairs of the painting brush cling together when taken out of water. This is because the water films formed on them tends to contract to a minimum area.

3) Needle floats on water due to surface tension.

Factors affecting surface tension of a liquid:

1) Presence of contamination

2) Presence of dissolved substances:
For example, a highly soluble substance like sodium chloride (NaCl) when dissolved in water (H_20) increases the surface tension of water. But the sparingly soluble substance like phenol or soap solution when mixed in water decreases the surface tension of water.

3) Electrification:
Surface tension decreases as the area of liquid surface increases.

4) Temperature:
Surface tension decreases with increase in temperature

$$\boldsymbol{T_t = T_0(1 - \alpha t)}$$

· Where, T_0 is the surface tension at temperature 0ºC and a is the temperature coefficient of surface tension. It is to be noted that at the critical temperature, the surface tension is zero as the interface between liquid and vapour disappear.

· Van der Wall suggested the important relation between the surface tension and the critical temperature as $T_t = T_0\left(1 - \frac{t}{t_c}\right)^n$

Where value of n differs for different liquids.

Surface energy and surface tension:

· The work done in increasing the surface area per unit area of the liquid against the surface tension force is called the surface energy of the liquid.

$$surface\ energy = \frac{work\ done\ in\ increasing\ surface\ area}{increase\ in\ surface\ area} = \frac{W}{\Delta A}$$

It is expressed in $J\ m^{-2}$or $N\ m^{-1}$.

Surface tension is the energy per unit area of the surface of a liquid

$$T = \frac{F}{l}$$

SI unit and dimensions of T are $Nm^{-1}$and MT^{-2}.

- Surface energy per unit area of surface is numerically equal to the surface tension.

- It should be remembered that a liquid drop has only one free surface. Therefore, the surface area of a spherical drop of radius r is equal to $4\pi r^2$, whereas, a bubble has two free surfaces and hence the surface area of a spherical bubble is equal to 2 ×$4\pi r^2$.

Angle of contact:

- The angle between the tangent to the liquid surface at the point of contact and the solid surface inside the liquid is known as the *angle of contact between the solid and the liquid*. It is denoted by θ.

- It is the factor which decides whether a liquid will spread on the surface of a chosen solid or it will form droplets on it.

- Soaps and detergents are wetting agents. When they are added to an aqueous solution, they will try to minimize the angle of contact and in turn penetrate well in the cloths and remove the dirt. On the other hand, water proofing paints are coated on the outer side of the building so that it will enhance the angle of contact between the water and the painted surface.

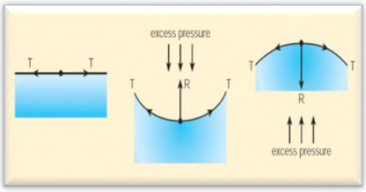
  (i) If $T_{sa}>T_{sl}$ (water-plastic interface) then the angle of contact θ is acute angle as cosθ is positive.

  (ii) If $T_{sa}<T_{sl}$ (water-leaf interface) then the angle of contact is obtuse angle as cosθ is negative.

  (iii) If $T_{sa}>T_{la}+ T_{sl}$ then there will be no equilibrium and liquid will spread over the solid.

Excess of pressure inside a liquid drop, a soap bubble and an air bubble:

- As a consequence of surface tension, the liquid-air interfaces have energy and for a given volume, the surface will have a minimum energy with least area. Due to this reason, the liquid drop becomes spherical (for a smaller radius).



1) When liquid is plane, resultant force acting on the molecule is zero. Thus pressure on both sides is equal.

2) For a convex surface, the resultant force is directed inwards towards the centre of curvature, whereas the resultant force is directed outwards from the centre of curvature for a concave surface. Thus, for a curved liquid surface in equilibrium, the pressure on its concave side is greater than the pressure on its convex side.

Excess pressure inside an air bubble is $\Delta P = P_2 - P_1 = \frac{2T}{R}$
Excess pressure inside a soap bubble is $\Delta P = \frac{4T}{R}$
Excess pressure inside liquid drop is $\Delta P = \frac{2T}{R}$, where T is the surface tension and R is radius of the bubble.

- The smaller the radius of a liquid drop, the greater is the excess of pressure inside the drop. It is due to this, the tiny fog droplets are rigid enough to behave like solids.

- When an ice-skater skate over the surface of the ice, some ice melts due to the pressure exerted by the sharp metal edges of the skates, the tiny droplets of water act as rigid ball- bearings and help the skaters to run along smoothly.

Capillarity:

- The tube having a very small diameter is called a ‘capillary tube’. When a glass capillary tube open at both ends is dipped vertically in water, the water in the tube will rise above the level of water in the vessel. In case of mercury, the liquid is depressed in the tube below the level of mercury in the vessel

- The rise or fall of a liquid in a narrow tube is called capillarity or capillary action. Depending on the diameter of the capillary tube, liquid rises or falls to different heights.

Contact angle	Strength of		Degree of wetting	Miniscus	Rise or fall of liquid in the capillary tube
	Cohesive force	Adhesive force			
θ=0	Weak	Strong	Perfect Wetting	Plane	Neither rises nor is depressed
θ<90	Weak	Strong	High	Concave	Rise of liquid
θ>90	Strong	Weak	Low	Convex	Fall of liquid

Practical applications of capillarity

- Due to capillary action, oil rises in the cotton within an earthen lamp. Likewise, sap rises from the roots of a plant to its leaves and branches.
- Absorption of ink by a blotting paper
- Capillary action is also essential for the tear fluid from the eye to drain constantly.
- Cotton dresses are preferred in summer because cotton dresses have fine pores which act as capillaries for sweat.

Surface tension by capillary rise method:

$$\textbf{surface tension}, \mathbf{T} = \frac{r\rho g h}{2\cos\theta}$$

Where h is the capillary rise, θ is angle of contact, ρ is density of liquid and r is radius of tube.

Applications of surface tension:

• Mosquitoes lay their eggs on the surface of water. To reduce the surface tension of water, oil is poured. This breaks the elastic film of water surface and eggs are killed by drowning.

• Chemical engineers must finely adjust the surface tension of the liquid, so it forms droplets of designed size and so it adheres to the surface without smearing. This is used in desktop printing, to paint automobiles and decorative items.

• Specks of dirt get removed when detergents are added to hot water while washing clothes because surface tension is reduced.

• A fabric can be made waterproof, by adding suitable waterproof material (wax) to the fabric. This increases the angle of contact

Bernoulli's theorem:

$$\mathbf{av = constant}$$

- where a is cross sectional area of pipe and v is velocity of fluid. The volume flux or flow rate remains constant throughout the pipe. It is called the equation of continuity and it is a statement of conservation of mass in the flow of fluids.

Bernoulli's theorem and its applications:

- In 1738, the Swiss scientist Daniel Bernoulli He proposed a theorem for the streamline flow of a liquid based on the law of conservation of energy. The sum of pressure energy, kinetic energy, and potential energy per unit mass of an incompressible, non-viscous fluid in a streamlined flow remains a constant.

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gh = constant$$

- Bernoulli's relation is strictly valid for fluids with zero viscosity or non-viscous liquids.

- When fluid flows through a horizontal pipe,

  h = 0. Thus, $\frac{p}{\rho} + \frac{1}{2}v^2 = constant$

ApplicationsofBernoulli's theorem:

1) Blowing of roofs during wind storm:

- Roofs are designed with a slope. During cyclones, the roof is blown off without damaging the other parts of the house. In accordance with the Bernoulli's principle, the high wind blowing over the roof creates a low-pressure P_1. The pressure under the roof P_2 is greater. Therefore, this pressure difference (P_2–P_1) creates an up thrust and the roof is blown off.

2) Aerofoil lift:

- The wings of an airplane (aerofoil) are so designed that its upper surface is more curved than the lower surface and the front edge is broader than the real edge.

As the aircraft moves, the air moves faster above the aerofoil than at the bottom.According to Bernoulli's Principle, the pressure of air below is greater than above, which creates an upthrust called the dynamic lift to the aircraft.

3) Bunsen burner:

- In this, the gas comes out of the nozzle with high velocity, hence the pressure in the stem decreases. So outside air reaches into the burner through an air vent and the mixture of air and gas gives a blue flame

4) Venturimeter:

- This device is used to measure the rate of flow (or say flow speed) of the incompressible fluid flowing through a pipe.

Pressure difference, $\Delta P = \rho \frac{v_1^2}{2} \frac{(A^2-a^2)}{a^2}$

Speed of flow at end of tube A, $v_1 = \sqrt{\frac{2(\Delta P)a^2}{\rho(A^2-a^2)}}$

Volume of liquid flowing out of second is

$$V = aA\sqrt{\frac{2(\Delta P)}{\rho(A^2 - a^2)}}$$

5) Other applications:

- This Bernoulli's concept is mainly used in the design of carburetor of automobiles, filter pumps, atomizers, and sprayers. The carburetorhas a very fi ne channel called nozzle through which the air is allowed to flow in larger speed. In this case, the pressure is lowered at the narrow neck and in turn, the required fuel or petrol is sucked into the chamber so as to provide the correct mixture of air and fuel necessary for ignition process.

More to know

A single strand of spider silk can stop flying insects which are tens and thousands times its mass. The young's modulus of the spider web is approximately $4.5 \times 10^9\ N\,m^{-2}$.

11th Standard - Volume (II)
UNIT 8: HEAT AND THERMODYNAMICS

Heat and temperature:

- Heat is not a quantity. When we use the word 'heat', it is the energy in transit but not energy stored in the body. Temperature is the degree of hotness or coolness of a body.

Boyle's law, Charles' law and ideal gas law:

- $P \propto \frac{1}{V}$ is known as Boyle's law.
- $V \propto T$ is known as Charles' law.
- $PV = \mu N_A kT = \mu RT$

- One mole of any substance is the amount of that substance which contains Avogadro number (N_A) of particles (such as atoms or molecules). The Avogadro's number N_A is defined as the number of carbon atoms contained in exactly 12 g of ^{12}C.

$$N_A = 6.023 \times 10^{23} \text{mol}^{-1}$$

Heat capacity and Specific Heat Capacity

- 'Heat capacity' is the amount of heat energy required to raise the temperature of the given body from T to T + ΔT.

  Heat capacity $S = \frac{\Delta Q}{\Delta T}$

- Specific heat capacity of a substance (s) is defined as the amount of heat energy required to raise the temperature of 1kg of a substance by 1 Kelvin or 1°C

$$s = \frac{1}{m}\left(\frac{\Delta Q}{\Delta T}\right)$$

- Where s depends only on the nature of substance and not on amount of substance

- ΔQ =Amount of heat energy ΔT = Change in temperature m = Mass of the substance. The SI unit for specific heat capacity is $kg^{-1}\,K^{-1}$. Heat capacity and specific heat capacity are always positive quantities. Water has the highest value of specific heat capacity. So it is used as coolant in power stations and reactors. When two objects of same mass are heated (or cooled) at equal rates, the object with smaller specific heat capacity will have a faster temperature increase (or drop).

- Molar specific heat capacity(C) is defined as heat energy required to increase the temperature of one mole of substance by 1K or 1℃

$$C = \frac{1}{\mu}\left(\frac{\Delta Q}{\Delta T}\right)$$

- μ is the number of moles in a substance. SI unit of specific heat capacity is $J\,mol^{-1}K^{-1}$. It is a positive quantity.

Thermal expansion of solid, liquid and Gas:

- Thermal expansion is the tendency of matter to change in shape, area, and volume due to a change in temperature.

- Railroad tracks and bridges have expansion joints to allow them to expand and contract freely with temperature changes. Liquids, have less intermolecular forces than solids and hence they expand more than solids. This is the principle behind the mercury thermometers. In hot air balloons when gas particles get heated, they expand and take up more space.

- The expansion in length is called linear expansion. Similarly the expansion in area is termed as area expansion and the expansion in volume is termed as volume expansion.

  - When the lid of a glass bottle is tight, keep the lid near the hot water which makes it easier to open. It is because the lid has higher thermal expansion than glass.

  - When the hot boiled egg is dropped in cold water, the egg shell can be removed easily. It is because of the different thermal expansions of the shell and egg.

$$\text{area expansion} \approx 2 \times \text{linear expansion}$$
$$\text{volume expansion} \approx 3 \times \text{linear expansion}$$

Anomalous expansion of water:

- The volume of given amount of water decreases as it is cooled but up to 4℃. Below 4℃ volume increases so density decreases. Water has maximum density at 4℃. this behaviour is called anomalous expansion of water.

- Since ice have lower density than water at 4℃ the ice will float at top of water. As water freezes only at top, species in bottom of the lake will be safe.

Change of state:

- Latent heat capacity of substance is defined as the amount of heat energy required to change the state of unit mass of the material.
- When heat is added or removed during a change of state, the temperature remains constant.

- The triple point of substance is the temperature and pressure at which the three phases (gas, liquid and solid) of that substance coexist in thermodynamic equilibrium. The triple point of water is at 273.1 K at a partial vapour pressure of 611.657 Pascal.

Calorimetry:

- A sample is heated at high temperature (T_1) and immersed into water at room temperature (T_2) in the calorimeter. After some time both reach a final equilibrium temperature T_f.

$$\mathbf{T_f = \frac{m_1 s_1 T_1 + m_2 s_2 T_2}{m_1 s_1 + m_2 s_2}}$$

Here s_1 and s_2 specific heat capacity of hot sample and water respectively.

Heat transfer:

Conduction:

- The quantity of heat transferred through a unit length of a material in a direction normal to unit surface area due to a unit temperature difference under steady state conditions is known as thermal conductivity of a material.

$$\mathbf{\frac{Q}{t} = \frac{KA\Delta T}{L}}$$

Where K is the coefficient of thermal conductivity. SI unit of thermal conductivity is $Js^{-1}m^{-1}K^{-1}$ or $W\ m^{-1}K^{-1}$

- Silver and aluminium are used to make cooking vessels as they have high thermal conductivity.

- The state at which temperature attains constant value everywhere and there is no further transfer of heat anywhere is called steady state.

Convection:

- Land has less specific heat capacity than water. This causes land breeze and sea breeze.

- Water in cooking pot is an example of convection. Water at bottom heats, become less dense and rises up. The water at top is cooler and denser, so falls to the bottom. This back and forth movement is convectional current.

- The air molecules near heater becomes hot, less dense and rises up. The cooler air at top is denser and comes down. This process is used in room heating.

Radiation:

- The visible radiation coming from the Sun is at the temperature of 5700 K and the Earth re emits the radiation in the infrared range into space which is at a temperature of around 300K.

Newton's law of cooling:

- Newton's law of cooling states that the rate of loss of heat of a body is directly proportional to the difference in the temperature between that body and its surroundings.

$$\frac{dQ}{dt} \propto -(T - T_s)$$

- The negative sign indicates that quantity of heat lost by liquid goes on decreasing with time. Where T= temperature of object T_s= temperature of surrounding

Laws of Heat transfer:

Prevost theory of heat exchange:

- Only at absolute zero temperature a body will stop emitting. Therefore Prevost theory states that all bodies emit thermal radiation at all temperatures above absolute zero irrespective of the nature of the surroundings. A body at high temperature radiates more heat to the surroundings than it receives from it.

Stefan Boltzmann Law:

- Stefan Boltzmann law states that, the total amount of heat radiated per second per unit area of a black body is directly proportional to the fourth power of its absolute temperature.

$$E = \sigma T^4$$

Stefan constant, σ = 5.67 × 10^{-8} $Wm^{-2}k^{-4}$

If a body is not a perfect black body, then

$$E = e\sigma T^4$$

- 'e' is emissivity of surface. Emissivity is defined as ratio of energy radiated from a material's surface to that radiated from a perfect black body at same temperature and wavelength.

Wien's displacement law:

- Wien's law states that, the wavelength of maximum intensity of emission of a black body radiation is inversely proportional to the absolute temperature of the black body.

$$\boldsymbol{\lambda_m = \frac{b}{T}}$$

Where Wien's constant, b = 2.898 × 10^{-3} $m\ K$

- It implies that if temperature of the body increases, maximal intensity wavelength (λ_m) shift s towards lower wavelength (higher frequency) of electromagnetic spectrum.

- The Sun is approximately taken as a black body. Since any object above 0 K will emit radiation, Sun also emits radiation. Its surface temperature is about 5700K.

$$\boldsymbol{\lambda_m = \frac{b}{T} = \frac{2.898 \times 10^{-3}}{5700} \approx 508\ nm}$$

- The humans evolved under the Sun by receiving its radiations. The human eye is sensitive only in the visible spectrum. Suppose if humans had evolved in a planet near the star Sirius (9940K), then they would have had the ability to see the Ultraviolet rays!

THERMODYNAMICS:

- A branch of physics which describes the laws governing the process of conversion of work into heat and conversion of heat into work is thermodynamics.

- A thermodynamic system is a finite part of the universe. It is a collection of large number of particles (atoms and molecules) specified by certain parameters called pressure (P), Volume (V) and Temperature (T). The remaining part of the universe is called surrounding.

Thermal equilibrium:

- Two systems are said to be in thermal equilibrium with each other if they are at the same temperature, which will not change with time.

- A system is said to be in mechanical equilibrium if no unbalanced force acts on the thermodynamic system or on the surrounding by thermodynamic system.

- There is no net chemical reaction between two thermodynamic systems in contact with each other then it is said to be in chemical equilibrium.

If two systems are set to be in thermodynamic equilibrium, then the systems are at thermal, mechanical and chemical equilibrium with each other. In a state of thermodynamic equilibrium the macroscopic variables such as pressure, volume and temperature will have fixed values and do not change with time.

Thermodynamic state variables:

- Heat and work are not state variables rather they are process variables.

- There are two types of thermodynamic variables: Extensive and Intensive

Extensive variable depends on the size or mass of the system.

Example: Volume, total mass, entropy, internal energy, heat capacity etc.
Intensive variables do not depend on the size or mass of the system.

Example: Temperature, pressure, specific heat capacity, density etc.

- The equation which connects the state variables in a specific manner is called equation of state. A thermodynamic equilibrium is completely specified by these state variables by the equation of state.

ZEROTH LAW OF THERMODYNAMICS:

- The zeroth law of thermodynamics states that if two systems, *A* and *B*, are in thermal equilibrium with a third system, *C*, then *A* and *B* are in thermal equilibrium with each other.

Example: Temperature of the thermometer will be same as the human body. This principle is used in finding the body temperature.

INTERNAL ENERGY (U):

- The internal energy of a thermodynamic system is the sum of kinetic and potential energies of all the molecules of the system with respect to the centre of mass of the system. The energy due to molecular motion including translational, rotational and vibrational motion is called internal kinetic energy (E_K) The energy due to molecular interaction is called internal potential energy (E_P). Example: Bond energy.

$$U = E_K + E_P$$

- Since ideal gas molecules are assumed to have no interaction with each other the internal energy consists of only kinetic energy part (E_K) which depends on the temperature, number of particles and is independent of volume. However this is not true for real gases like Van der Waals gases.

- Internal energy is a state variable. It depends only on the initial and final states of the thermodynamic system and not the way it is arrived at.

- Internal energy of a thermodynamic system is associated with only the kinetic energy of the individual molecule due to its random motion and the potential energy of molecules which depends on their chemical nature.The bulk kinetic energy of the entire system or gravitational potential energy of the system should not be mistaken as a part of internal energy.

Heat does not always increase the internal energy.

Joule's Mechanical Equivalent of Heat:

- In the eighteenth century, Joule showed that mechanical energy can be on converted into internal energy and vice versa. In fact, Joule was able to show that the mechanical work has the same effect as giving heat. He found that to raise 1 g of an object by 1°C, 4.186 J of energy is required.

$$1\,cal = 4.186\,J$$

First Law of Thermodynamics:

- This law states that 'Change in internal energy (ΔU) of the system is equal to heat supplied to the system (Q) minus the work done by the system (W) on the surroundings'.

$$\Delta U = Q - W$$

System gains heat	Q is positive	Internal energy increase
System loses heat	Q is negative	Internal energy decreases
Work done on the system	W is negative	Internal energy increase
Work done by the system	W is positive	Internal energy decreases

- This law is applicable to solid, liquid and gases.

Quasi static process:

- A quasi-static process is an infinitely slow process in which the system changes its variables (P,V,T) so slowly such that it remains in thermal, mechanical and chemical equilibrium with its surroundings throughout.

Work Done in Volume changes:

$$W = \int_{V_i}^{V_f} PdV$$

- If work is done on the system $V_i > V_f$ and W is negative. The area under the PV diagram will give the work done during expansion or compression.

SPECIFIC HEAT CAPACITY OF A GAS:

Specific heat capacity at constant pressure ($\boldsymbol{s_p}$):

- The amount of heat energy required to raise the temperature of one kg of a substance by 1 K or 1°C by keeping the pressure constant is called specific heat capacity of at constant pressure.

- In this process a part of the heat energy is used for doing work (expansion) and the remaining part is used to increase the internal energy of the gas.

Specific heat capacity at constant volume ($\boldsymbol{s_v}$):

- The amount of heat energy required to raise the temperature of one kg of a substance by 1 K or 1°C by keeping the volume constant. If the volume is kept constant, then the supplied heat is used to increase only the internal energy. No work is done by the gas.

$\boldsymbol{s_p}$ is always greater than $\boldsymbol{s_v}$.

- The amount of heat required to raise the temperature of one mole of a substance by 1K or 1°C at constant volume is called molar specific heat capacity at constant volume (C_v). If pressure is kept constant, it is called molar specific heat capacity at constant pressure (C_p).

$$C_v = \frac{1}{\mu}\frac{dU}{dT}$$

Meyer's Relation:

$$C_p - C_v = R$$

THERMODYNAMIC PROCESS

Isothermal process (constant temperature):

$$\Delta U = 0$$
$$Q = W$$

So, the heat supplied to a gas is used to do only external work.

Examples:

(i) When water is heated, at the boiling point, the temperature will not increase unless the water completely evaporates. Similarly, at the freezing point, when the ice melts to water, the temperature of ice will not increase even when heat is supplied to ice.

(ii) All biological processes occur at constant body temperature (37°C).

Adiabatic process:

- This is a process in which no heat flows into or out of the system (Q=0). But the gas can expand by spending its internal energy or gas can be compressed through some external work.

$$\Delta U = W$$

The adiabatic process can be achieved by the following methods

- Thermally insulating the system from surroundings.
- If the process occurs so quickly that there is no time to exchange heat with surroundings even though there is no thermal insulation.

Example: When the warm air rises from the surface of the Earth, it adiabatically expands. As a result the water vapour cools and condenses into water droplets forming a cloud.

$$PV^{\gamma} = constant$$

Here γ is adiabatic exponent and $\gamma = {C_p}/{C_v}$ which depends on nature of gas.

- The PV diagram for an adiabatic process is also called *adiabat.* The PV diagram for isothermal and adiabatic processes look similar. But the adiabatic curve is steeper than isothermal curve.

$$TV^{\gamma-1} = constant$$
$$T^{\gamma}P^{1-\gamma} = constant$$

Work done in adiabatic process,

$$W_{adia} = \frac{\mu R}{\gamma - 1}[T_i - T_f]$$

- In adiabatic expansion, work done is positive and $T_i > T_f$ and gas cools.
- In adiabatic compression, work done is negative and $T_i < T_f$ and temperature of gas increases.

Isobaric Process (constant pressure):

Examples for Isobaric process:

- v When the gas is heated and pushes the piston so that it exerts a force equivalent to atmospheric pressure plus the force due to gravity.

- v When the food is cooked in an open vessel, the pressure above the food is always at atmospheric pressure.

Work done in an isobaric process,

$$W = P\Delta V = \mu R T_f \left(1 - \frac{T_i}{T_f}\right)$$
$$\Delta U = Q - P\Delta V$$

Isochoric Process (constant volume):

$$\Delta V = 0 \; and \; W = 0. \; So, \qquad \Delta U = Q$$

Examples:

- When food is being cooked in closed position, after a certain time you can observe the lid is being pushed upwards by the water steam. This is because when the lid is closed, the volume is kept constant. As the heat continuously supplied, the pressure increases and water steam tries to push the lid upwards

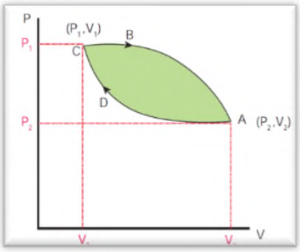
- In automobiles the petrol engine undergoes four processes. First the piston is adiabatically compressed to some volume as shown in the Figure (a). In the second process (Figure (b)), the volume of the air-fuel mixture is kept constant and heat is being added. As a result the temperature and pressure are increased. This is an isochoric process. For a third stroke (Figure (c)) there will be an adiabatic expansion, and fourth stroke again isochoric process by keeping the piston immoveable (Figure (d)).

Cyclic process:

- The thermodynamic system returns to its initial state after undergoing a series of changes. The change in the internal energy is zero. From the first law of thermodynamics, the net heat transferred to the system is equal to work done by the gas.

$$Q_{net} = Q_{in} - Q_{out} = W$$

PV diagram for cyclic process:



- The total work done is green shaded area in the figure. If the net work done is positive, then work done by the system is greater than the work done on the system. If the net work done is negative then the work done by the system is less than the work done on the system.

- Further, in a cyclic process the net work done is positive if the process goes clockwise and network done is negative if the process goes anti-clockwise.

Limitations of First Law of Thermodynamics:

- The first law of thermodynamics explains well the inter convertibility of heat and work. But it does not indicate the direction of change.

Process		Heat Q	Temperature & internal	Pressure	Volume	Equation of state	Work done (ideal gas)
Isothermal	Expansion	$\boldsymbol{Q} > 0$	Constant	Decrea	Increas	$\boldsymbol{PV} = \boldsymbol{constant}$	$W = \mu RT\, ln\left(\frac{V_f}{V_i}\right) > 0$
	Compressi	• $Q < 0$	Constant	Increas	Decrea		$\boldsymbol{W = \mu RT\, ln\left(\frac{V_f}{}\right) < 0}$
Isobaric	Expansion	$\boldsymbol{Q} > 0$	Increases	Consta	Increas	$\boldsymbol{\frac{V}{T} = constant}$	$\boldsymbol{W = P[V_f - V_i]}$
	Compressi	$\boldsymbol{Q} < 0$	Decrease	Consta	Decrea		$\boldsymbol{W = P[V_f - V_i]}$
Isochoric	Expansion	$\boldsymbol{Q} > 0$	Increase	Increas	Consta	$\boldsymbol{\frac{P}{T} = constant}$	Zero
	Compressi	$\boldsymbol{Q} < 0$	Decrease	Decrea	Consta		
Adiabatic	Expansion	$\boldsymbol{Q = 0}$	Decrease	Decrea	Increas	$\boldsymbol{PV^{\gamma} = constant}$	$W_{adia} = \frac{\mu R}{\gamma - 1}[T_i - T_f] > 0$
	Compressi	$\boldsymbol{Q = 0}$	Increase	Increas	Decrea		$W_{adia} = \frac{\mu R}{\gamma - 1}[T_i - T_f] < 0$

For example,

- According to first law, it is possible for the energy to flow from hot object to cold object or from cold object to hot object. But in nature the direction of heat flow is always from higher temperature to lower temperature

- Heat produced against friction is not reconverted to the kinetic energy of the car.

Reversible Process:

- A thermodynamic process can be considered reversible only if it possible to retrace the path in the opposite direction in such a way that the system and surroundings pass through the same states as in the initial, direct process.

Example: A quasi–static isothermal expansion of gas, slow compression and expansion of a spring. Conditions for reversible process:

1. The process should proceed at an extremely slow rate.

2. The system should remain in mechanical, thermal and chemical equilibrium state at all the times with the surroundings, during the process.

3. No dissipative forces such as friction, viscosity, electrical resistance should be present.

Irreversible process:

- All natural processes are irreversible. Irreversible process cannot be plotted in PV diagram.

- According to second law of thermodynamics “Heat always flows from hotter object to colder object spontaneously”. This is known as the Clausius form of second law of thermodynamics.

HEAT ENGINE:

- Heat engine is a device which takes heat as input and converts this heat in to work by undergoing a cyclic process.

- A heat engine has three parts:

(a) Hot reservoir (or) Source: It is maintained at a high temperature T_H

(b) Working substance

- It is a substance like gas or water, which converts the heat supplied into work.
- The working substance in steam engine is water which absorbs heat from the burning of coal. The heat converts the water into steam.

- This steam is does work by rotating the wheels.(c) Cold reservoir (or) Sink: It is maintained at lower temperature T_L

Reservoir:

- It is defined as a thermodynamic system which has very large heat capacity. By taking in heat from reservoir or giving heat to reservoir, the reservoir's temperature does not change.

- The heat engine works in a cyclic process. After a cyclic process it returns to the same state. Since the heat engine returns to the same state after it ejects heat, the change in the internal energy of the heat engine is zero.

$$efficiency, \eta = \frac{output}{input} = \frac{W}{Q_H} = \frac{Q_H - Q_L}{Q_H} = 1 - \frac{Q_L}{Q_H}$$

- Since $Q_L < Q_H$, the efficiency (η) always less than 1. This implies that heat absorbed is not completely converted into work.

Kelvin-Planck statement:

It is impossible to construct a heat engine that operates in a cycle, whose sole effect is to convert the heat completely into work. This implies that no heat engine in the universe can have 100% efficiency.

Carnot's ideal Heat Engine:

- A reversible heat engine operating in a cycle between two temperatures in a particular way is called a Carnot Engine.

- The carnot engine has four parts.

i Source: It is at T_H. Any amount of heat can be extracted, without changing temperature.

ii Sink: It is maintained at T_L. It can absorb any amount of heat.

iii Insulating stand: It is made of perfectly non-conducting material.

iv Working substance: It is an ideal gas enclosed in a cylinder with perfectly non-conducting walls and perfectly conducting bottom. A non-conducting and frictionless piston is fitted in it.

The working substance is subjected to four successive reversible processes forming what is called Carnot's cycle.

a) Quasi-static Isothermal Expansion
b) Quasi- static Adiabatic Expansion
c) Quasi-static Isothermal compression
d) Quasi-static Adiabatic Compression

- After one cycle the working substance returns to the initial temperature T_H. This implies that the change in internal energy of the working substance after one cycle is zero.

Efficiency of Carnot Engine:

$$\text{efficiency}, \eta = 1 - \frac{T_L}{T_H}$$

a) It can be 100% only when $T_L = 0\,K$ which is impossible.

b) Efficiency is independent of working substance.

c) When $T_L = T_H, \eta = 0$. No carnot engine can have source and sink at same temperature.

d) Carnot theorem is stated as 'Between two constant temperatures reservoirs, only Carnot engine can have maximum efficiency. All real heat engines will have efficiency less than the Carnot engine'

e) The efficiency depends on the ratio of the two temperature and not on the difference in the temperature. The engine which operates in lower temperature has highest efficiency.

Entropy and second law of thermodynamics:

$$\textbf{entropy} = \frac{\mathbf{Q}}{\mathbf{T}}$$

- Change in entropy of Carnot Engine in one cycle is zero. "For all the processes that occur in nature (irreversible process), the entropy always increases. For reversible process entropy will not change".
- Entropy determines the direction in which natural process should occur.
- Entropy is also called 'measure of disorder'. All natural process occur such that the disorder should always increases.
- **Example:** a drop of ink diffusing in water.

Refrigerator:

- A refrigerator is a Carnot's engine working in the reverse order.
- The working substance (gas) absorbs quantity of heat Q_L from cold body (sink) at lower temperature T_L. A certain amount of work W is done on the working substance by the compressor and a quantity of heat Q_H is ejected to the hot body (source) i.e., atmosphere at T_H.

$$Q_L + \mathbf{W} = \mathbf{Q_H}$$

As a result, cold reservoir gets further cooled down and surroundings are heated more.

$$\text{coefficient of performance, COP} = \beta = \frac{Q_L}{W} = \frac{Q_L}{Q_H - Q_L} = \frac{T_L}{T_H - T_L}$$

1. The greater the COP, the better is the condition. A refrigerator has COP around 5 to 6.

2. Lesser the difference in the temperatures of the cooling chamber and the atmosphere, higher is the COP of a refrigerator.

3. In the refrigerator the heat is taken from cold object to hot object by doing external work. It is not a violation of second law of thermodynamics, because the heat is ejected to surrounding air and total entropy of (refrigerator + surrounding) is always increased.

Greenhouse effect:

- Top of the atmosphere is at -19°C and bottom of the atmosphere is at +14°C. The increase in 33°C from top to bottom is due to Greenhouse gases and this effect is called Greenhouse effect.

- The greenhouse gases are mainly CO_2, water vapour, Ne, He, NO_2,
- CH_4, Xe, Kr, ozone and NH_3. Except CO_2 and water vapour, all others are present only in very small amount in the atmosphere. The radiation from the Sun is mainly in the visible region of the spectrum. The earth absorbs these radiations and reradiate in the infrared region. Carbon dioxide and water Vapour are good absorbers of infrared radiation since they have more vibrational degree of freedom compared to nitrogen and oxygen which keeps earth warmer.
- The amount of CO_2 present in the atmosphere is increased from 20% to 40% due to human activities since 1900s. The major emission of CO_2 comes from burning of fossil fuels in automobiles. Due to this increase in the CO_2 content in the atmosphere, the average temperature of the earth increases by 1°C. This effect is called global warming. It has serious influence and alarming effect on ice glaciers. In addition, the CO_2 content is also increasing in ocean which is very dangerous to species in the oceans.
- Another very important greenhouse gas is Chloro flouro carbon(CFC) which is used as coolant in refrigerators. In the human made greenhouse gases CO_2 is 55%, CFCs are 24%. Nitrogen oxide is 6% and methane is 15%. CFCs also has made huge damage to ozone layer.

FAST FACTS

a) When the piston is compressed so quickly that there is no time to exchange heat to the surrounding, the temperature of the gas increases rapidly. This principle is used in the diesel engine. The air-gasoline mixture is compressed so quickly (adiabatic compression) that the temperature increases enormously, which is enough to produce a spark.

b) All reversible processes are quasi-static but all quasi- static processes need not be reversible. For example when we push the piston very slowly, if there is friction between cylinder wall and piston some amount of energy is lost to surroundings, which cannot be retrieved back.

c) The efficiency of diesel engines has maximum up to 44% and the efficiency of petrol engines are maximum up to 30%. Now a days typical bikes give a mileage of 50 km per Liter of petrol. This implies only 30% of 1 Liter of petrol is converted into mechanical work and the remaining 70% goes out as wasted heat.

d) In earthern pot, the cooling process is not due to any cyclic process. The cooling occurs due to evaporation of water molecules which oozes out through pores of the pot. Even though the heat flows from cold water to open atmosphere, it is not a violation of second law of thermodynamics. The water inside the pot is an open thermodynamic system, so the entropy of water + surrounding always increases.

KINETIC THEORY OF GASES

KINETIC THEORY

Introduction

Thermodynamics is basically a macroscopic science. We discussed macroscopic parameters like pressure, temperature and volume of thermodynamical systems in unit 8. In this unit we discuss the microscopic origin of pressure and temperature by considering a thermodynamic system as collection of particles or molecules. Kinetic theory relates pressure and temperature to molecular motion of sample of a gas and it is a bridge between Newtonian mechanics and thermodynamics. The present chapter introduces the kinetic nature of gas molecules.

Postulates of kinetic theory of gases

Kinetic theory is based on certain assumptions which makes the mathematical treatment simple. None of these assumptions are strictly true yet the model based on these assumptions can be applied to all gases.

All the molecules of a gas are identical, elastic spheres.
The molecules of different gases are different.
The number of molecules in a gas is very large and the average separation between them is larger than size of the gas molecules.
The molecules of a gas are in a state of continuous random motion.
The molecules collide with one another and also with the walls of the container.
These collisions are perfectly elastic so that there is no loss of kinetic energy during collisions.
Between two successive collisions, a molecule moves with uniform velocity.
The molecules do not exert any force of attraction or repulsion on each other except during collision. The molecules do not possess any potential energy and the energy is wholly kinetic.
The collisions are instantaneous. The time spent by a molecule in each collision is very small compared to the time elapsed between two consecutive collisions.
These molecules obey Newton’s laws of motion even though they move randomly.

PRESSURE EXERTED BY A GAS

Consider a monoatomic gas of N molecules each having a mass m inside a cubical container of side l

(a) Container of gas molecules

(b) Collision of a molecule with the wall

The molecules of the gas are in random motion. They collide with each other and also with the walls of the container. As the collisions are elastic in nature, there is no loss of kinetic energy, but a change in momentum occurs.

The molecules of the gas exert pressure on the walls of the container due to collision on it. During each collision, the molecules impart certain momentum to the wall. Due to transfer of momentum, the walls experience a continuous force. The force experienced per unit area of the walls of the container determines the pressure exerted by the gas. It is essential to determine the total momentum transferred by the molecules in a short interval of time.
A molecule of mass m moving with a velocity $\vec{V}$ having components (v_x, v_y, v_z) hits the right side wall. Since we have assumed that the collision is elastic, the particle rebounds with same speed and its x-component is reversed. (b). The components of velocity of the molecule after collision are ($-v_x$, v_y, v_z).

The x-component of momentum of the molecule before collision = mv_x
The x-component of momentum of the molecule after collision = $-mv_x$
The change in momentum of the molecule in x direction = Final momentum – initial momentum = $-mv_x - mv_x = -2mv_x$

According to law of conservation of linear momentum, the change in momentum of the wall = $2mv_x$

NOTE
In x direction, the total momentum of the system before collision is equal to momentum of the molecule (mv_x) since the momentum of the wall is zero. According to the law of conservation of momentum the total momentum of system after the collision must be equal to total momentum of system before collision. The momentum of the molecule (in x direction) after the collision is $-mv_x$ and the momentum of the wall after the collision is $2mv^x$. So total momentum of the system after the collision is ($2mvx - mv_x$) = mv_x which is same as the total momentum of the system before collision.

The number of molecules hitting the right side wall in a small interval of time Δt is calculated as follows.

The molecules within the distance of $v_x \Delta t$ from the right side wall and moving towards the right will hit the wall in the time interval Δt.

The number of molecules that will hit the right side wall in a time interval Δt is equal to the product of volume ($Av_x \Delta t$) and number density of the molecules (n). Here A is area of the wall and n is number of molecules per unit volume $\frac{N}{V}$

We have assumed that the number density is the same throughout the cube.

Number of molecules hitting the wall

Not all the n molecules will move to the right, therefore on an average only half of the n molecules move to the right and the other half moves towards left side.

The number of molecules that hit the right side wall in a time interval Δt

$$= \frac{n}{2} A v_x \Delta t$$

In the same interval of time Δt, the total momentum transferred by the molecules

$$\Delta p = \frac{n}{2} A v_x \Delta t \times 2 m v_x = A v_x^2 m n \Delta t$$

From Newton's second law, the change in momentum in a small interval of time gives rise to force.

The force exerted by the molecules on the wall (in magnitude)

$$F = \frac{\Delta p}{\Delta t} = nmAv_x^2$$

Pressure, P = force divided by the area of the wall

$$P = \frac{F}{A} = nmv_x^2$$

Since all the molecules are moving completely in random manner, they do not have same speed. So we can replace the term v_x^2 by the average v_x^2 in equation

$$P = mn\overline{v_X^2}$$

Since the gas is assumed to move in random direction, it has no preferred direction of motion (the effect of gravity on the molecules is neglected). It implies that the molecule has same average speed in all the three direction. So, $\overline{v_X^2} = \overline{v_y^2} = \overline{v_z^2}$

The mean square speed is written as

$$\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} = 3v_x^2$$

$$\overline{v_x^2} = \frac{1}{3}\overline{v^2}$$

Using this in equation, we get

$$P = \frac{1}{3}nm\overline{v^2} \text{ or } P = \frac{1}{3}\frac{N}{V}m\overline{v^2}$$

$$as \left[n = \frac{N}{V}\right]$$

The following inference can be made from the above equation. The pressure exerted by the molecules depends on

Number density= $\frac{N}{V}$. It implies that if the number density increases then pressure will increase. For example when we pump air inside the cycle tyre or car tyre essentially the number density increases and as a result the pressure increases.

Mass of the molecule Since the pressure arises due to momentum transfer to the wall, larger mass will have larger momentum for a fixed speed. As a result the pressure will increase.

Mean square speed For a fixed mass if we increase the speed, the average speed will also increase. As a result the pressure will increase.

For simplicity the cubical container is taken into consideration. The above result is true for any shape of the container as the area A does not appear in the final expression. Hence the pressure exerted by gas molecules on the wall is independent of area of the wall (A).

Kinetic interpretation of temperature

To understand the microscopic origin of temperature in the same way,

Rewrite the equation

$$P = \frac{1}{3}\frac{N}{V}m\overline{v^2}$$

$$PV = \frac{1}{3}Nm\overline{v^2}$$

Comparing the equation with ideal gas quation PV=NkT

$$NkT = \frac{1}{3}Nmv^2$$

$$kT = \frac{1}{3}m\overline{v^2}$$

Multiply the above equation by 3/2 on both sides.

$$\frac{3}{2}kT = \frac{1}{2}m\overline{v^2}$$

R.H.S of the equation (9.9) is called average kinetic energy of a single molecule (KE).

The average kinetic energy per molecule

$$\overline{KE} = \in = \frac{3}{2}kT$$

Equation implies that the temperature of a gas is a measure of the average translational kinetic energy per molecule of the gas.

Equation is a very important result from kinetic theory of gas. We can infer the following from this equation.

The average kinetic energy of the molecule is directly proportional to absolute temperature of the gas. The equation gives the connection between the macroscopic world (temperature) to microscopic world (motion of molecules).
The average kinetic energy of each molecule depends only on temperature of the gas not on mass of the molecule. In other words, if the temperature of an ideal gas is measured using thermometer, the average kinetic energy of each molecule can be calculated without seeing the molecule through naked eye.

By multiplying the total number of gas molecules with average kinetic energy of each molecule, the internal energy of the gas is obtained.
Internal energy of ideal gas U= N$\left(\frac{1}{2}m\overline{v^2}\right)$
By using equation

$$U = \frac{3}{2}NkT$$

From equation, we understand that the internal energy of an ideal gas depends only on absolute temperature and is independent of pressure and volume.

A football at 27°C has 0.5 mole of air molecules. Calculate the internal energy of air in the ball.

Solution
The internal energy of ideal gas = $\frac{3}{2}NkT$
The number of air molecules is given in terms of number of moles so, rewrite the expression as follows $U = \frac{3}{2}\mu RT$
Since Nk = μR. Here μ is number of moles.
Gas constant R = 8.31$\frac{J}{molk}$
Temperature T = 273 + 27 =300k

$$U = \frac{3}{2} \times 0.5 \times 8.31 \times 300 = 1869.75J$$

This is approximately equivalent to the kinetic energy of a man of 57 kg running with a speed of 8 m s^{-1}.

Relation between pressure and mean kinetic energy

From earlier section, the internal energy of the gas is given by

$$U = \frac{3}{2}NkT$$

He above equation can also be written as

$$U = \frac{3}{2}PV$$

Since PV = NkT

$$P = \frac{2}{3}\frac{U}{V} = \frac{2}{3}u$$

From the equation, we can state that the pressure of the gas is equal to two thirds of internal energy per unit volume or internal energy density $\left(u = \frac{U}{v}\right)$.

Writing pressure in terms of mean kinetic energy density using equation.

$$P = \frac{1}{3}nm\overline{v^2} = \frac{1}{3}p\overline{v^2}$$

where ρ = nm = mass density (Note n is number density)

Multiply and divide R.H.S of equation by 2, we get

$$P = \frac{2}{3}\left(\frac{p}{2}\overline{v^2}\right)$$

$$P = \frac{2}{3}\overline{KE}$$

From the equation, pressure is equal to 2/3 of mean kinetic energy per unit volume.

Some elementary deductions from kinetic theory of gases

Boyle’s law:

From equation, we know that $PV = \frac{2}{3}U$

But the internal energy of an ideal gas is equal to N times the average kinetic energy (∈) of each molecule.

$$U = N \in$$

For a fixed temperature, the average translational kinetic energy ∈ will remain constant. It implies that

$$PV = \frac{2}{3}N \in \ Thus\ PV = constant$$

Therefore, pressure of a given gas is inversely proportional to its volume provided the temperature remains constant. This is Boyle’s law.

Charles' law:

From the equation, we get $\boldsymbol{PV} = \frac{2}{3}\boldsymbol{U}$

For a fixed pressure, the volume of the gas is proportional to internal energy of the gas or average kinetic energy of the gas and the average kinetic energy is directly proportional to absolute temperature. It implies that

$$V \alpha T \text{ or } \frac{T}{V} = constant$$

This is Charles' law.

Avogadro's law:

This law states that at constant temperature and pressure, equal volumes of all gases contain the same number of molecules. For two different gases at the same temperature and pressure, according to kinetic theory of gases,

$$P = \frac{1}{3}\frac{N}{V}m_1\overline{v_1^2} = \frac{1}{3}\frac{N_2}{V}m_2\overline{v_2^2}$$

where $\overline{V_1^2}$ and $V_2^2$are the mean square speed for two gases and N_1 and N_2 are the number of gas molecules in two different gases.

At the same temperature, average kinetic energy per molecule is the same for two gases.

$$\frac{1}{2}m_1\overline{v_1^2} = \frac{1}{2}m_2\overline{v_2^2}$$

Dividing the equation (9.15) by (9.16) we get N1 = N2

This is Avogadro's law. It is sometimes referred to as Avogadro's hypothesis or Avogadro's Principle.

Root mean square speed (v_{rms})

Root mean square speed (v_{rms}) is defined as the square root of the mean of the square of speeds of all molecules. It is denoted by $v_{rms} = \sqrt{\overline{v^2}}$

Equation can be re-written as,

mean square speed $\overline{v^2} = \frac{3kT}{m}$

Root mean square speed,

$$v_{rms} = \sqrt{\frac{3kT}{m}} = 1.73\sqrt{\frac{kT}{m}}$$

From the equation we infer the following

(i) rms speed is directly proportional to square root of the temperature and inversely proportional to square root of mass of the molecule. At a given temperature the molecules of lighter mass move faster on an average than the molecules with heavier masses.
Example: Lighter molecules like hydrogen and helium have high 'vrms' than heavier molecules such as oxygen and nitrogen at the same temperature.
(ii) Increasing the temperature will increase the r.m.s speed of molecules
We can also write the vrms in terms of gas constant R. Equation (9.18) can be rewritten as follows

$v_{rms} = \sqrt{\frac{3N_1kT}{N_1m}}$ where N_A is avogadra number.

Since $N_Ak = R$ and $N_Am = M$ (molar mass)

The root mean square speed or r.m.s speed

$$v_{rms} = \sqrt{\frac{3RT}{M}}$$

The equation can also be written in term of rms speed $p = \frac{1}{3}nmv^2{}_{rms}$ since $v^2_{rms} = \overline{v^2}$

Impact of v_{rms} in nature:

1. Moon has no atmosphere.

The escape speed of gases on the surface of Moon is much less than the root mean square speeds of gases due to low gravity. Due to this all the gases escape from the surface of the Moon.

2. No hydrogen in Earth's atmosphere.

As the root mean square speed of hydrogen is much greater than that of nitrogen, it easily escapes from the earth's atmosphere.

In fact, the presence of nonreactive nitrogen instead of highly combustible hydrogen deters many disastrous consequences.

A room contains oxygen and hydrogen molecules in the ratio 3:1. The temperature of the room is 27°C. The molar mass of 0_2 is 32 g mol^{-1} and of H_2 is 2 g mol^{-1}. The value of gas constant R is 8.32 J mol^{-1} K^{-1}

Calculate

(a) rms speed of oxygen and hydrogen molecule
(b) Average kinetic energy per oxygen molecule and per hydrogen Molecule
(c) Ratio of average kinetic energy of oxygen molecules and hydrogen molecules

SOLUTION
(a) absolute temperature

$$T = 27°C = 27 + 273 = 300K$$

Gas constant R = 8.32 J $mol^{-1}k^{-1}$

For oxygen molecule molar mass $M = 32g = 32 \times 10^{-3} kg\ mol^{-1}$

rms speed $v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3\times8032\times300}{32\times10^{-3}}}$

$$= 483.73 m\ s^{-1} \approx 484 m\ s^{-1}$$

For hydrogen molecule:

Molar mass $M = 2 \times 10^{-3} kg\ mol^{-1}$

$$rms\ speed\ v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 8.32 \times 300}{2 \times 10^{-3}}}$$

$$= 1934 m\ s^{-1} = 1.93 km\ s^{-1}$$

Note that the rms speed is inversely proportional to M and the molar mass of oxygen is 16 times higher than molar mass of hydrogen. It implies that the rms speed of hydrogen is 4 times greater than rms speed of oxygen at the same temperature.

$$\frac{1934}{484} = 4$$

(b) The average kinetic energy per molecule is $\frac{3}{2}kT$. It depends only on absolute temperature of the gas and is independent of the nature of molecules. Since both the gas molecules are at the same temperature, they have the same average kinetic energy per molecule. k is Boltzmaan constant.

$$\frac{3}{2}kT = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21} J$$

(c) Average kinetic energy of total oxygen molecules = $\frac{3}{2}\boldsymbol{N_o kT}$ where N_o - number of oxygen molecules in the room

Average kinetic energy of total hydrogen molecules = $\frac{3}{2}\boldsymbol{N_o kT}$ where N_H - number of hydrogen molecules in the room.

It is given that the number of oxygen molecules is 3 times more than number of hydrogen molecules in the room. So the ratio of average kinetic energy of oxygen molecules with average kinetic energy of hydrogen molecules is 3:1.

Mean (or) average speed ($\overline{v}$)

It is defined as the mean (or) average of all the speeds of molecules
If $v_1, v_2, v_3 \ldots . v_N$ are the individual speeds of molecules then

$$\overline{v} = \frac{v_1 + v_2 + v_3 \ldots\ldots + v_n}{N} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8kT}{\pi m}}$$

Here M- molar mass and m – mass of the molecule.

$$\overline{v} = 1.60\sqrt{\frac{kT}{m}}$$

Most probable speed (V_{mp})

It is defined as the speed acquired by most of the molecules of the gas.

$$v_{mp} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2kT}{m}}$$

$$v_{mp} = 1.41\sqrt{\frac{kT}{m}}$$

The derivation of equations is beyond the scope of the book

Comparison of v_{rms} , $\overline{v}$ and v_{mp}

Among the speeds v_{rms} is the largest and v_{mp} is the least

$$v_{rms} > \overline{v} > v_{mp}$$

Ratio-wise,

$$v_{rms} : \bar{v} : v_{mp} = \sqrt{3} : \sqrt{8/\pi} : \sqrt{2} = \mathbf{1.732 : 1.6 : 1.414}$$

Ten particles are moving at the speed of 2, 3, 4, 5, 5, 5, 6, 6, 7 and 9 m s^{-1}. Calculate rms speed, average speed and most probable speed.

Solution
The average speed

$$\bar{v} = \frac{2 + 3 + 4 + 5 + 5 + 6 + 6 + 7 + 9}{10} = 5.2 m\, s^{-1}$$

To find the rms speed, first calculation the mean square speed $\overline{v^2}$

$$\overline{v^2} = \frac{2^2 + 3^2 + 4^2 + 5^2 + 5^2 + 5^2 + 6^2 + 6^2 + 7^2 + 9^2}{10}$$

$$30.6 m^2 s^{-1}$$

The rms speed

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{30.6} = 5.53 m s^{-1}$$

The most probable speed is $5 m s^{-1}$ because three of the particles have that speed.

Calculate the rms speed, average speed and the most probable speed of mole of hydrogen molecules at 300 K. Neglect the mass of electron.

Solution

The hydrogen atom has one proton and one electron. The mass of electron is negligible compared to the mass of proton.

Mass of one proton = $1.67 \times 10^{-27} kg$

One hydrogen molecule = 2 hydrogen atoms = $2 \times 1.67 \times 10^{-27} kg$.
The average speed

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}} = 1.60 \sqrt{\frac{kT}{m}}$$

$$= 1.60\sqrt{\frac{(1.38 \times 10^{-23}) \times (300)}{2(1.67 \times 10^{-27})}} = 1.78 \times 10^{3} ms^{-1}$$

$$(Boltzmann\ Constant\ k = 1.38 \times 10^{-23} JK^{-1})$$

The rms speed $v_{rms} = \sqrt{\frac{3kT}{m}} = 1.73\sqrt{\frac{kT}{m}}$

$$= 1.73\sqrt{\frac{(1.38 \times 10^{-23}) \times (300)}{2(1.67 \times 10^{-27})}} = 1.93 \times 10^{3} ms^{-1}$$

Most probable speed $v_{rms} = \sqrt{\frac{2kT}{m}} = 1.41\sqrt{\frac{kT}{m}}$

$$= 1.41\sqrt{\frac{(1.38 \times 10^{-23}) \times (300)}{2(1.67 \times 10^{-27})}} = 1.57 \times 10^{3} ms^{-1}$$

Note that $v_{rms} > \overline{v} > v_{rms}$

Maxwell-Boltzmann speed distribution function

In a classroom, the air molecules are moving in random directions. The speed of each molecule is not the same even though macroscopic parameters like temperature and pressure are fixed. Each molecule collides with every other molecule and they exchange their speed. In the previous section we calculated the rms speed of each molecule and not the speed of each molecule which is rather difficult. In this scenario we can find the number of gas molecules that move with the speed of 5 m s^{-1} to 10 m s^{-1} or 10 m $s^{-1}$to 15 m s^{-1} etc. In general our interest is to find how many gas molecules have the range of speed from v to v + dv. This is given by Maxwell's speed distribution function.

$$N_v = 4\pi N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2kT}}$$

The above expression is graphically shown as follows

It is clear that, for a given temperature the number of molecules having lower speed increases parabolically (v^2) but decreases exponentially $e^{\frac{mv^2}{2kT}}$ after reaching most probable speed. The rms speed, average speed and most probable speed are indicated. It can be seen that the rms speed is greatest among the three.

To know the number of molecule in the range of speed between 50m s-1 and 60m s-1, we need to integrate $\int_{30}^{*60} 4\pi N \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} v^2 e^{\frac{mv^2}{2kT}} dv = N(50\ to\ 60m\ s^{-1})$. In general the number of molecules within the range of speed v and v+dv is given by $\int_{V}^{*V+DV} 4\pi N \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{\frac{mv^2}{2kT}} dv = N(v + dv)$.

The exact integration is beyond the scope of the book. But we can infer the behavior of gas molecules from the graph.

The area under the graph will give the total number of gas molecules in the system

The speed distribution graph for two different temperatures. As temperature increases, the peak of the curve is shifted to right. It implies that the average speed of each molecule will increase. But the area under each graph is same since it represents the total number of gas molecules.

DEGREES OF FREEDOM

Definition

The minimum number of independent coordinates needed to specify the position and configuration of a thermo-dynamical system in space is called the degree of freedom of the system.

Example:
A free particle moving along x-axis needs only one coordinate to specify it completely. So its degree of freedom is one.
Similarly a particle moving over a plane has two degrees of freedom.
A particle moving in space has three degrees of freedom.

Suppose if we have N number of gas molecules in the container, then the total number of degrees of freedom is f = 3N.

But, if the system has q number of constraints (restrictions in motion) then the degrees of freedom decreases and it is equal to f = 3N-q where N is the number of particles.

Monoatomic molecule

A monoatomic molecule by virtue of its nature has only three translational degrees of freedom.

Therefore f = 3

Example: Helium, Neon, Argon

Diatomic molecule

There are two cases.
1. At Normal temperature

A molecule of a diatomic gas consists of two atoms bound to each other by a force of attraction. Physically the molecule can be regarded as a system of two point masses fixed at the ends of a massless elastic spring.
The center of mass lies in the center of the diatomic molecule. So, the motion of the center of mass requires three translational degrees of freedom. In addition, the diatomic molecule can rotateabout three mutually perpendicular axes. But the moment of inertia about its own axis of rotation is negligible (about y axis in the figure 9.5). Therefore, it has only two rotational degrees of freedom (one rotation is about Z axis and another rotation is about X axis). Therefore totally there are five degrees of freedom.

f = 5
2. At High Temperature

At a very high temperature such as 5000 K, the diatomic molecules possess additional two degrees of freedom due to vibrational motion[one due to kinetic energy of vibration and the other is due to potential energy] (Figure 9.5c). So totally there are seven degrees of freedom.

f = 7

Examples: Hydrogen, Nitrogen, Oxygen.

Triatomic molecules

There are two cases.
Linear triatomic molecule

In this type, two atoms lie on either side of the central atom.

Linear triatomic molecule has three translational degrees of freedom. It has two rotational degrees of freedom because it is similar to diatomic molecule except there is an additional atom at the center. At normal temperature, linear triatomic molecule will have five degrees of freedom. At high temperature it has two additional vibrational degrees of freedom. So a linear triatomic molecule has seven degrees of freedom.

Non-linear triatomic molecule

In this case, the three atoms lie at the vertices of a triangle.
It has three translational degrees of freedom and three rotational degrees of freedom about three mutually orthogonal axes. The total degrees of freedom,
f = 6

Example: Water, Sulphurdioxide.

LAW OF EQUIPARTITION OF ENERGY

That the average kinetic energy of a molecule moving in x direction is $\frac{1}{2}\boldsymbol{m\overline{v_x^2}} = \frac{1}{2}\boldsymbol{kT}$.

Similarly, then the motion is in y direction $\frac{1}{2}\boldsymbol{m\overline{v_y^2}} = \frac{1}{2}\boldsymbol{kT}$ and and for the motion along z direction, $\frac{1}{2}\boldsymbol{m\overline{v_z^2}} = \frac{1}{2}\boldsymbol{kT}$.

According to kinetic theory, the average kinetic energy of system of molecules in thermal equilibrium at temperature T is uniformly distributed to all degrees of freedom (x or y or z directions of motion) so that each degree of freedom will get $\frac{1}{2}\boldsymbol{kT}$ of energy. This is called law of equipartition of energy.
Average kinetic energy of a monatomic molecule $(\boldsymbol{with\, f = 3}) = \boldsymbol{3} \times \frac{1}{2}\boldsymbol{kT} = \frac{3}{2}\boldsymbol{kT}$

Average kinetic energy of diatomic molecule at low temperature $(\boldsymbol{with\, f = 5}) = \boldsymbol{5} \times \frac{1}{2}\boldsymbol{kT} = \frac{5}{2}\boldsymbol{kT}$

Average kinetic energy of diatomic molecule at high temperature $(\boldsymbol{with\, f = 7}) = \boldsymbol{7} \times \frac{1}{2}\boldsymbol{kT} = \frac{7}{2}\boldsymbol{kT}$

Average kinetic energy of linear triatomic molecule $(\boldsymbol{with\, f = 7}) = \boldsymbol{7} \times \frac{1}{2}\boldsymbol{kT} = \frac{7}{2}\boldsymbol{kT}$

Average kinetic energy of non linear triatomic molecule ($\boldsymbol{with\ f = 6}$) = $\boldsymbol{6 \times \frac{1}{2} kT = 3kT}$

Application of law of equipartition energy in specific heat of a gas

Meyer's relation $C_P - C_V = R$ connects the two specific heats for one mole of an ideal gas.

Equipartition law of energy is used to calculate the value of $C_P - C_V$ and the ratio between them $\boldsymbol{y = \frac{C_P}{C_V}}$ Here γ is called adiabatic exponent.

i) Monatomic molecule

Average kinetic energy of a molecule

$$\left[\frac{3}{2} kT\right]$$

Total energy of a mole of gas

$$= \frac{3}{2} kT \times N_A = \frac{3}{2} RT$$

For one mole, the molar specific heat at constant volume

$$C_V = \frac{dU}{dT} = \frac{d}{dT}\left[\frac{3}{2} RT\right]$$

$$C_V = \left[\frac{3}{2} R\right]$$

$$C_P = C_v + R = \frac{3}{2} R + R = \frac{5}{2} R$$

The ratio of specific heats,

$$\gamma = \frac{C_P}{C_P} = \frac{\frac{5}{2} R}{\frac{3}{2} R} = \frac{5}{3} = 1.67$$

ii) Diatomic molecule

Average kinetic energy of a diatomic molecule at low temperature = $\boldsymbol{\frac{5}{2} kT}$

Total energy of one mole of gas $= \frac{5}{2} kT \times N_A = \frac{5}{2} RT$

(Here, the total energy is purely kinetic)

For one mole Specific heat at constant volume

$$C_P = \frac{dU}{dT} = \left[\frac{5}{2}RT\right] = \frac{5}{2}R$$

$$But\ C_P = C_V + R = \frac{5}{2}R + R = \frac{7}{2}R$$

$$\therefore \gamma = \frac{C_P}{C_V} = \frac{\frac{7}{2}R}{\frac{5}{2}R} = {}^{7}/_{5} = 1.40$$

Energy of a diatomic molecule at high temperature is equal to $\frac{7}{2}RT$

$$C_V = \frac{dU}{dT} = \left[\frac{7}{2}RT\right] = \frac{7}{2}R$$

$$\therefore C_P = C_V + R = {}^{7}/_{2}R + R$$

$$C_P = {}^{9}/_{2}R$$

Note that the C_V and C_P are higher for diatomic molecules than the mono atomic molecules. It implies that to increase the temperature of diatomic gas molecules by 1°C it require more heat energy than monoatomic molecules.

$$\therefore \gamma = \frac{C_P}{C_V} = \frac{{}^{9}/_{2}R}{{}^{7}/_{2}R}{}^{9}/_{7} = 1.28$$

iii) Triatomic molecule

a) Linear molecule

Energy of the one mole $= \frac{7}{2}kT \times N_A = \frac{7}{2}RT$

$$C_P = \frac{dU}{dT} = \frac{d}{dT}\left[\frac{7}{2}RT\right]$$

$$C_V = \frac{7}{2}R$$

$$C_P = C_V + R = \frac{7}{2}R + R = \frac{9R}{2}$$

$$\therefore \gamma = \frac{C_P}{C_V} = \frac{\frac{9}{2}R}{\frac{7}{2}R} = \frac{9}{7} = 1.28$$

b) Non-linear molecule

Energy of a mole = $\frac{6}{2}kT \times N_A = \frac{6}{2}RT = 3RT$

$$C_V = \frac{dU}{dT} = 3R$$

$$C_P = C_V + R = 3R + R = 4R$$

$$\therefore \gamma = \frac{C_P}{C_V} = \frac{4R}{3R} = \frac{4}{3} = 1.33$$

Note that according to kinetic theory model of gases the specific heat capacity at constant volume and constant pressure are independent of temperature. But in reality it is not sure. The specific heat capacity varies with the temperature.

Find the adiabatic exponent γ for mixture of μ1 moles of monoatomic gas and μ2 moles of a diatomic gas at normal temperature (27°C).

Solution

The specific heat of one mole of a monoatomic gas $C_V = \frac{3}{2}R$

$$for\ \mu_1\ mole\ C_V \frac{3}{2}\mu_1 R\ \ C_P = \frac{5}{2}\mu_1 R$$

The specific heat of one mole of a diatomic gas

$$C_V = \frac{5}{2}R$$

$$for\ \mu_2\ mole\ C_V\ \frac{5}{2}\ \mu_2\ R\ \ C_P = \frac{7}{2}\mu_2\ R$$

The specific heat of the mixture at constant volume $C_V = \frac{3}{2}\mu_1 R + \frac{5}{2}\mu_2 R$

The specific heat of the mixture at constant pressure$C_P = \frac{5}{2}\mu_1 R + \frac{7}{2}\mu_2 R$

The adiabatic exponent $\gamma = \frac{C_P}{C_P} = \frac{5\mu_1+7\mu_2}{3\mu_1+5\mu_2}$

MEAN FREE PATH

Usually the average speed of gas molecules is several hundred meters per second even at room temperature (27°C). Odour from an open perfume bottle takes some time to reach us even if we are closer to the room. The time delay is because the odour of the molecules cannot travel straight to us as it undergoes a lot of collisions with the nearby air molecules and moves in a zigzag path. This average distance travelled by the molecule between two successive collisions is called mean free path (λ). We can calculate the mean free path based on kinetic theory.

Expression for mean free path

We know from postulates of kinetic theory that the molecules of a gas are in random motion and they collide with each other. Between two successive collisions, a molecule moves along a straight path with uniform velocity. This path is called mean free path.

Consider a system of molecules each with diameter d. Let n be the number of molecules per unit volume.
Assume that only one molecule is in motion and all others are at rest.

If a molecule moves with average speed v in atime t, the distance travelled is vt. In this time t, consider the molecule to move in an imaginary cylinder of volume πd2vt. It collides with any molecule whose center is within this cylinder. Therefore, the number of collisions is equal to the number of molecules in the volume of the imaginary cylinder. It is equal to $\boldsymbol{\pi d2\ vtn}$. The total path length divided by the number of collisions in time t is the mean free path.

$$\boldsymbol{mean\ free\ path, \lambda = \frac{distance\ travelled}{number\ of\ collisions}}$$

$$\boldsymbol{\lambda = \frac{vt}{n\pi d^2 vt} = \frac{1}{n\pi d^2}}$$

Though we have assumed that only one molecule is moving at a time and other molecules are at rest, in actual practice all the molecules are in random motion. So the average relative speed of one molecule with respect to other molecules has to be taken into account. After some detailed calculations (you will learn in higher classes) the correct expression for mean free path

$$\therefore \boldsymbol{\lambda = \frac{1}{\sqrt{2} n\pi d^2}}$$

The equation implies that the mean free path is inversely proportional to number density. When the number density increases the molecular collisions increases so it decreases the distance travelled by the molecule before collisions.

Case1: Rearranging the equation using ‘m’ (mass of the molecule)

$$\therefore \lambda = \frac{m}{\sqrt{2}\pi d^2 mn}$$

But mn=mass per unit volume = ρ(density of the gas)

$$\therefore \lambda = \frac{m}{\sqrt{2}\pi d^2 \rho}$$

Also we know that PV = NkT

$$p = \frac{N}{V}kT = nkT$$

$$\therefore n = \frac{p}{kT}$$

Substituting $n = \frac{P}{kT}$

$$\lambda = \frac{kT}{\sqrt{2}\pi d^2 P}$$

1. Mean free path increases with increasing temperature. As the temperature increases, the average speed of each molecule will increase. It is the reason why the smell of hot sizzling food reaches several meter away than smell of cold food.

2. Mean free path increases with decreasing pressure of the gas and diameter of the gas molecules.

An oxygen molecule is travelling in air at 300 K and 1 atm, and the diameter of oxygen molecule is $1.2 \times 10^{-10} m$. Calculate the mean free path of oxygen molecule.

Solution

$$\lambda = \frac{1}{\sqrt{2}\pi n d^2}$$

We have to find the number density n By using ideal gas law

$$n = \frac{N}{V} = \frac{P}{kT} = \frac{101.3 \times 10^3}{1.381 \times 10^{-23} \times 300}$$

$$= 2.449 \times 10^{25}\ molecues/m^3$$

$$\lambda = \frac{1}{\sqrt{2} \times \pi \times 2.449 \times 10^{25} \times (1.2 \times 10^{-10})^2}$$

$$= \frac{1}{15.65 \times 10^5}$$

$$\lambda = 0.63 \times 10^{-6} m$$

BROWNIAN MOTION

In 1827, Robert Brown, a botanist reported that grains of pollen suspended in a liquid moves randomly from one place to other. The random (Zig - Zag path) motion of pollen suspended in a liquid is called Brownian motion. In fact we can observe the dust particle in water moving in random directions. This discovery puzzled scientists for long time. There were a lot of explanations for pollen or dust to move in random directions but none of these explanations were found adequate. After a systematic study, Wiener and Gouy proposed that Brownian motion is due to the bombardment of suspended particles by molecules of the surrounding fluid. But during 19th century people did not accept that every matter is made up of small atoms or molecules. In the year 1905, Einstein gave systematic theory of Brownian motion based on kinetic theory and he deduced the average size of molecules.

According to kinetic theory, any particle suspended in a liquid or gas is continuously bombarded from all the directions so that the mean free path is almost negligible. This leads to the motion of the particles in a random and zig–zag manner as shown in Figure 9.9. But when we put our hand in water it causes no random motion because the mass of our hand is so large that the momentum transferred by the molecular collision is not enough to move our hand.

Factors affecting Brownian Motion

Brownian motion increases with increasing temperature.

1. Brownian motion decreases with bigger particle size, high viscosity and density of the liquid (or) gas.

UNIT - 10 OSCILLATIONS

INTRODUCTION

Have you seen the Thanjavur Dancing Doll (In Tamil, it is called ‘Thanjavur thalayattibommai’)?. It is a world famous Indiancultural doll (Figure 10.1). What does this doll do when disturbed? It will dance such that the head and body move continuously in a to and fro motion, until the movement gradually stops. Similarly, when we walk on the road, our hands and legs will move front and back. Again similarly, when a mother swings a cradle to make her child sleep, the cradle is made to move in to and fro motion. All these motions are diff erent from the motion that we have discussed so far. Th ese motions are shown in Figure 10.2. Generally, they are known as oscillatory motion or vibratory motion. A similar motion occurs even at atomic levels. When the temperature is raised, the atoms in a solid vibrate about their rest position (mean position or equilibrium position). Th e study of vibrational motion is very important in engineering applications, such as, designing the structure of building, mechanical equipments, etc.

Periodic and nonperiodic motion

Motion in physics can be classified as repetitive (periodic motion) and non-repetitive (non-periodic motion).

Periodic motion

Any motion which repeats itself in a fixed time interval is known as periodic motion.

Examples : Hands in pendulum clock, swing of a cradle, the revolution of the Earth around the Sun, waxing and waning of Moon, etc.

Non-Periodic motion

Any motion which does not repeat itself after a regular interval of time is known as non-periodic motion.

Example : Occurance of Earth quake, eruption of volcano, etc.

E X A M P L E

Classify the following motions as periodic and non-periodic motions?.

a. Motion of Halley's comet.
b. Motion of clouds.
c. Moon revolving around the Earth

Solution

a. Periodic motion
b. Non-periodic motion
c. Periodic motion

EXAMPLE

Which of the following functions of time represent periodic and non-periodic motion?.

a. $\sin \omega t + \cos \omega t$
b. $\ln \omega t$

Solution

a. Periodic
b. Non-periodic

Oscillatory motion

When an object or a particle moves back and forth repeatedly for some duration of time its motion is said to be oscillatory (or vibratory). Examples; our heart beat, swinging motion of the wings of an insect, grandfather's clock (pendulum clock), etc. Note that all oscillatory motion are periodic whereas all periodic motions need not be oscillation in nature. see Figure 10.3

SIMPLE HARMONIC MOTION (SHM)

Simple harmonic motion is a special type of oscillatory motion in which the acceleration or force on the particle is directly proportional to its displacement from a fixed point and is always directed towards that fixed point. In one dimensional case, let x be the displacement of the particle and a_x be the acceleration of the particle, then

$$a_x \propto x$$

$$a_x = -\,b\,x$$

where b is a constant which measures acceleration per unit displacement and dimensionally it is equal to T^{-2}. By multiplying by mass of the particle on both sides of equation (10.2) and from Newton's second law, the force is

$$F_x = -\,k\,x$$

where k is a force constant which is defined as force per unit length. The negative sign indicates that displacement and force (or acceleration) are in opposite directions. This means that when the displacement of the particle is taken towards right of equilibrium position (x takes positive value), the force (or acceleration) will point towards equilibrium (towards left) and similarly, when the displacement of the particle is taken towards left of equilibrium position (x takes negative value), the force (or acceleration) will point towards equilibrium (towards right). This type of force is known as restoring force because it always directs the particle executing simple harmonic motion to restore to its original (equilibrium or mean) position. This force (restoring force) is central and attractive whose center of attraction is the equilibrium position.

In order to represent in two or three dimensions, we can write using vector notation

$$\vec{F} = -k\vec{r}$$

where $\vec{r}$ the displacement of the particle from the chosen origin. Note that the force and displacement have a linear relationship. This means that the exponent of force $\vec{F}$ and the exponent of displacement $\vec{r}$ are unity. The sketch between cause (magnitude of force $|\vec{F}|$) and effect (magnitude of displacement $|\vec{r}|$) is a straight line passing through second and fourth quadrant as shown in. By measuring slope $\frac{1}{k}$ one can find the numerical value of force constant k.

The projection of uniform circular motion on a diameter of SHM

Consider a particle of mass m moving with uniform speed v along the circumference of a circle whose radius is r in anti-clockwise direction (as shown in Figure 10.6). Let us assume that the origin of the coordinate system coincides with the center O of the circle. If ω is the angular velocity of the particle and θ the angular displacement of the particle at any instant of time t, then $\theta = \omega t$. By projecting the uniform circular motion on its diameter gives a simple harmonic motion. This means that we can associate a map (or a relationship) between uniform circular (or revolution) motion to vibratory motion. Conversely, any vibratory motion or revolution can be mapped to uniform circular motion. In other words, these two motions are similar in nature.

Let us first project the position of a particle moving on a circle, on to its vertical diameter or on to a line parallel to vertical diameter as shown in Figure 10.7. Similarly, we can do it for horizontal axis or a line parallel to horizontal axis.

As a specific example, consider a spring mass system (or oscillation of pendulum) as shown in Figure 10.8. When the spring moves up and down (or pendulum moves to and fro), the motion of the mass or bob is mapped to points on the circular motion.

Thus, if a particle undergoes uniform circular motion then the projection of the particle on the diameter of the circle (or on a line parallel to the diameter) traces straight line motion which is simple harmonic in nature. The circle is known as reference circle of the simple harmonic motion. The simple harmonic motion can also be defined as the motion of the projection of a particle on any diameter of a circle of reference.

Displacement, velocity, acceleration and its graphical representation – SHM

The distance travelled by the vibrating particle at any instant of time t from its mean position is known as displacement. Let P be the position of the particle on a circle of radius A at some instant of time t as shown in Figure 10.9. Then its displacement y at that instant of time t can be derived as follows In ΔOPN

$$\sin\theta = \frac{ON}{OP} \Rightarrow ON = OP\sin\theta$$

But $\theta = \omega t$, ON $= y$ and OP $= A$

$$y = A\sin\omega t$$

The displacement y takes maximum value (which is equal to A) when $\sin\omega t=1$.This maximum displacement from the mean positionis known as amplitude (A) of the vibrating particle. For simple harmonic motion, the amplitude is constant. But, in general, for any motion other than simple harmonic, the amplitude need not be constant, it may vary with time.

Velocity

The rate of change of displacement is velocity. Taking derivative of equation (10.6) with respect to time, we get

$$v = \frac{dy}{dt} = \frac{d}{dt}(\text{A} \sin \omega t)$$

For circular motion (of constant radius), amplitude A is a constant and further, for uniform circular motion, angular velocity ω is a constant. Therefore,

$$v = \frac{dy}{dt} = \text{A}\ \omega \cos \omega t$$

Using trigonometry identity,

$$\sin^2 \omega t + \cos^2 \omega t = 1 \Rightarrow \cos \omega t = \sqrt{1 - \sin^2 \omega t}$$

we get

$$v = A\omega\sqrt{1 - \sin^2 \omega t}$$

From equation (10.6),

$$\sin \omega t = \frac{y}{A}$$

$$v = A\omega\sqrt{1 - \left(\frac{y}{A}\right)^2}$$

$$v = \omega\sqrt{A^2 - y^2}$$

From equation (10.8), when the displacement y = 0, the velocity v = ωA (maximum) and for the maximum displacement y = A, the velocity v = 0 (minimum).

As displacement increases from zero to maximum, the velocity decreases from maximum to zero. This is repeated.

Since velocity is a vector quantity, equation (10.7) can also be deduced by resolving in to components.

Acceleration

The rate of change of velocity is acceleration.

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(A\omega\cos\omega t\right)$$

$$a = -\omega^2 A\sin\omega t = -\omega^2 y$$

$$\therefore \quad a = \frac{d^2 y}{dt^2} = -\omega^2 y$$

From the Table 10.1 and figure 10.10, we observe that at the mean position

Table 10.1 Displacement, velocity and acceleration at different instant of time.

Time	0	$\frac{T}{4}$	$\frac{2T}{4}$	$\frac{3T}{4}$	$\frac{4T}{4} = T$
ωt	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Displacement $y = A\sin\omega t$	0	A	0	$-A$	0
Velocity $v = A\omega\cos\omega t$	$A\omega$	0	$-A\omega$	0	$A\omega$
Acceleration $a = -A\omega^2\sin\omega t$	0	$-A\omega^2$	0	$A\omega^2$	0

($y = 0$), velocity of the particle is maximum but the acceleration of the particle is zero. At the extreme position ($y = \pm A$), the velocity of the particle is zero but the acceleration is maximum $\pm A\omega^2$ acting in the opposite direction.

E X A M P L E

Which of the following represent simple harmonic motion?

a. $x = A\sin\omega t + B\cos\omega t$
b. $x = A\sin\omega t + B\cos 2\omega t$
c. $x = A e^{i\omega t}$
d. $x = A\ln\omega t$

Solution

a. $x = A\sin\omega t + B\cos\omega t$

$$\frac{dx}{dt} = A\,\omega \cos \omega t - B\,\omega \sin \omega t$$

$$\frac{d^2x}{dt^2} = -\,\omega^2\,(A \sin \omega t + B \cos \omega t)$$

$$\frac{d^2x}{dt^2} = -\,\omega^2 x$$

This differential equation is similar to the differential equation of SHM (equation 10.10). Therefore, x = A sin ωt + B cos ωt represents SHM.

b. x =A sin ωt + B cos2ωt

$$\frac{dx}{dt} = A\,\omega \cos \omega t - B\,(2\omega) \sin 2\omega t$$

$$\frac{d^2x}{dt^2} = -\,\omega^2\,(A \sin \omega t + 4B \cos 2\omega t)$$

$$\frac{d^2x}{dt^2} \neq -\,\omega^2 x$$

This differential equation is not like the differential equation of a SHM (equation 10.10). Therefore, x = A sin ωt + B cos 2ωt does not represent SHM.

c. x = A $e^{i\omega t}$

$$\frac{dx}{dt} = A\,\omega e^{i\omega t}$$

$$\frac{d^2x}{dt^2} = -A\,\omega^2\, e^{i\omega t} = -\omega^2 x$$

This differential equation is like the differential equation of SHM (equation 10.10). Therefore, x = A $e^{i\omega t}$ represents SHM.

d. x = A ln ωt

$$\frac{dx}{dt} = \left(\frac{A}{\omega t}\right)\omega = \frac{A}{t}$$

$$\frac{d^2x}{dt^2} = -\frac{A}{t^2} \Rightarrow \frac{d^2x}{dt^2} \neq -\omega^2 x$$

This differential equation is not like the differential equation of a SHM (equation 10.10). Therefore, $x = A \ln \omega t$ does not represent SHM.

E X A M P L E

Consider a particle undergoing simple harmonic motion. The velocity of the particle at position x_1 is v_1 and velocity of the particle at position x_2 is v_2. Show that the ratio of time period and amplitude is

$$\frac{T}{A} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 x_2^2 - v_2^2 x_1^2}}$$

Solution

$$v = \omega\sqrt{A^2 - x^2} \Rightarrow v^2 = \omega^2 (A^2 - x^2)$$

Therefore, at position x_1,

$$v_1^2 = \omega^2(A^2 - x_1^2)$$

Similarly, at position x_2,

$$v_2^2 = \omega^2(A^2 - x_2^2)$$

$$v_1^2 - v_2^2 = \omega^2(A^2 - x_1^2) - \omega^2(A^2 - x_2^2)$$

$$= \omega^2(x_2^2 - x_1^2)$$

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} \Rightarrow T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

$$\frac{v_1^2}{v_2^2} = \frac{\omega^2\left(A^2 - x_1^2\right)}{\omega^2\left(A^2 - x_2^2\right)} \Rightarrow A = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$$

$$\frac{T}{A} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 x_2^2 - v_2^2 x_1^2}}$$

Time period, frequency, phase, phase difference and epoch in SHM.
Time period

The time period is defined as the time taken by a particle to complete one oscillation. It is usually denoted by T. For one complete revolution, the time taken is t = T, therefore

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$$

Then, the displacement of a particle executing simple harmonic motion can be written either as sine function or cosine function.

$$y(t) = A \sin \frac{2\pi}{T} t \quad \text{or} \quad y(t) = A \cos \frac{2\pi}{T} t$$

where T represents the time period. Suppose the time t is replaced by t + T, then the function

$$\begin{aligned} y(t+T) &= A \sin \frac{2\pi}{T}(t+T) \\ &= A \sin\left(\frac{2\pi}{T} t + 2\pi\right) \\ &= A \sin \frac{2\pi}{T} t = y(t) \\ y(t+T) &= y(t) \end{aligned}$$

Thus, the function repeats after one time period. This y(t) is an example of periodic function.

Frequency and angular frequency

The number of oscillations produced by the particle per second is called frequency. It is denoted by f. SI unit for frequency is s^{-1} or hertz (In symbol, Hz). Mathematically, frequency is related to time period by

$$f = \frac{1}{T}$$

The number of cycles (or revolutions) per second is called angular frequency. It is usually denoted by the Greek small letter 'omega', ω. Comparing equation (10.11) and equation (10.12), angular frequency and frequency are related by

$$\omega = 2\pi f$$

SI unit for angular frequency is rad s^{-1}. (read it as radian per second)

Phase

The phase of a vibrating particle at any instant completely specifies the state of the particle. It expresses the position and direction of motion of the particle at that instant with respect to its mean position (Figure 10.11).

$$y = A \sin(\omega t + \varphi_0)$$

where $\omega t + \varphi_0 = \varphi$ is called the phase of the vibrating particle. At time t = 0 s (initial time), the phase $\varphi = \varphi_0$ is called epoch (initial phase) where φ_0 is called the angle of epoch. Phase difference: Consider two particles executing simple harmonic motions. Their equations are $y_1 = A \sin(\omega t + \varphi_1)$ and $y_2 = A \sin(\omega t + \varphi_2)$, then the phase difference $\Delta\varphi = (\omega t + \varphi_2) - (\omega t + \varphi_1) = \varphi_2 - \varphi_1$.

E X A M P L E

A nurse measured the average heart beats of a patient and reported to the doctor in terms of time period as 0.8 s. Express the heart beat of the patient in terms of number of beats measured per minute.

Solution

Let the number of heart beats measured be f. Since the time period is inversely proportional to the heart beat, then

$$f = \frac{1}{T} = \frac{1}{0.8} = 1.25 s^{-1}$$

One minute is 60 second,

$$(1 \text{ second} = \frac{1}{60} \text{ minute} \Rightarrow 1 \text{ s}^{-1} = 60 \text{ min}^{-1})$$

$$f = 1.25 \text{ s}^{-1} \Rightarrow f = 1.25 \times 60 \text{ min}^{-1} = 75 \text{ beats per minute}$$

E X A M P L E

Calculate the amplitude, angular frequency, frequency, time period and initial phase for the simple harmonic oscillation given below

a. $y = 0.3 \sin(40\pi t + 1.1)$
b. $y = 2 \cos(\pi t)$
c. $y = 3 \sin(2\pi t - 1.5)$

Solution

Simple harmonic oscillation equation is $y = A \sin(\omega t + \varphi_0)$ or $y = A \cos(\omega t + \varphi_0)$

a. For the wave, $y = 0.3 \sin(40p\,t + 1.1)$
Amplitude is A = 0.3 unit
Angular frequency $\omega = 40p$ rad s^{-1}
Frequency $f = \frac{w}{2p} = \frac{40p}{2p} = 20\,Hz$
Time period $T = \frac{1}{f} = \frac{1}{20} = 0.05s$
Initial phase is $\varphi_0 = 1.1$ rad

b. For the wave, $y = 2 \cos(p\,t)$
Amplitude is A = 2 unit
Angular frequency $\omega = p$ rad s^{-1}
Frequency $f = \frac{w}{2p} = \frac{p}{2p} = 0.5\,Hz$
Time period $T = \frac{1}{f} = \frac{1}{0.5} = 2s$
Initial phase is $\varphi_0 = 0$ rad

c. For the wave, $y = 3 \sin(2p\,t + 1.5)$
Amplitude is A = 3 unit
Angular frequency $\omega = 2p$ rad s^{-1}
Frequency $f = \frac{w}{2p} = \frac{2p}{2p} = 1Hz$
Time period $T = \frac{1}{f} = \frac{1}{1} = 1s$
Initial phase is $\varphi_0 = 1.5$ rad

E X A M P L E

Show that for a simple harmonic motion, the phase difference between

a. displacement and velocity is $\frac{\pi}{2}$ radian or 90°.

b. velocity and acceleration is $\frac{\pi}{2}$ radian or 90

c. displacement and acceleration is π radian or 180°.

Solution

a. The displacement of the particle executing simple harmonic motion y=A sinωt

Velocity of the particle is

$$v = A\omega\cos\omega t = A\omega\sin\left(\omega t + \frac{\pi}{2}\right)$$

The phase difference between displacement and velocity is $\frac{\pi}{2}$

b. The velocity of the particle is v = A ω cos ωt

Acceleration of the particle is

$$a = -A\omega^2\sin\omega t = A\omega^2\cos\left(\omega t + \frac{\pi}{2}\right)$$

The phase difference between velocity and acceleration is $\frac{\pi}{2}$

c. The displacement of the particle is y = A sinωt

Acceleration of the particle is

$$a = -A\omega^2\sin\omega t = A\omega^2\sin(\omega t + \pi)$$

The phase difference between displacement and acceleration is π.

ANGULAR SIMPLE HARMONIC MOTION

Time period and frequency of angular SHM

When a body is allowed to rotate freely about a given axis then the oscillation is known as the angular oscillation. The point at which the resultant torque acting on the body is taken to be zero is called mean position. If the body is displaced from the mean position, then the resultant torque acts such that it is proportional to the angular displacement and this torque has a tendency to bring the body towards the mean position. (Note: Torque is explained in unit 5)

Let $\vec{\theta}$ be the angular displacement of the body and the resultant torque $\vec{\tau}$ acting on the body is

$$\vec{\tau} \propto \vec{\theta}$$

$$\vec{\tau} = -\kappa\vec{\theta}$$

K is the restoring torsion constant, which is torque per unit angular displacement. If I is the moment of inertia of the body and $\vec{\alpha}$ is the angular acceleration then

$$\vec{\tau} = I\vec{\alpha} = -\kappa\vec{\theta}$$

But $\vec{\alpha} = \frac{d^2\vec{\theta}}{dt^2}$ and therefore,

$$\frac{d^2\vec{\theta}}{dt^2} = -\frac{\kappa}{I}\vec{\theta}$$

This differential equation resembles simple harmonic differential equation.

So, comparing equation (10.17) with simple harmonic motion given in equation (10.10), we have

$$\omega = \sqrt{\frac{\kappa}{I}}\, rad\, s^{-1}$$

The frequency of the angular harmonic motion (from equation 10.13) is

$$f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}\, Hz$$

The time period (from equation 10.12) is

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \text{ second}$$

Comparison of Simple Harmonic Motion and Angular Simple Harmonic Motion

In linear simple harmonic motion, the displacement of the particle is measured in terms of linear displacement $\vec{r}$ The restoring force is $\vec{F} = -k\vec{r}$, where k is a spring constant or force constant which is force per unit displacement. In this case, the inertia factor is mass of the body executing simple harmonic motion.

In angular simple harmonic motion, the displacement of the particle is measured in terms of angular displacement $\vec{q}$. Here, the spring factor stands for torque constant i.e., the moment of the couple to produce unit angular displacement or the restoring torque per unit angular displacement. In this case, the inertia factor stands for moment of inertia of the body executing angular simple harmonic oscillation.

Table 10.2 Comparision of simple harmonic motion and angular harmonic motion

S.No	Simple Harmonic Motion	Angular Harmonic Motion
1.	The displacement of the particle is measured in terms of linear displacement $\vec{r}$.	The displacement of the particle is measured in terms of angular displacement $\vec{\theta}$ (also known as angle of twist).
2.	Acceleration of the particle is $\vec{a}=-\omega^2\vec{r}$	Angular acceleration of the particle is $\vec{\alpha}=-\omega^2\vec{\theta}$.
3.	Force, $\vec{F}=m\vec{a}$, where m is called mass of the particle.	Torque, $\vec{\tau}=I\vec{\alpha}$, where I is called moment of inertia of a body.
4.	The restoring force $\vec{F}=-k\vec{r}$, where k is restoring force constant.	The restoring torque $\vec{\tau}=-\kappa\vec{\theta}$, where the symbol κ (Greek alphabet is pronounced as '*kappa*') is called restoring torsion constant. It depends on the property of a particular torsion fiber.
5.	Angular frequency, $\omega=\sqrt{\frac{k}{m}}$ rad s^{-1}	Angular frequency, $\omega=\sqrt{\frac{\kappa}{I}}$ rad s^{-1}

LINEAR SIMPLE HARMONIC OSCILLATOR (LHO)
Horizontal oscillationsof a spring-mass system

Consider a system containing a block of mass m attached to a massless spring with stiffness constant or force constant or spring constant k placed on a smooth horizontal surface (frictionless surface) as shown in Figure 10.13. Let x_0 be the equilibrium position or mean position of mass m when it is left undisturbed. Suppose the mass is displaced through a small displacement x towards right from its equilibrium position and then released, it will oscillate back and forth about its mean position x_0. Let F be the restoring force (due to stretching of the spring) which is proportional to the amount of displacement of block. For one dimensional motion, mathematically, we have

$$F \propto x$$
$$F = -k x$$

where negative sign implies that the restoring force will always act opposite to the direction of the displacement. This equation is called Hooke's law (refer to unit 7). Notice that, the restoring force is linear with the displacement (i.e., the exponent of force and displacement are unity). This is not always true; in case if we apply a very large stretching force, then the amplitude of oscillations becomes very large (which means, force is proportional to displacement containing higher powers of x) and therefore, the oscillation of the system is not linear and hence, it is called non-linear oscillation. We restrict ourselves only to linear oscillations throughout our discussions, which means Hooke's law is valid (force and displacement have a linear relationship).

From Newton's second law, we can write the equation for the particle executing simple harmonic motion

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Comparing the equation (10.21) with simple harmonic motion equation (10.10), we get

$$\omega^2 = \frac{k}{m}$$

which means the angular frequency or natural frequency of the oscillator is

$$\omega = \sqrt{\frac{k}{m}} \text{ rad } s^{-1}$$

The frequency of the oscillation is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hertz}$$

and the time period of the oscillation is

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \text{ seconds}$$

Notice that in simple harmonic motion, the time period of oscillation is independent of amplitude. This is valid only if the amplitude of oscillation is small. The solution of the differential equation of a SHM may be written as

$$x(t) = A\sin(\omega t + \varphi)$$

or

$$x(t) = A\cos(\omega t + \varphi)$$

where A, ω and ϕ are constants. General solution for differential equation 10.21 is $x(t) = A\sin(\omega t + \varphi) + B\cos(\omega t + \varphi)$ where A and B are contants.

Vertical oscillations of a spring

Let us consider a massless spring with stiff ness constant or force constant k attached to a ceiling as shown in Figure 10.15. Let the length of the spring before loading mass m be L. If the block of mass m is attached to the other end of spring, then the spring elongates by a length l. Let F1 be the restoring force due to stretching of spring. Due to mass m, the gravitational force acts vertically downward. We can draw free-body diagram for this system as shown in Figure 10.15. When the system is under equilibrium,

$$F_1 + mg = 0$$

But the spring elongates by small displacement l, therefore

$$F_1 \propto l \Rightarrow F_1 = -k\,l$$

Substituting equation (10.28) in equation (10.27), we get

$$-k\,l + mg = 0$$

$$mg = kl$$

or

$$\frac{m}{k} = \frac{l}{g}$$

Suppose we apply a very small external force on the mass such that the mass further displaces downward by a displacement y, then it will oscillate up and down. Now, the restoring force due to this stretching of spring (total extension of spring is y + l) is

$$F_2 \propto (y + l)$$
$$F_2 = -k(y + l) = -ky - kl$$

Since, the mass moves up and down with acceleration $\frac{d^2y}{dt^2}$ by drawing the free body diagram for this case, we get

$$-ky - kl + mg = m\frac{d^2y}{dt^2}$$

The net force acting on the mass due to this stretching is

$$F = F_2 + mg$$
$$F = -ky - kl + mg$$

The gravitational force opposes the restoring force. Substituting equation (10.29) in equation (10.32), we get

$$F = -ky - kl + kl = -ky$$

Applying Newton's law, we get

$$m\frac{d^2y}{dt^2} = -ky$$
$$\frac{d^2y}{dt^2} = -\frac{k}{m}y$$

The above equation is in the form of simple harmonic differential equation. Therefore, we get the time period as

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ second}$$

The time period can be rewritten using equation (10.29)

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}} \text{ second}$$

The acceleration due to gravity g can be computed from the formula

$$g = 4\pi^2\left(\frac{l}{T^2}\right) \text{ m s}^{-2}$$

E X A M P L E

A spring balance has a scale which ranges from 0 to 25 kg and the length of the scale is 0.25m. It is taken to an unknown planet X where the acceleration due to gravity is 11.5 m s^{-1}. Suppose a body of mass M kg is suspended in this spring and made to oscillate with a period of 0.50 s. Compute the gravitational force acting on the body.

Solution

Let us first calculate the stiff ness constant of the spring balance by using equation (10.29),

$$k = \frac{mg}{l} = \frac{25 \times 11.5}{0.25} = 1150\ N m^{-1}$$

The time period of oscillations is given by $T = 2p\sqrt{\frac{M}{k}}$ where M is the mass of the body. Since, M is unknown, rearranging, we get

$$M = \frac{kT^2}{4\pi^2} = \frac{(1150)(0.5)^2}{4\pi^2} = 7.3kg$$

The gravitational force acting on the body is W = Mg = 7.3 × 11.5 = 83.95 N ≈ 84 N

Combinations of springs

Spring constant or force constant, also called as stiffness constant, is a measure of the stiffness of the spring. Larger the value of the spring constant, stiffer is the spring. This implies that we need to apply more force to compress or elongate the spring. Similarly, smaller the value of spring constant, the spring can be stretched (elongated) or compressed with lesser force. Springs can be connected in two ways. Either the springs can be connected end to end, also known as series connection, or alternatively, connected in parallel. In the following subsection, we compute the effective spring constant when

a. Springs are connected in series
b. Springs are connected in parallel

Springs connected in series

When two or more springs are connected in series, we can replace (by removing) all the springs in series with an equivalent spring (effective spring) whose net effect is the same as if all the springs are in series connection. Given the value of individual spring constants k_1, k_2, k_3,... (known quantity), we can establish a mathematical relationship to find out an effective (or equivalent) spring constant k_s (unknown quantity). For simplicity, let us consider only two springs whose spring constant are k_1 and k_2 and which can be attached to a mass m as shown in Figure 10.17. The results thus obtained can be generalized for any number of springs in series.

Let F be the applied force towards right as shown in Figure 10.18. Since the spring constants for different spring are different and the connection points between them is not rigidly fixed, the strings can stretch in different lengths. Let x1 and x2 be the elongation of springs from their equilibrium position (un-stretched position) due to the applied force F. Then, the net displacement of the mass point is

$$x = x_1 + x_2$$

$$F = -k_s(x_1 + x_2) \Rightarrow x_1 + x_2 = -\frac{F}{k_s} \quad (10.38)$$

For springs in series connection

$$-k_1 x_1 = -k_2 x_2 = F$$

$$\Rightarrow x_1 = -\frac{F}{k_1} \text{ and } x_2 = -\frac{F}{k_2}$$

Therefore, substituting equation (10.39) in equation (10.38), the effective spring constant can be calculated as

$$-\frac{F}{k_1}-\frac{F}{k_2}=-\frac{F}{k_s}$$

$$\frac{1}{k_s}=\frac{1}{k_1}+\frac{1}{k_2}$$

Or

$$k_s=\frac{k_1k_2}{k_1+k_2}\ \text{Nm}^{-1}$$

Suppose we have n springs connected in series, the effective spring constant in series is

$$\frac{1}{k_s}=\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\ldots+\frac{1}{k_n}=\sum_{i=1}^{n}\frac{1}{k_i}$$

$$\frac{1}{k_s}=\frac{n}{k}\Rightarrow k_s=\frac{k}{n}$$

This means that the effective spring constant reduces by the factor n. Hence, for springs in series connection, the effective spring constant is lesser than the individual spring constants.

From equation (10.39), we have,

$$k_1x_1 = k_2x_2$$

Then the ratio of compressed distance or elongated distance x_1 and x_2 is

$$\frac{x_2}{x_1}=\frac{k_1}{k_2}$$

The elastic potential energy stored in first and second springs are $V_1=\frac{1}{2}k_1x_1^2$ and $V_2=\frac{1}{2}k_2x_2^2$ respectively. Then, their ratio is

$$\frac{V_1}{V_2}=\frac{\frac{1}{2}k_1x_1^2}{\frac{1}{2}k_2x_2^2}=\frac{k_1}{k_2}\left(\frac{x_1}{x_2}\right)^2=\frac{k_2}{k_1}$$

E X A M P L E

Consider two springs whose force constants are 1 N m^{-1} and 2 N m^{-1} which are connected in series. Calculate the effective spring constant (k_s) and comment on k_s.

Solution

$$k_1 = 1\,\text{N m}^{-1},\ k_2 = 2\,\text{N m}^{-1}$$
$$k_s = \frac{k_1 k_2}{k_1 + k_2}\ \text{N m}^{-1}$$
$$k_s = \frac{1\times 2}{1+2} = \frac{2}{3}\ \text{N m}^{-1}$$
$$k_s < k_1 \text{ and } k_s < k_2$$

Therefore, the effective spring constant is lesser than both k_1 and k_2.

Springs connected in parallel

When two or more springs are connected in parallel, we can replace (by removing) all these springs with an equivalent spring (effective spring) whose net effect is same as if all the springs are in parallel connection. Given the values of individual spring constants to be k_1,k_2,k_3, ... (known quantities), we can establish a mathematical relationship to find out an effective (or equivalent) spring constant k_p (unknown quantity). For simplicity, let us consider only two springs of spring constants $k_1$and k_2 attached to a mass m as shown in Figure 10.19. The results can be generalized to any number of springs in parallel

Let the force F be applied towards right as shown in Figure 10.20. In this case, both the springs elongate or compress by the same amount of displacement. Therefore, net force for the displacement of mass m is

$$F = -k_p x$$

where k_p is called effective spring constant. Let the first spring be elongated by a displacement x due to force F_1 and second spring be elongated by the same displacement x due to force F_2, then the net force

$$F = -k_1x - k_2x$$

Equating equations (10.46) and (10.45), we get

$$k_p = k_1 + k_2$$

Generalizing, for n springs connected in parallel,

$$k_p = \sum_{i=1}^{n} k_i$$

If all spring constants are identical i.e., $k_1 = k_2 = ... = k_n = k$ then

$$kp = n\,k$$

This implies that the effective spring constant increases by a factor n. Hence, for the springs in parallel connection, the effective spring constant is greater than individual spring constant.

E X A M P L E

Consider two springs with force constants 1 N m^{-1} and 2 N m^{-1} connected in parallel. Calculate the effective spring constant (k_p) and comment on k_p.

Solution

$$k_1 = 1\text{ N m}^{-1}, k_2 = 2\text{ N m}^{-1}$$
$$k_p = k_1 + k_2\text{ N m}^{-1}$$
$$k_p = 1 + 2 = 3\text{ N m}^{-1}$$
$$k_p > k_1 \text{ and } k_p > k_2$$

Therefore, the effective spring constant is greater than both k_1 and k_2.

E X A M P L E

Calculate the equivalent spring constant for the following systems and also compute if all the spring constants are equal:

Solution

a. Since k_1 and k_2 are parallel, $k_u = k_1 + k_2$ Similarly, k_3 and k_4 are parallel, therefore, $k_d = k_3 + k_4$But k_u and k_d are in series,

therefore, $k_{eq} = \frac{k_u k_d}{k_u + k_d}$ If all the spring constants are equal then, $k_1 = k_2 = k_3 = k_4 = k$

Which means, $k_u = 2k$ and $k_d = 2k$

Hence, $k_{eq} = \frac{4k^2}{4k} = k$

b. Since k_1 and k_2 are parallel, $k_A = k_1 + k_2$ Similarly, k_4 and k_5 are parallel, therefore, $k_B = k_4 + k_5$ But k_A, k_3, k_B, and k_6 are in series,
therefore,

$$\frac{1}{k_{eq}} = \frac{1}{k_A} + \frac{1}{k_3} + \frac{1}{k_B} + \frac{1}{k_6}$$

If all the spring constants are equal then, $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k$ which means, $k_A = 2_k$ and $k_B = 2_k$

$$\frac{1}{k_{eq}} = \frac{1}{2k} + \frac{1}{k} + \frac{1}{2k} + \frac{1}{k} = \frac{3}{k}$$

$$k_{eq} = \frac{k}{3}$$

E X A M P L E

A mass m moves with a speed v on a horizontal smooth surface and collides with a nearly massless spring whose spring constant is k. If the mass stops after collision, compute the maximum compression of the spring.

Solution

When the mass collides with the spring, from the law of conservation of energy “the loss in kinetic energy of mass is gain in elastic potential energy by spring”.

Let x be the distance of compression of spring, then the law of conservation of energy

$$\frac{1}{2}m v^2 = \frac{1}{2} k x^2 \Rightarrow x = v\sqrt{\frac{m}{k}}$$

Oscillations of a simple pendulum in SHM and laws of simple pendulum
Simple pendulum

A pendulum is a mechanical system which exhibits periodic motion. It has a bob with mass m suspended by a long string (assumed to be massless and inextensible string) and the other end is fixed on a stand as shown in Figure 10.21 (a). At equilibrium, the pendulum does not oscillate and hangs vertically downward. Such a position is known as mean position or equilibrium position. When a pendulum is displaced through a small displacement from its equilibrium position and released, the bob of the pendulum executes to and fro motion. Let l be the length of the pendulum which is taken as the distance between the point of suspension and the centre of gravity of the bob. Two forces act on the bob of the pendulum at any displaced position, as shown in the Figure 10.21 (d),

1. The gravitational force acting on the body $(\vec{F} = m\vec{g})$ which acts vertically downwards.
2. The tension in the string $\vec{T}$ which acts along the string to the point of suspension

Resolving the gravitational force into its components:

Normal component:

The component along the string but in opposition to the direction of tension, $F_{as} = mg\cos\theta$.

Tangential component:

The component perpendicular to the string i.e., along tangential direction of arc of swing, $F_{ps} = mg\sin\theta$.
Therefore, The normal component of the force is, along the string,

$$T - W_{as} = m\frac{v^2}{l}$$

Here v is speed of bob

$$T - mg\cos\theta = m\frac{v^2}{l}$$

From the Figure 10.21, we can observe that the tangential component W_{ps} of the gravitational force always points towards the equilibrium position i.e., the direction in which it always points opposite to the direction of displacement of the bob from the mean position. Hence, in this case, the tangential force is nothing but the restoring force. Applying Newton's second law along tangential direction, we have

$$m\frac{d^2s}{dt^2} + F_{ps} = 0 \Rightarrow m\frac{d^2s}{dt^2} = -F_{ps}$$

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad (10.51)$$

where, s is the position of bob which is measured along the arc. Expressing arc length in terms of angular displacement i.e.,

$$s = l\theta$$

then its acceleration,

$$\frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$$

Substituting equation (10.53) in equation (10.51), we get

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

Because of the presence of sin θ in the above differential equation, it is a non-linear differential equation (Here, homogeneous second order). Assume "the small oscillation approximation", $\sin\theta \approx \theta$, the above differential equation becomes linear differential equation.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

This is the well known oscillatory diff erential equation. Therefore, the angular frequency of this oscillator (natural frequency of this system) is

$$\omega^2 = \frac{g}{l}$$

$$\Rightarrow \quad \omega = \sqrt{\frac{g}{l}} \text{ in rad } s^{-1}$$

The frequency of oscillations is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \text{ in Hz}$$

and time period of oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}} \text{ in second}$$

Laws of simple pendulum

The time period of a simple pendulum

a. Depends on the following laws
Law of length

For a given value of acceleration due to gravity, the time period of a simple pendulum is directly proportional to the square root of length of the pendulum.

$$T \propto \sqrt{l}$$

Law of acceleration

For a fi xed length, the time period of a simple pendulum is inversely proportional to square root of acceleration due to gravity.

$$T \propto \frac{1}{\sqrt{g}}$$

b. Independent of the following factors
Mass of the bob

Th e time period of oscillation is independent of mass of the simple pendulum. This is similar to free fall. Therefore, in a pendulum of fixed length, it does not matter whether an elephant swings or an ant swings. Both of them will swing with the same time period.

Amplitude of the oscillations

For a pendulum with small angle approximation (angular displacement is very small), the time period is independent of amplitude of the oscillation.

E X A M P L E

In simple pendulum experiment, we have used small angle approximation . Discuss the small angle approximation.

θ (in degrees)	θ(in radian)	sin θ
0	0	0
5	0.087	0.087
10	0.174	0.174
15	0.262	0.256
20	0.349	0.342
25	0.436	0.422
30	0.524	0.500
35	0.611	0.574
40	0.698	0.643
45	0.785	0.707

For θ in radian, $\sin\theta \approx \theta$ for very small angles

This means that "for θ as large as 10 degrees, sin θ is nearly the same as θ when θ is expressed in radians". As θ increases in value sinθ gradually becomes different from θ

Pendulum length due to effect of temperature

Suppose the suspended wire is affected due to change in temperature. The rise in temperature affects length by

$$l = l_o(1 + \alpha\,\Delta t)$$

where lo is the original length of the wire and l is final length of the wire when the temperature is raised. Let Δt is the change in temperature and α is the co-efficient of linear expansion.

Then, $T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{l_0(1+\alpha\Lambda t)}{g}}$

$$= 2\pi\sqrt{\frac{l_0}{g}}\sqrt{(1+\alpha\Delta t)}$$

$$T = T_0\,(1+\alpha\,\Delta t)^{\frac{1}{2}} \approx T_0\,(1+\frac{1}{2}\,\alpha\,\Delta t)$$

$$\Rightarrow \quad \frac{T}{T_0} - 1 = \frac{T-T_0}{T_0} = \frac{\Lambda T}{T_0} = \frac{1}{2}\alpha\,\Delta t$$

where ΔT is the change in time period due to the effect of temperature and T_0 is the time period of the simple pendulum with original length l_0.

E X A M P L E

If the length of the simple pendulum is increased by 44% from its original length, calculate the percentage increase in time period of the pendulum.

Solution

Since

$$T \propto \sqrt{l}$$

Therefore,

$$T = \text{constant } \sqrt{l}$$

$$\frac{T_f}{T_i} = \sqrt{\frac{l + \frac{44}{100}l}{l}} = \sqrt{1.44} = 1.2$$

Therefore, $T_f = 1.2\ T_i = T_i + 20\%\ T_i$

Oscillation of liquid in a U-tube:

Consider a U-shaped glass tube which consists of two open arms with uniform crosssectional area A. Let us pour a non-viscous uniform incompressible liquid of density ρ in the U-shaped tube to a height h as shown in the Figure 10.22. If the liquid and tube are not disturbed then the liquid surface will be in equilibrium position O. It means the pressure as measured at any point on the liquid is the same and also at the surface on the arm (edge of the tube on either side), which balances with the atmospheric pressure. Due to this the level of liquid in each arm will be the same. By blowing air one can provide sufficient force in one arm, and the liquid gets disturbed from equilibrium position O, which means, the pressure at blown arm is higher than the other arm. This creates difference in pressure which will cause the liquid to oscillate for a very short duration of time about the mean or equilibrium position and finally comes to rest.

Time period of the oscillation is

$$T = 2\pi\sqrt{\frac{l}{2g}} \text{ second}$$

ENERGY IN SIMPLE HARMONIC MOTION

a. Expression for Potential Energy

For the simple harmonic motion, the force andthe displacement are related by Hooke's law

$$\vec{F} = -k\vec{r}$$

Since force is a vector quantity, in three dimensions it has three components. Further, the force in the above equation is a conservative force field; such a force can be derived from a scalar function which has only one component. In one dimensional case

$$F = -k\,x$$

As we have discussed in unit 4 of volume I, the work done by the conservative force field is independent of path. The potential energy U can be calculated from the following expression.

$$F = -\frac{dU}{dx}$$

Comparing (10.63) and (10.64), we get

$$-\frac{dU}{dx} = -\,kx$$

$$dU = k\,x\,dx$$

This work done by the force F during a small displacement dx stores as potential energy

$$U(x) = \int_0^x k\,x'\,dx' = \frac{1}{2}k\left(x'\right)^2\Big|_0^x = \frac{1}{2}kx^2$$

From equation (10.22), we can substitute the value of force constant k = m ω2 in equation (10.65),

$$U(x) = \frac{1}{2}m\omega^2 x^2$$

where ω is the natural frequency of the oscillating system. For the particle executing simple harmonic motion from equation (10.6), we get

$$x = A\sin\omega t$$

$$U(t) = \frac{1}{2}m\omega^2 A^2 \sin^2\omega t$$

b. Expression for Kinetic Energy

Kinetic energy

$$KE = \frac{1}{2}\,mv_x^2 = \frac{1}{2}\,m\left(\frac{dx}{dt}\right)^2$$

Since the particle is executing simple harmonic motion, from equation (10.6)

$$x = A \sin \omega t$$

Therefore, velocity is

$$v_x = \frac{dx}{dt} = A\omega \cos \omega t$$
$$= A\omega\sqrt{1-\left(\frac{x}{A}\right)^2}$$
$$v_x = \omega\sqrt{A^2 - x^2}$$

Hence,

$$KE = \frac{1}{2}mv_x^2 = \frac{1}{2}m\omega^2\left(A^2 - x^2\right)$$
$$KE = \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t$$

c. Expression for Total Energy

Total energy is the sum of kinetic energy and potential energy

$$E = KE + U$$
$$E = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2$$

Hence, cancelling x^2 term,

$$E = \frac{1}{2}m\omega^2 A^2 = \text{ constant}$$

Alternatively, from equation (10.67) and equation (10.72), we get the total energy as

$$E = \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t + \frac{1}{2}m\omega^2 A^2 \cos^2 \omega t$$
$$= \frac{1}{2}m\omega^2 A^2 (\sin^2 \omega t + \cos^2 \omega t)$$

From trigonometry identity, $(\sin^2 \omega t + \cos^2 \omega t) = 1$

$$E = \frac{1}{2}m\omega^2 A^2 = \text{constant}$$

Thus the amplitude of simple harmonic oscillator, can be expressed in terms of total energy.

$$A=\sqrt{\frac{2E}{m\omega^2}}=\sqrt{\frac{2E}{k}}$$

E X A M P L E

Write down the kinetic energy and total energy expressions in terms of linear momentum, For one-dimensional case.

Solution

Kinetic energy is $KE=\frac{1}{2}mv_x^2$

Multiply numerator and denominator by m

$$KE=\frac{1}{2m}m^2 v_x^2=\frac{1}{2m}(\mathrm{m}\,v_x)^2=\frac{1}{2m}p_x^2$$

where, px is the linear momentum of the particle executing simple harmonic motion.

Total energy can be written as sum of kinetic energy and potential energy, therefore, from equation (10.73) and also from equation (10.75), we get

$$E=KE+U(x)=\frac{1}{2m}p_x^2+\frac{1}{2}m\omega^2x^2=\text{constant}$$

E X A M P L E

Compute the position of an oscillating particle when its kinetic energy and potential energy are equal.

Solution

Since the kinetic energy and potential energy of the oscillating particle are equal,

$$\frac{1}{2}m\omega^2(\mathrm{A}^2-x^2)-\frac{1}{2}m\omega^2x^2$$

$$A^2 - x^2 = x^2$$

$$2x^2 = A^2$$

$$\Rightarrow x = \pm\frac{A}{\sqrt{2}}$$

TYPES OF OSCILLATIONS:

Free oscillations

When the oscillator is allowed to oscillate by displacing its position from equilibrium position, it oscillates with a frequency which is equal to the natural frequency of the oscillator. Such an oscillation or vibration is known as free oscillation or free vibration. In this case, the amplitude, frequency and the energy of the vibrating object remains constant.

Examples

a. Vibration of a tuning fork.
b. Vibration in a stretched string.
c. Oscillation of a simple pendulum.
d. Oscillationsof a spring-mass system.

Damped oscillations

During the oscillation of a simple pendulum (in previous case), we have assumed that the amplitude of the oscillation is constant and also the total energy of the oscillator is constant. But in reality, in a medium, due to the presence of friction and air drag, the amplitude of oscillation decreases as time progresses. It implies that the oscillation is not sustained and the energy of the SHM decreases gradually indicating the loss of energy. Th e energy lost is absorbed by the surrounding medium. Th is type of oscillatory motion is known as damped oscillation. In other words, if an oscillator moves in a resistive medium, its amplitude goes on decreasing and the energy of the oscillator is used to do work against the resistive medium. Th e motion of the oscillator is said to be damped and in this case, the resistive force (or damping force) is proportional to the velocity of the oscillator.

Examples

a. Th e oscillations of a pendulum (including air friction) or pendulum oscillating inside an oil fi lled container.
b. Electromagnetic oscillations in a tank circuit.
c. Oscillations in a dead beat and ballistic galvanometers.

Maintained oscillations

While playing in swing, the oscillations will stop aft er a few cycles, this is due to damping. To avoid damping we have to supply a push to sustain oscillations. By supplying energy from an external source, the amplitude of the oscillation can be made constant. Such vibrations are known as maintained vibrations.

Example:

The vibration of a tuning fork getting energyfrom a battery or from external powersupply.

Forced oscillations

Any oscillator driven by an external periodic agency to overcome the damping is known as forced oscillator or driven oscillator. In this type of vibration, the body executing vibration initially vibrates with its natural frequency and due to the presence of external periodic force, the body later vibrates with the frequency of the applied periodic force. Such vibrations are known as forced vibrations.

Example:

Sound boards of stringed instruments.

Resonance

It is a special case of forced vibrations where the frequency of external periodic force (or driving force) matches with the natural frequency of the vibrating body (driven). As a result the oscillating body begins to vibrate such that its amplitude increases at each step and ultimately it has a large amplitude. Such a phenomenon is known as resonance and the corresponding vibrations are known as resonance vibrations.

Example

The breaking of glass due to sound

Soliders are not allowed to march on a bridge. This is to avoid resonant vibration of the bridge. While crossing a bridge, if the period of stepping on the ground by marching soldiers equals the natural frequency of the bridge, it may result in resonance vibrations. This may be so large that the bridge may collapse.

11th vol II
WAVES

INTRODUCTION

In the previous chapter, we have discussed the oscillation of a particle. Consider a medium which consists of a collection of particles. If the disturbance is created at one end, itpropagates and reaches the other end. That is, the disturbance produced at the first mass point is transmitted to the next neighbouring mass point, and so on. Notice that here, only the disturbance is transmitted, not the mass points. Similarly, the speech we deliver is due to the vibration of our vocal chord inside the throat. This leads to the vibration of the surrounding air molecules and hence, the effect of speech (information) is transmitted from one point in space to another point in space without the medium carrying the particles. Thus, the disturbance which carries energy and momentum from one point in space to another point in space without the transfer of the medium is known as a wave.

Standing near a beach, one can observe tides in the ocean reaching the seashore with a similar wave pattern; hence they are called ocean waves. A rubber band when plucked vibrates like a wave which is an example of a standing wave. These are shown in Figure 11.2. Other examples of waves are light waves (electromagnetic waves), through which we observe and enjoy the beauty of nature and sound waves using which we hear and enjoy pleasant melodious songs. Day to day applications of waves are numerous, as in mobile phone communication, laser surgery, etc.

Ripples and wave formation on the water surface

Suppose we drop a stone in a trough of still water, we can see a disturbance produced at the place where the stone strikes the water surface as shown in Figure 11.3. We find that this disturbance spreads out (diverges out) in the form of concentric circles of ever increasing radii (ripples) and strike the boundary of the trough. This is because some of the kinetic energy of the stone is transmitted to the water molecules on the surface. Actually the particles of the water (medium) themselves do not move outward with the disturbance. This can be observed by keeping a paper strip on the water surface. The strip moves up and down when the disturbance (wave) passes on the water surface. This shows that the water molecules only undergo vibratory motion about their mean positions.

Formation of waves on stretched string

Let us take a long string and tie one end of the string to the wall as shown in Figure 11.4 (a). If we give a quick jerk, a bump (like pulse) is produced in the string as shown in Figure 11.4 (b). Such a disturbance is sudden and it lasts for a short duration, hence it is known as a wave pulse. If jerks are given continuously then the waves produced are standing waves. Similar waves are produced by a plucked string in a guitar.

Formation of waves in a tuning fork

When we strike a tuning fork on a rubber pad, the prongs of the tuning fork vibrate about their mean positions. The prong vibrating about a mean position means moving outward and inward, as indicated in the Figure 11.5. When a prong moves outward, it pushes the layer of air in its neighbourhood which means there is more accumulation of air molecules in this region. Hence, the density and also the pressure increase. These regions are known as compressed regions or compressions. This compressed air layer moves forward and compresses the next neighbouring layer in a similar manner. Thus a wave of compression advances or passes through air. When the prong moves inwards, the particles of the medium are moved to the right. In this region both density and pressure are low. It is known as a rarefaction or elongation.

Characteristics of wave motion

- For the propagation of the waves, the medium must possess both inertia and elasticity, which decide the velocity of the wave in that medium.
- In a given medium, the velocity of a wave is a constant whereas the constituent particles in that medium move with different velocities at different positions. Velocity is maximum at their mean position and zero at extreme positions.
- Waves undergo reflections, refraction, interference, diffraction and polarization.

Point to ponder
1. The medium possesses both inertia and elasticity for propagation of waves.
2. Light is an electromagnetic wave. what is the medium for its transmission?

Mechanical wave motion and its types

Wave motion can be classified into two types

a. Mechanical wave – Waves which require a medium for propagation are known as mechanical waves.
   Examples: sound waves, ripples formed on the surface of water, etc.

b. Non mechanical wave – Waves which do not require any medium for propagation are known as non-mechanical waves.
   Example: light

Further, waves can be classified into two types

a. Transverse waves
b. Longitudinal waves

Transverse wave motion

In transverse wave motion, the constituents of the medium oscillate or vibrate about their mean positions in a direction perpendicular to the direction of propagation (direction of energy transfer) of waves.

Example: light (electromagnetic waves)

Longitudinal wave motion

In longitudinal wave motion, the constituent of the medium oscillate or vibrate about their mean positions in a direction parallel to the direction of propagation (direction of energy transfer) of waves as shown in Figure 11.7.

Example: Sound waves travelling in air.

Discuss with your Teacher

- Tsunami (pronounced soo-nah-mee in Japanese) means Harbour waves. A tsunami is a series of huge and giant waves which come with great speed and huge force. What happened on 26th December2004 in southern part of India? - Discuss
- Gravitational waves - LIGO (Laser Interferometer Gravitational wave Observatory) experiment Nobel Prize winners in Physics 2017
    i. Prof. Rainer Weiss
    ii. Prof. Barry C. Barish
    iii. Prof. Kip S. Thorne

  "For decisive contributions to the LIGO detector and observation of gravitational forces"

Comparison of transverse and longitudinal waves		
S.No	Transverse waves	Longitudinal waves
1.	The direction of vibration of particles of the medium is perpendicular to the direction of propagation of waves.	The direction of vibration of particles of the medium is parallel to the

		direction of propagation of waves.
2.	The disturbances are in the form of crests and troughs.	The disturbances are in the form of compressions and rarefactions
3.	Transverse waves are possible in elastic medium.	Longitudinal waves are possible in all types of media (solid, liquid and gas).

NOTE:
1. Absence of medium is also known as vacuum. Only electromagnetic waves can travel through vacuum.
2. Rayleigh waves are considered to be mixture of transverse and longitudinal.

TERMS AND DEFINITIONS USED IN WAVE MOTION

Suppose we have two waves as shown in Figure 11.8. Are these two waves identical?. No. Though, the two waves are both sinusoidal, there are many difference between them. Therefore, we have to define some basic terminologies to distinguish one wave from another.

Consider a wave produced by a stretched string as shown in Figure 11.9.

If we are interested in counting the number of waves created, let us put a reference level (mean position) as shown in Figure 11.9. Here the mean position is the horizontal line shown. The highest point in the shaded portion is called crest. With respect to the reference level, the lowest point on the un-shaded portion is called trough. This wave contains repetition of a section O to B and hence we define the length of the smallest section without repetition as one wavelength as shown in Figure 11.10. In Figure 11.10 the length OB or length BD is one wave lengh. A Greek letter lambda λ is used to denote one wavelength.

For transverse waves (as shown in Figure 11.11), the distance between two neighbouring crests or troughs is known as the wavelength. For longitudinal waves, (as shown in Figure 11.12) the distance between two neighbouring compressions or rarefactions is known as the wavelength. The SI unit of wavelength is meter.

E X A M P L E

Which of the following has longer wavelength?

In order to understand frequency and time period, let us consider waves (made of three wavelengths) as shown in Figure 11.13 (a). At time t = 0 s, the wave reaches the point A from left. After time t = 1 s (shown in figure 11.13(b)), the number of waves which have crossed the point A is two. Therefore, the frequency is defined as “the number of waves crossing a point per second” It is measured in hertz whose symbol is Hz. In this example,

$$f = 2 \text{ Hz}$$

wave consisting of three wavelengths passing a point A at time (a) t = 0 s and (b) after time t = 1 sin

If two waves take one second (time) to cross the point A then the time taken by one wave to cross the point A is half a second. This defines the time period T as

$$T = \frac{1}{2} = 0.5\, s$$

From equation (11.1) and equation (11.2), frequency and time period are inversely related i.e.

$$T = \frac{1}{f}$$

Time period is defined as the time taken by one wave to cross a point.in

E X A M P L E

Three waves are shown in the figure below

Write down

(a) the frequency in ascending order
(b) the wavelength in ascending order

Solution

$$f_c < f_a < f_b$$
$$\lambda_b < \lambda_a < \lambda_c$$

From the example 11.2, we observe that the frequency is inversely related to the wavelength, $f\, a\, \frac{1}{l}$

Then, $f\lambda$ is equal to what?

$$[(\text{i.e})\ \ f\lambda = ?\ ]$$

A simple dimensional argument will help us to determine this unknown physical quantity. Dimension of wavelength is, [λ] = L

Frequency $f = \dfrac{1}{Time\ period}$ which implies that the dimension of frequency is,

$$[f] = \frac{1}{[T]} = T^{-1}$$
$$\Rightarrow [\lambda f] = [\lambda][f] = LT^{-1} = [\text{velocity}]$$

Therefore,

Velocity, $\lambda f = v$

where v is known as the wave velocity or phase velocity. This is the velocity with which the wave propagates. Wave velocity is the distance travelled by a wave in one second.

Note:

The number of cycles (or revolutions) per unit time is called angular frequency. Angular frequency, $w = \dfrac{2p}{T} = 2p\, f$ (unit is radians / second)

The number of cycles per unit distance or number of waves per unit distance is called wave number. wave number, $k = \dfrac{2p}{l}$ (unit is radians / meter In two, three or higher dimensional case, the wave number is the magnitude of a vector called \ wave vector. The points in space of wave vectors are called reciprocal vectors, $\overset{\text{'}}{k}$

Example

The average range of frequencies at which human beings can hear sound waves varies from 20 Hz to 20 kHz. Calculate the wavelength of the sound wave in these limits. (Assume the speed of sound to be 340 m s^{-1}.

Solution

$$\lambda_1 = \frac{v}{f_1} = \frac{340}{20} = 17\text{m}$$

$$\lambda_2 = \frac{v}{f_2} = \frac{340}{20\times10^3} = 0.017\text{m}$$

Therefore, the audible wavelength region is from 0.017 m to 17 m when the velocity of sound in that region is 340 m s^{-1}.

Example

A man saw a toy duck on a wave in an ocean. He noticed that the duck moved up and down 15 times per minute. He roughly measured the wavelength of the ocean wave as 1.2 m. Calculate the time taken by the toy duck for going one time up and down and also the velocity of the ocean wave.

Solution

Given that the number of times the toy duck moves up and down is 15 times per minute. This information gives us frequency (the number of times the toy duck moves up and down)

$$f = \frac{\text{15 times toy duck moves up and down}}{\text{one minute}}$$

But one minute is 60 second, therefore, expressing time in terms of second

$$f = \frac{15}{60} = \frac{1}{4} = 0.25\text{Hz}$$

The time taken by the toy duck for going one time up and down is time period which is inverse of frequency

$$T = \frac{1}{f} = \frac{1}{0.25} = 4\text{ s}$$

The velocity of ocean wave is

$$v = \lambda f = 1.2 \times 0.25 = 0.3\text{ m s}^{-1}.$$

Amplitude of a wave:

The waves shown in thesame wavelength, same frequency andsame time period and also move with samevelocity. The only difference between twowaves is the height of either crest or trough.This means, the height of the crest or troughalso signifies a wave character. So we definea quantity called an amplitude of the wave, as the maximum displacement of the medium with respect to a reference axis (for example in this case x-axis). Here, it is denoted by A.

Example

Consider a string whose one end is attached to a wall. Then compute the following in both situations given in figure (assume waves crosses the distance in one second)

(a) Wavelength, (b) Frequency and (c) Velocity

Solution

	First Class	Second Class
(a) Wavelength	$\lambda = 6$ m	$\lambda = 2$ m
(b)Frequency	f = 2 Hz	f = 6 Hz
(c) Velocity	v = 6 × 2 = 12 m s-1	v = 2 × 6 = 12 m s-1

This means that the speed of the wave along a string is a constant. Higher the frequency, shorter the wavelength and vice versa, and their product is velocity which remain the same.

Velocity of Waves in different Media

Suppose a hammer is stroked on long rails at a distance and when a person keeps his ear near the rails at the other end he/she will hear two sounds, at different instants. The sound that is heard through the rails (solid medium) is faster than the sound we hear through the air (gaseous medium). This implies the velocity of sound is different in different media.

In this section, we shall derive the velocity of waves in two different cases:

1. The velocity of a transverse waves along a stretched string.
2. The velocity of a longitudinal waves in an elastic medium.

Velocity of transverse waves in a stretched string

Let us compute the velocity of transverse travelling waves on a string. When a jerk is given at one end (left end) of the rope, the wave pulses move towards right end with a velocity v. This means that the pulses move with a velocity v with respect to an observer who is at rest frame. Suppose an observer also moves with same velocity v in the direction of motion of the wave pulse, then that observer will notice that the wave pulse is stationary and the rope is moving with pulse with the same velocity v.

Consider an elemental segment in the string as shown in the Figure. Let A andB be two points on the string at an instant of time. Let dl and dm be the length and mass of the elemental string, respectively. By definition, linear mass density, μ is

$$\mu = \frac{dm}{dl}$$

$$dm = \mu \, dl$$

The elemental string AB has a curvature which looks like an arc of a circle with centre at O, radius R and the arc subtending an angle θ at the origin O as shown in Figure. The angle θ can be written in terms of arc length and radius as $\theta = \frac{dl}{R}$. The centripetal acceleration supplied by the tension in the string is

$$a_{cp} = \frac{v^2}{R}$$

Then, centripetal force can be obtained when mass of the string (dm) is included in equation.

$$F_{cp} = \frac{(dm)v^2}{R}$$

The centripetal force experienced by elemental string can be calculated by substituting equation

$$\frac{(dm)v^2}{R} = \frac{\mu v^2 \, dl}{R}$$

The tension T acts along the tangent of the elemental segment of the string at A and B. Since the arc length is very small, variation in the tension force can be ignored. We can resolve T into horizontal component $T\cos\left(\frac{\theta}{2}\right)$ and vertical component $T\sin\left(\frac{\theta}{2}\right)$. The horizontal components at A and B are equal in magnitude but opposite in direction; therefore, they cancel each other. Since the elemental arc length AB is taken to be very small, the vertical components at A and B appears to acts vertical towards the centre of the arc and hence, they add up. The net radial force Fr is

$$F_r = 2T\sin\left(\frac{\theta}{2}\right)$$

Since the amplitude of the wave is very small when it is compared with the length of the string, the sine of small angle is approximated as $\sin\left(\frac{\theta}{2}\right)\approx\frac{\theta}{2}$. Hence, equation can be written as

$$F_r = 2T\times\frac{\theta}{2} = T\theta$$

But $\theta=\frac{dl}{R}$ therefore substituting in equation (11.11), we get

$$F_r = T\frac{dl}{R}$$

Applying Newton's second law to the elemental string in the radial direction, under equilibrium, the radial component of the force is equal to the centripetal force. Hence equating equation (11.9) and equation (11.12), we have $T\frac{dl}{R}=\mu v^2\frac{dl}{R}$

$$v=\sqrt{\frac{T}{\mu}} \text{ measured in m s}^{-1}$$

Observations:

- The velocity of the string is

a. directly proportional to the square root of the tension force

b. inversely proportional to the square root of linear mass density

c. independent of shape of the waves.

Example

Calculate the velocity of the travelling pulse as shown in the figure below. The linear mass density of pulse is 0.25 kg m-1. Further, compute the time taken by the travelling pulse to cover a distance of 30 cm on the string.

Solution

The tension in the string is T = m g = 1.2 × 9.8 = 11.76 N
The mass per unit length is μ = 0.25 kg m-1
Therefore, velocity of the wave pulse is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{11.76}{0.25}} = 6.858\ m\ s^{-1} = 6.8\ m\ s^{-1}$$

The time taken by the pulse to cover the distance of 30 cm is

$$t = \frac{d}{v} = \frac{30 \times 10^{-2}}{6.8} = 0.044\,s = 44\ ms \text{ where,}$$

ms = milli second

Velocity of longitudinal waves in an elastic medium

Consider an elastic medium (here we assume air) having a fixed mass contained in a long tube (cylinder) whose cross sectional area is A and maintained under a pressure P. One can generate longitudinal waves in the fluid either by displacing the fluid using a piston or by keeping a vibrating tuning fork at one end of the tube. Let us assume that the direction of propagation of waves coincides with theaxis of the cylinder. Let ρ be the density of the fluid which is initially at rest. At t = 0, the piston at left end of the tube is set in motion toward the right with a speed u.

Let u be thevelocity of the piston and v be the velocity of the elastic wave. In time interval Δt, the distance moved by the piston $\Delta d = u\ \Delta t$. Now, the distance moved by the elastic disturbance is $\Delta x = v\Delta t$. Let Δm be the mass of the air that has attained a velocity v in a time Δt . Therefore,

$$\Delta m = \rho\ A\ \Delta x = \rho\ A\ (v\ \Delta t)$$

Then, the momentum imparted due to motion of piston with velocity u is

$$\Delta p = [\rho\ A\ (v\ \Delta t)]\,u$$

But the change in momentum is impulse. The net impulse is

$$I = (\Delta P\, A)\Delta t$$

Or $$(\Delta P\, A)\Delta t = [\rho\, A\, (v\, \Delta t)]u$$

$$\Delta P = \rho\, v\, u \qquad (1$$

When the sound wave passes through air, the small volume element (ΔV) of theair undergoes regular compressions and rarefactions. So, the change in pressure can also be written as

$$\Delta P = B\frac{\Delta V}{V}$$

where, V is original volume and B is known as bulk modulus of the elastic medium.

But V = A Δx = A v Δt and

ΔV = A Δd =A u Δt

Therefore,

$$\Delta P = B\frac{A\, u\, \Delta t}{A\, v\, \Delta t} = B\frac{u}{v}$$

$$\rho v u = B\frac{u}{v} \text{ or } v^2 = \frac{B}{\rho}$$

$$\Rightarrow v = \sqrt{\frac{B}{\rho}}$$

In general, the velocity of a longitudinal wave in elastic medium is $v = \sqrt{\frac{E}{r}}$ where E is the modulus of elasticity of the medium.

Cases: For a solid:

(i) one dimension rod (1D)

$$v = \sqrt{\frac{Y}{\rho}}$$

where Y is the Young's modulus of the material of the rod and ρ is the density of the rod. The 1D rod will have only Young's modulus.

(ii) Three dimension rod (3D) The speed of longitudinal wave in a solid is

$$v = \sqrt{\frac{K + \frac{4}{3}\eta}{\rho}}$$

where η is the modulus of rigidity, K is the bulk modulus and ρ is the density of the rod.

Cases: For liquids:

$$v = \sqrt{\frac{K}{\rho}}$$

where, K is the bulk modulus and ρ is the density of the rod.

E X A M P L E

Calculate the speed of sound in a steel rod whose Young's modulus Y = 2×10^{11} N m^{-2} and ρ = 7800 kg m^{-3}.

Solution

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2\times10^{11}}{7800}} = \sqrt{0.2564\times10^{8}}$$

$$= 0.506\times10^{4} ms^{-1} = 5\times10^{3} ms^{-1}$$

Therefore, longitudinal waves travel faster in a solid than in a liquid or a gas. Now you may understand why a shepherd checks before crossing railway track by keeping his ears on the rails to safeguard his cattle.

E X A M P L E

An increase in pressure of 100 kPa causes a certain volume of water to decrease by 0.005% of its original volume.

(a) Calculate the bulk modulus of water?.

(b) Compute the speed of sound (compressional waves) in water?.

Solutions

a) Bulk modules

$$B = V\left|\frac{\Delta P}{\Delta V}\right| = \frac{100\times10^3}{0.005\times10^{-2}} =$$

$$= \frac{100\times10^3}{5\times10^{-5}} = 2000\ MPa,$$ where MP_a

Mega Pascal

(b) Speed of sound in water is

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2000\times10^6}{1000}} = 1414\,ms^{-1}$$

The velocities of both transverse waves and longitudinal waves depend on elastic property (like string tension T or bulk modulus B) and inertial property (like density or mass per unit length) i.e., $v = \sqrt{\frac{\text{Elastic property}}{\text{Inertial property}}}$

Speed of Sound in Various media

S.No	Medium	Speed in ms $^{-1}$
1	Rubber	1600
2	Gold	3240
3	Brass	4700
4	Copper	5010
5	Iron	5950
6	Aluminium	6420
		Liquids at 25^{0}C
1	Kerosene	1324
2	Mercury	1450
3	Water	1493
4	Sea water	1533
		Gas (at 0^{0}C)
1	Oxygen	317
2	Air	337
3	Helium	972
4	Hydrogen	1286
		Gas (at 20°C)
1	Air	343

PROPAGATION OF SOUND WAVES

We know that sound waves are longitudinal waves, and when they propagate compressions and rarefactions are formed. In the following section, we compute the speed of sound in air by Newton's method and also discuss the Laplace correction and the factors affecting sound in air.

Newton's formula for speed of sound waves in air

Sir Isaac Newton assumed that when sound propagates in air, the formation of compression and rarefaction takes placein a very slow manner so that the process is isothermal in nature. That is, the heat produced during compression (pressure increases, volume decreases), and heat lost during rarefaction (pressure decreases, volume increases) occur over a period of time such that the temperature of the medium remains constant. Therefore, by treating the air molecules to form an ideal gas, the changes in pressure and volume obey Boyle's law, Mathematically

$$PV = \text{Constant} \qquad (11.20)$$

Differentiating equation (11.20), we get

$$PdV + VdP - 0$$

$$\text{or,} \quad P = -V\frac{dP}{dV} = B_T \qquad (11.21)$$

where, B_T is an isothermal bulk modulus of air. Substituting equation (11.21) in equation (11.16), the speed of sound in air is

$$v_T = \sqrt{\frac{B_T}{\rho}} = \sqrt{\frac{P}{\rho}} \qquad (11.22)$$

Since P is the pressure of air whose value at NTP (Normal Temperature and Pressure) is 76 cm of mercury, we have

$$P = (0.76 \times 13.6 \times 10^3 \times 9.8)\ \text{N m}^{-2}$$

$\rho = 1.293$ kg m^{-3}. here ρ is density of air

Then the speed of sound in air at Normal Temperature and Pressure (NTP) is

$$v_T = \sqrt{\frac{(0.76 \times 13.6 \times 10^3 \times 9.8)}{1.293}}$$

$$= 279.80 \text{ m s}^{-1} \approx 280 \text{ ms}^{-1} \text{ (theoretical value)}$$

But the speed of sound in air at 0°C is experimentally observed as 332 m s^{-1} which is close upto 16% more than theoretical value (Percentage error is $\frac{(332-280)}{332} \times 100\% \approx 15.6\%)$. This error is not small.

Laplace's correction

In 1816, Laplace satisfactorily corrected this discrepancy by assuming that when the sound propagates through a medium, the particles oscillate very rapidly such that the compression and rarefaction occur very fast. Hence the exchange of heat produced due to compression and cooling effect due to rarefaction do not take place, because, air (medium) is a bad conductor of heat. Since, temperature is no longer considered as a constant here, sound propagation is an adiabatic process. By adiabatic considerations, the gas obeys Poisson's law (not Boyle's law as Newton assumed), which is

$$PV^{\gamma} = \text{constant} \qquad (11.23)$$

where, $\gamma = \frac{C_P}{C_V}$, which is the ratio between specific heat at constant pressure and specific heat at constant volume.

Differentiating equation (11.23) on both the sides, we get

$$V^{\gamma} \, dP + P\,(\gamma V^{\gamma-1} \, dV) = 0$$

$$\text{or, } \gamma P = -V \frac{dp}{dV} = B_A$$

where, B_A is the adiabatic bulk modulus of air. Now, substituting equation (11.24) in equation (11.16), the speed of sound in air is

$$v_A = \sqrt{\frac{B_A}{\rho}} = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma} v_T$$

Since air contains mainly, nitrogen, oxygen, hydrogen etc, (diatomic gas), we take γ = 1.47. Hence, speed of sound in air is $v_A = \left(\sqrt{1.4}\right)\left(280 ms^{-1}\right) = 331.30 ms^{-1}$ which is very much closer to experimental data.

Factors affecting speed of sound in gases

Let us consider an ideal gas whose equation of state is

$$PV = n R T$$

where, P is pressure, V is volume, T is temperature, n is number of mole and R is universal gas constant. For a given mass of a molecule, equation (11.26) can be written as

$$\frac{PV}{T} = \text{constant}$$

For a fixed mass m, density of the gas inversely varies with volume. i.e.,

$$\rho \propto \frac{1}{V}, V = \frac{m}{\rho}$$

Substituting equation (11.28) in equation (11.27), we get

$$\frac{P}{\rho} = cT$$

where c is constant.

The speed of sound in air given in equation (11.25) can be written as

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma c T}$$

From the above relation we observe the following

(a) Effect of pressure:

For a fixed temperature, when the pressure varies, correspondingly density also varies such that the ratio $\left(\frac{P}{\rho}\right)$ becomes constant. This means that the speed of sound is independent of pressure for a fixed temperature. If the temperature remains same at the top and the bottom of a mountain then the speed of sound will remain same at these two points. But, in practice, the temperatures are not same at top and bottom of a mountain; hence, the speed of sound is different at different points.

(b) Effect of temperature:

Since, $v \propto \sqrt{T}$, the speed of sound varies directly to the square root of temperature in kelvin.

Let v_0 be the speed of sound at temperature at 0° C or 273 K and v be the speed of sound at any arbitrary temperature T (in kelvin), then

$$\frac{v}{v_0} = \sqrt{\frac{T}{273}} = \sqrt{\frac{273+t}{273}}$$

$$v = v_0\sqrt{1+\frac{t}{273}} \cong v_0\left(1+\frac{t}{546}\right)$$

(using binomial expansion)

Since v_0 = 331m s^{-1} at 0^0C, v at any temperature in t^0C is

v = (331 + 0.60t) m s-1

Thus the speed of sound in air increases by 0.61 m s^{-1} per degree celcius rise in temperature. Note that when the temperature is increased, the molecules will vibrate faster due to gain in thermal energy and hence, speed of sound increases.

(c) Effect of density:

Let us consider two gases with different densities having same temperature and pressure. Then the speed of sound in the two gases are

$$v_1 = \sqrt{\frac{\gamma_1 P}{\rho_1}}$$

and

$$v_2 = \sqrt{\frac{g_2 P}{r_2}}$$

Taking ratio of equation (11.31) and equation (11.32), we get

$$\frac{v_1}{v_2} = \frac{\sqrt{\frac{g_1 P}{r1}}}{\sqrt{\frac{g_2 P}{r_2}}} = \sqrt{\frac{g_1 r_2}{g_2 r_1}}$$

For gases having same value of γ,

$$\frac{v_1}{v_2} = \sqrt{\frac{r_2}{r_1}}$$

(e) Effect of wind:

The speed of sound is also affected by blowing of wind. In the direction along the wind blowing, the speed of sound increases whereas in the direction opposite to wind blowing, the speed of sound decreases.

Example

The ratio of the densities of oxygen andnitrogen is 16:14. Calculate the temperaturewhen the speed of sound in nitrogen gasat 17°C is equal to the speed of sound inoxygen gas.

Solution

From equation (11.25), we have

$$v = \sqrt{\frac{gP}{r}}$$

But, $r = \frac{M}{V}$

Therefore, $v = \sqrt{\frac{gPV}{M}}$

Using equation (11.26)

$$v = \sqrt{\frac{gRT}{M}}$$

Where, R is the universal gas constant and M is the molecular mass of the gas. The speed of sound in nitrogen gas at 17°C is

$$v_N = \sqrt{\frac{gR(273K + 17K)}{M_N}}$$
$$= \sqrt{\frac{gR(290K)}{M_N}}$$

Similarly, the speed of sound in oxygen gas at t in K is

$$v_0 = \sqrt{\frac{gR(273K + t)}{M_0}}$$

Given that the value of γ is same for both the gases, the two speeds must be equal. Hence, equating equation (1) and (2), we get

$$v_o = v_N$$
$$\sqrt{\frac{gR(273+t)}{M_0}} = \sqrt{\frac{gR(290)}{M_N}}$$

Squaring on both sides and cancelling γ R term and rearranging, we get

$$\frac{M_0}{M_N} = \frac{273+t}{290}$$

Since the densities of oxygen and nitrogen is 16:14,

$$\frac{r_0}{r_N} = \frac{16}{14}$$

$$\frac{\rho_0}{\rho_N} = \frac{\frac{M_O}{V}}{\frac{M_N}{V}} = \frac{M_0}{M_N} \Rightarrow \frac{M_0}{M_N} = \frac{16}{14} \qquad (5)$$

Substituting equation (5) in equation (3), we get

$$\frac{273+t}{290} = \frac{16}{14} \Rightarrow 3822 + 14t = 4640$$

$$\Rightarrow \ t = 58.4\,\text{K}$$

REFLECTION OF SOUND WAVES

When sound wave passes from one medium to another medium, the following things can happen

(a) Reflection of sound: If the medium is highly dense (highly rigid), the sound can be reflected completely (bounced back) to the original medium.

(b) Refraction of sound: When the sound waves propagate from one medium to another medium such that there can be some energy loss due to absorption by the second medium.

In this section, we will consider only the reflection of sound waves in a medium when it experiences a harder surface. Similar to light, sound can also obey the laws of reflection, which states that

(i) The angle of incidence of sound is equal to the angle of reflection.

(ii) When the sound wave is reflected by a surface then the incident wave, reflected wave and the normal at the point of incidence all lie in the same plane.

Similar to reflection of light from a mirror, sound also reflects from a harder flat surface, This is called as specular reflection.

Specular reflection is observed only when the wavelength of the source is smaller than dimensions of the reflecting surface, as well as smaller than surface irregularities.

Reflection of sound through the plane surface

When the sound waves hit the plane wall, they bounce off in a manner similar to that of light. Suppose a loudspeaker is kept at an angle with respect to a wall (plane surface), then the waves coming from the source (assumed to be a point source) can be treated as spherical wave fronts (say, compressions moving like a spherical wave front). Therefore, the reflected wave front on the plane surface is also spherical, such that its centre of curvature (which lies on the other side of plane surface) can be treated as the image of the sound source (virtual or imaginary loud speaker) which can be assumed to be at a position behind the plane surface.

Reflection of sound through the curved surface

The behaviour of sound is different when it is reflected from different surfaces-convex or concave or plane. The sound reflected from a convex surface is spread out and so it is easily attenuated and weakened. Whereas, if it is reflected from the concave surface it will converge at a point and this can be easilyamplified. The parabolic reflector (curved reflector) which is used to focus the sound precisely to a point is used in designing the parabolic mics which are known as high directional microphones.

We know that any surface (smooth or rough) can absorb sound. For example, the sound produced in a big hall or auditorium or theatre is absorbed by the walls, ceilings, floor, seats etc. To avoid such losses, a curved sound board (concave board) is kept in front of the speaker, so that the board reflects the sound waves of the speaker towards the audience. This method will minimize the spreading of sound waves in all possible direction in that hall and also enhances the uniform distribution of sound throughout the hall. That is why a person sitting at any position in that hall can hear the sound without any disturbance.

Applications of reflection of sound waves

(a) Stethoscope: It works on the principle of multiple reflections.

It consists of three main parts:
(i) Chest piece
(ii) Ear piece
(iii) Rubber tube

(i) Chest piece: It consists of a small disc-shaped resonator (diaphragm) which is very sensitive to sound and amplifies the sound it detects.

(ii) Ear piece: It is made up of metal tubes which are used to hear sounds detected by the chest piece.

(iii) Rubber tube: This tube connects both chest piece and ear piece. It is used to transmit the sound signal detected by the diaphragm, to the ear piece. The sound of heart beats (or lungs) or any sound produced by internal organs can be detected, and it reaches the ear piece through this tube by multiple reflections.

(b) Echo: An echo is a repetition of sound produced by the reflection of sound waves from a wall, mountain or other obstructing surfaces. The speed of sound in air at 20°C is 344 m s^{-1}. If we shout at a wall which is at 344 m away, then the sound will take 1 second to reach the wall. After reflection, the sound will take one more second to reach us. Therefore, we hear the echo after two seconds.
Scientists have estimated that we can hear two sounds properly if the time gap or time interval between each sound is $\left(\frac{1}{10}\right)^{th}$ of a second (persistence of hearing) i.e., 0.1 s. Then,

$$\text{velocity} = \frac{\text{Distance travelled}}{\text{time taken}} = \frac{2d}{t}$$

$$2d = 344 \times 0.1 = 34.4 \text{ m}$$
$$d = 17.2 \text{ m}$$

The minimum distance from a sound reflecting wall to hear an echo at 20°C is 17.2 meter.

(c) SONAR: SOund NAvigation and Ranging. Sonar systems make use of reflections of sound waves in water to locate the position or motion of an object. Similarly, dolphins and bats use the sonar principle to find their way in the darkness.

(d) Reverberation: In a closed room the sound is repeatedly reflected from the walls and it is even heard long after the sound source ceases to function. The residual sound remaining in an enclosure and the phenomenon of multiple reflections of sound is called reverberation. The duration for which the sound persists is called reverberation time. It should be noted that the reverberation time greatly affects the quality of sound heard in a hall. Therefore, halls are constructed with some optimum reverberation time.

Example

Suppose a man stands at a distance from a cliff and claps his hands. He receives an echo from the cliff after 4 second. Calculate the distance between the man and the cliff. Assume the speed of sound to be 343 m s-1.

Solution

The time taken by the sound to come back as echo is $2t = 4 \Rightarrow t = 2$ s
$\therefore$The distance is $d = vt = (343 \text{ m s}^{-1})(2 \text{ s}) = 686$ m.

Note: Classification of sound waves: Sound waves can be classified in three groups according to their range of frequencies:

(1) Infrasonic waves:
Sound waves having frequencies below 20 Hz are called infrasonic waves. These waves are produced during earthquakes. Human beings cannot hear these frequencies. Snakes can hear these frequencies.

(2) Audible waves:
Sound waves having frequencies between 20 Hz to 20,000 Hz (20kHz) are called audible waves. Human beings can hear these frequencies.

(3) Ultrasonic waves:
Sound waves having frequencies greater than 20 kHz are known as ultrasonic waves. Human beings cannot hear these frequencies. Bats can produce and hear these frequencies.

(1.) Supersonic speed:
An object moving with a speed greater than the speed of sound is said to move with a supersonic speed.

(2.) Mach number:
It is the ratio of the velocity of source to the velocity of sound.

$$\text{Mach number} = \frac{\text{velocity of source}}{\text{velocity of sound}}$$

PROGRESSIVE WAVES (OR) TRAVELLING WAVES

If a wave that propagates in a medium is continuous then it is known as progressive wave or travelling wave.

Characteristics of progressive waves

1. Particles in the medium vibrate about their mean positions with the same amplitude.
2. The phase of every particle ranges from 0 to 2π.
3. No particle remains at rest permanently. During wave propagation, particles come to the rest position only twice at the extreme points.

4. Transverse progressive waves are characterized by crests and troughs whereas longitudinal progressive waves are characterized by compressions and rarefactions.
5. When the particles pass through the mean position they always move with the same maximum velocity.
6. The displacement, velocity and acceleration of particles separated from each other by $n\lambda$ are the same, where n is an integer, and λ is the wavelength.

Equation of a plane progressive wave

Suppose we give a jerk on a stretched string at time t = 0 s. Let us assume that the wave pulse created during this disturbance moves along positive x direction with constant speed v (a). We can represent the shape of the wave pulse, mathematically as y = y(x, 0) = f(x) at time t = 0 s. Assume that the shape of the wave pulse remains the same during the propagation. After some time t, the pulse moving towards the right and any point on it can be represented by x' (read it as x prime) (b). Then,

$$y(x, t) = f(x') = f(x - vt)$$

Similarly, if the wave pulse moves towards left with constant speed v, then y = f(x + vt). Both waves y = f(x + vt) and y = f(x − vt) will satisfy the following one dimensional differential equation known as the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

where the symbol ∂ represent partial derivative (read $\frac{\P y}{\P x}$ as partial y by partial x). Not all the solutions satisfying this differential equation can represent waves, because any physical acceptable wave must take finite values for all values of x and t. But if the function represents a wave then it must satisfy the differential equation. Since, in one dimension (one independent variable), the partial derivative with respect to x is the same as total derivative in coordinate x, we write

$$\frac{d^2 y}{dx^2} = \frac{1}{v^2}\frac{d^2 y}{dt^2}$$

This can be extended to more than one dimension (two, three, etc.). Here, for simplicity, we focus only on the one dimensional wave equation.

Example
Sketch y = x −a for different values of a.

Solution

This implies, when increasing the value of a, the line shifts towards right side. For a = vt, y = x − vt satisfies the differential equation. Though this function satisfies the differential equation, it is not finite for all values of x and t. Hence, it does not represent a wave.

Example

How does the wave y = sin(x − a) for a = 0, $a=\frac{\pi}{4}, a=\frac{\pi}{2}, a=\frac{3\pi}{2}$ and a = π look like? Sketch this wave.

Solution

From the above picture we observe that y = sin (x−a) for a = 0, $a=\frac{\pi}{4}, a=\frac{\pi}{2}, a=\frac{3\pi}{2}$ and a = π, the function y = sin (x−a) shifts towards right. Further, we can take a = vt and $v=\frac{\pi}{4}$, and sketching for different times t = 0s, t = 1s, t = 2s etc., we once again observe that y = sin(x−vt) moves towards the right. Hence, y = sin(x−vt) is a travelling (or progressive) wave moving towards the right. If y = sin(x+vt) then the travelling (or progressive) wave moves towards the left. Thus, any arbitrary function of type y = f(x−vt) characterising the wave must move towards right and similarly, any arbitrary function of type y = f(x+vt) characterizing the wave must move towards left.

Example

Check the dimensional of the wave y = sin(x−vt). If it is dimensionally wrong, write the above equation in the correct form.

Solution

Dimensionally it is not correct. we know that y = sin(x−vt) must be a dimensionless quantity but x−vt has dimension. The correct equation is y = sin (k x−ωt), where k and ω have the dimensions of inverse of length and inverse of time respectively. The sine functions and cosine functions are periodic functions with period 2π. Therefore, the correct expression is $y=\sin\left(\frac{2\pi}{\lambda}x-\frac{2\pi}{T}t\right)$ where λ and T are wavelength and time period, respectively. In general, y(x,t)=A sin(k x−ωt).

Graphical representation of the wave

Let us graphically represent the two forms of the wave variation
(a) Space (or Spatial) variation graph
(b) Time (or Temporal) variation graph

(a) Space variation graph

By keeping the time fixed, the change in displacement with respect to x is plotted. Let us consider a sinusoidal graph, y = A sin(kx), where k is a constant. Since the wavelength λ denotes the distance between any two points in the same state of motion, the displacement y is the same at both the ends y = x and y = x + λ, i.e.,

$$y = A \sin(kx) = A \sin(k(x + \lambda))$$
$$= A \sin(kx + k\lambda)$$

The sine function is a periodic function with period 2π. Hence,

$$y = A \sin(kx + 2\pi) = A \sin(kx)$$

Comparing equation, we get. kx + k λ = kx + 2π

That implies

$$k = \frac{2p}{l} radm^{-1}$$

where k is called wave number. This measures how many wavelengths are present in 2π radians.

The spatial periodicity of the wave is $l = \frac{2p}{k}$ in m,

Then,
At t = 0 s y(x, 0) = y(x + λ, 0)
and
At any time t, y(x, t) = y(x + λ, t)

Example

The wavelength of two sine waves are λ1 = 1m and λ2 = 6m. Calculate the corresponding wave numbers.

Solution

$$k_1 = \frac{2\pi}{k} = 6.28 rad m^{-1}$$

$$k_1 = \frac{2\pi}{6} = 1.05 rad m^{-1}$$

(b) Time variation graph

By keeping the position fixed, the change in displacement with respect to time is plotted. Let us consider a sinusoidal graph, y =A sin(ωt), where ω is angular frequency of the wave which measures how quickly wave oscillates in time or number of cycles per second.

The temporal periodicity or time period is

$$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}$$

The angular frequency is related to frequency f by the expression ω = 2 πf, where the frequency f is defined as the number of oscillations made by the medium particle per second. Since inverse of frequency is time period, we have,

$$T = \frac{1}{f} \text{ in seconds}$$

This is the time taken by a medium particle to complete one oscillation. Hence, we can define the speed of a wave (wave speed, v) as the distance traversed by the wave per second

$$v = \frac{\lambda}{T} = \lambda f \text{ in } ms^{-1}$$

which is the same relation as we obtained in equation (11.4).

Particle velocity and wave velocity

In a plane progressive harmonic wave, the constituent particles in the medium oscillate simple harmonically about their equilibrium positions. When a particle is in motion, the rate of change of displacement at any instant of time is defined as velocity of the particle at that instant of time. This is known as particle velocity.

$$v_P = \frac{dy}{dt} ms^{-1}$$

But y(x, t)= A sin(k x - ω t)

Therefore, $\frac{dy}{dt} = -\omega A \cos(k x - \omega t)$

Similarly, we can define velocity (here speed) for the travelling wave (or progressive wave). In order to determine the velocity of a progressive wave, let us consider a progressive wave moving towards right. This can be mathematically represented as a sinusoidal wave. Let P be any point on the phase of the wave and y_P be its displacement with respect to the mean position. The displacement of the wave at an instant t is

$$y = y(x,t) = A \sin(k x - \omega t)$$

At the next instant of time $t' = t + \Delta t$ the position of the point P is $x' = x + \Delta x$. Hence, the displacement of the wave at this instant is

$$y = y(x', t') = y(x + \Delta x, t + \Delta t)$$
$$= A \sin[k (x + \Delta x) - \omega (t + \Delta t)]$$

Since the shape of the wave remains the same, this means that the phase of the wave remains constant (i.e., the y- displacement of the point is a constant). Therefore, equating equation (11.42) and equation (11.44), we get

y(x',t') = y(x,t), which implies $A \sin[k (x + \Delta x) - \omega (t + \Delta t)] = A \sin(k x - \omega t)$ Or

$$k (x + \Delta x) - \omega (t + \Delta t) = k x - \omega t = \text{constant}$$

On simplification of equation (11.45), we get

$$v = \frac{\Delta x}{\Delta t} = \frac{w}{k} = v_p$$

where v_p is called wave velocity or phase velocity.

By expressing the angular frequency and wave number in terms of frequency and wave length, we obtain

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$k = \frac{2\pi}{\lambda}$$

$$v = \frac{\omega}{k} = \lambda f$$

Example

A mobile phone tower transmits a wave signal of frequency 900MHz. Calculate the length of the waves transmitted from the mobile phone tower.

Solution

Frequency, f = 900MHz = 900 $\times 10^{6}$ Hz

The speed of wave is c = 3 $\times$ 10^{8}m s^{-1}

$$\lambda = \frac{v}{f} = \frac{3 \times 10^{8}}{900 \times 10^{6}} = 0.33m$$

SUPERPOSITION PRINCIPLE

When a jerk is given to a stretched string which is tied at one end, a wave pulse is produced and the pulse travels along the string. Suppose two persons holding the stretched string on either side give a jerk simultaneously, then these two wave pulses move towards each other, meet at some point and move away from each other with their original identity. Their behaviour is very different only at the crossing/meeting points; this behaviour depends on whether the two pulses have the same or different shape. When the pulses have the same shape, at the crossing, the total displacement is the algebraic sum of their individual displacements and hence its net amplitude is higher than the amplitudes of the individual pulses. Whereas, if the two pulses have same amplitude but shapes are 180° out of phase at the crossing point, the net amplitude vanishes at that point and the pulses will recover their identities after crossing. Only waves can possess such a peculiar property and it is called superposition of waves. This means that the principle of superposition explains the net behaviour of the waves when they overlap. Generalizing to any number of waves i.e, if two are more waves in a medium move simultaneously, when they overlap, their total displacement is the vector sum of the individual displacements. We know that the waves satisfy the wave equation which is a linear second order homogeneous partial differential equation in both space coordinates and time. Hence, their linear combination (often called as linear superposition of waves) will also satisfy the same differential equation.

To understand mathematically, let us consider two functions which characterize the displacement of the waves, for example,

$$y_1 = A_1 \sin(kx - \omega t)$$

and

$$y_2 = A_2 \cos(kx - \omega t)$$

Since, both y1 and y2 satisfy the wave equation (solutions of wave equation) then their algebraic sum

$$y = y_1 + y_2$$

also satisfies the wave equation. This means, the displacements are additive. Suppose we multiply y_1 and y_2 with some constant then their amplitude is scaled by that constant.

Further, if C_1 and C_2 are used to multiply the displacements y_1 and y_2, respectively, then, their net displacement y is

$$y = C_1 y_1 + C_2 y_2$$

This can be generalized to any number of waves. In the case of n such waves in more than one dimension the displacements are written using vector notation. Here, the net displacement $\vec{y}$ is

$$\vec{y} = \sum_{i=1}^{n} C_i \vec{y}_i$$

The principle of superposition can explain the following :

(a) Space (or spatial) Interference (also known as Interference)

(b) Time (or Temporal) Interference (also known as Beats)

(c) Concept of stationary waves

Waves that obey principle of superposition are called linear waves (amplitude is much smaller than their wavelengths). In general, if the amplitude of the wave is not small then they are called non-linear waves. These violate the linear superposition principle, e.g. laser. In this chapter, we will focus our attention only on linear waves.

We will discuss the following in different subsections:

Interference of waves

Interference is a phenomenon in which two waves superimpose to form a resultant wave of greater, lower or the same amplitude.

Consider two harmonic waves having identical frequencies, constant phase difference φ and same wave form (can be treated as coherent source), but having amplitudes A_1 and A_2, then

$$y_1 = A_1 \sin(kx - \omega t)$$
$$y_2 = A_2 \sin(kx - \omega t + \varphi)$$

Suppose they move simultaneously in a particular direction, then interference occurs (i.e., overlap of these two waves). Mathematically

$$y = y1 + y2$$

Therefore, substituting equation (11.47) and equation (11.48) in equation (11.49), we get $y = A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \varphi)$

Using trigonometric identity $\sin(\alpha+\beta) = (\sin\alpha \cos\beta + \cos\alpha \sin\beta)$, we get
$y = A_1 \sin(kx - \omega t) + A_2 [\sin(kx - \omega t) \cos\varphi + \cos(kx - \omega t) \sin\varphi]$

$$y = \sin(kx - \omega t)(A_1 + A_2 \cos\varphi) + A_2 \sin\varphi \cos(kx - \omega t)$$

Let us re-define

$$A \cos\theta = (A_1 + A_2 \cos\varphi)$$
$$\text{and } A \sin\theta = A_2 \sin\varphi$$

then equation (11.50) can be rewritten as $y = A \sin(kx-\omega t) \cos\theta + A \cos(kx-\omega t) \sin\theta$

$$y = A (\sin(kx-\omega t) \cos\theta + \sin\theta \cos(kx-\omega t))$$
$$y = A \sin(kx-\omega t + \theta) \quad (11.53)$$

By squaring and adding equation (11.51) and equation (11.52), we get

$$A_2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos\varphi \quad (11.54)$$

Since, intensity is square of the amplitude ($I = A^2$), we have

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos j$$

This means the resultant intensity at any point depends on the phase difference at that point.

(a) For constructive interference:

When crests of one wave overlap with crests of another wave, their amplitudes will add up and we get constructive interference. The resultant wave has a larger amplitude than the individual waves.

The constructive interference at a point occurs if there is maximum intensity at that point, which means that

$$\cos\varphi = +1 \Rightarrow \phi = 0,\ 2\pi, 4\pi, \ldots = 2n\pi, \text{ where } n = 0,1,2,\ldots$$

This is the phase difference in which two waves overlap to give constructive interference.

Therefore, for this resultant wave,

$$I_{\max imum} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = \left(A_1 + A_2\right)^2$$

Hence, the resultant amplitude $A = A_1 + A_2$

(b) For destructive interference:

When the trough of one wave overlaps with the crest of another wave, their amplitudes "cancel" each other and we get destructive interference as shown in Figure 11.29 (b). The resultant amplitude is nearly zero. The destructive interference occurs if there is minimum intensity at that point, which means $\cos\varphi = -1 \Rightarrow \phi = \pi, 3\pi, 5\pi, \ldots = (2n-1)\pi$, where n = 0,1,2,…. i.e. This is the phase difference in which two waves overlap to give destructive interference. Therefore,

$$I_{\max imum} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = \left(A_1 + A_2\right)^2$$

Hence, the resultant amplitude

$$A = |A1 - A2|$$

Let us consider a simple instrument to demonstrate the interference of sound waves as shown in Figure 11.30.

Figure 11.30 Simple instrument to demonstrate interference of sound waves

A sound wave from a loudspeaker S is sent through the tube P. This looks like a T-shaped junction. In this case, half of the sound energy is sent in one direction and the remaining half is sent in the opposite direction. Therefore, the sound waves that reach the receiver R can travel along either of two paths. The distance covered by the sound wave along any path from the speaker to receiver is called the path length . From the Figure 11.30, we notice that the lower path length is fixed but the upper path length can be varied by sliding the upper tube i.e., is varied. The difference in path length is known as path difference,

$$\Delta r = |r2 - r1|$$

Suppose the path difference is allowed to be either zero or some integer (or integral) multiple of wavelength λ. Mathematically, we have

$$\Delta r = n\lambda \text{ where, } n = 0, 1, 2, 3, \ldots$$

Then the two waves arriving from the paths r_1 and r_2 reach the receiver at any instant are in phase (the phase difference is 0° or 2π) and interfere constructively as shown in Figure 11.31.

Therefore, in this case, maximum sound intensity is detected by the receiver. If the path difference is some half-odd-integer (or half-integral) multiple of wavelength λ, mathematically, $\Delta r = n\frac{\lambda}{2}$

where, n = 1,3,... (n is odd) then the two waves arriving from the paths r_1 and r_2 and reaching the receiver at any instant are out of phase (phase difference of π or 180°). They interfere destructively as shown in Figure 11.32. They will cancel each other.

Therefore, the amplitude is minimum or zero amplitude which means no sound. No sound intensity is detected by the receiver in this case. The relation between path difference and phase difference is phase difference = $\frac{2\pi}{\lambda}$ (path difference) (11.56) i.e., $\Delta\varphi = \frac{2\pi}{\lambda}\Delta r$ or $\Delta r = \frac{\lambda}{2\pi}\Delta\varphi$

Example

Consider two sources A and B as shown in the figure below. Let the two sources emit simple harmonic waves of same frequency but of different amplitudes, and both are in phase (same phase). Let O be any point equidistant from A and B as shown in the figure. Calculate the intensity at points O, Y and X. (X and Y are not equidistant from A & B)

Solution

The distance between OA and OB are the same and hence, the waves starting from A and B reach O after covering equal distances (equal path lengths). Thus, the path difference between two waves at O is zero.

$$OA - OB = 0$$

Since the waves are in the same phase, at the point O, the phase difference between two waves is also zero. Thus, the resultant intensity at the point O is maximum. Consider a point Y, such that the path difference between two waves is λ. Then the phase difference at Y is

$$\Delta\varphi = \frac{2\pi}{\lambda} \times \Delta r = \frac{2\pi}{\lambda} \times \lambda = 2\pi$$

Therefore, at the point Y, the two waves from A and B are in phase, hence, the intensity will be maximum.

Consider a point X, and let the path difference the between two waves be $\frac{\lambda}{2}$.

Then the phase difference at X is

$$\Delta\varphi = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

Therefore, at the point X, the waves meet and are in out of phase, Hence, due to destructive interference, the intensity will be minimum.

Example

Two speakers C and E are placed 5 m apart and are driven by the same source. Let a man stand at A which is 10 m away from the mid point O of C and E. The man walks towards the point O which is at 1 m (parallel to OC) as shown in the figure. He receives the first minimum in sound intensity at B. Then calculate the frequency of the source. (Assume speed of sound = 343 m s^{-1})

Solution

The first minimum occurs when the two waves reaching the point B are 180° (out of phase). The path difference $\Delta x = \frac{\lambda}{2}$.

In order to calculate the path difference, we have to find the path lengths x_1 and x_2. In a right triangle BDC,

DB = 10m and OC = $\frac{1}{2}$(5) = 2.5m

CD = OC −1 = (2.5 m)−1 m = 1.5 m

$$x_1 = \sqrt{(10)^2 + (1.5)^2} = \sqrt{100 + 2.25}$$
$$= \sqrt{102.25} = 10.1m$$

In a right triangle EFB,

DB = 10m and OE = $\frac{1}{2}$(5) = 2.5m = FA

FB = FA + AB = (2.5 m) + 1 m = 3.5 m

$$x_2 = \sqrt{(10)^2 + (3.5)^2} = \sqrt{100 + 12.25}$$
$$= \sqrt{112.25} = 10.6m$$

The path difference Δ x = x2 − x1 = 10.6 m−10.1 m = 0.5 m. Required that this path difference

$$\Delta x = \frac{\lambda}{2} = 0.5 \Rightarrow \lambda = 1.0m$$

To obtain the frequency of source, we use

$$v = \lambda f \Rightarrow f = \frac{v}{\lambda} = \frac{343}{1} = 343\,Hz$$
$$= 0.3\,kHz$$

If the speakers were connected such that already the path difference is $\frac{\lambda}{2}$. Now, the path difference combines with a path difference of $\frac{\lambda}{2}$. This gives a total path difference of λ which means, the waves are in phase and there is a maximum intensity at point B.

Formation of beats

When two or more waves superimpose each other with slightly different frequencies, then a sound of periodically varying amplitude at a point is observed. This phenomenon is known as beats. The number of amplitude maxima per second is called beat frequency. If we have two sources, then their difference in frequency gives the beat frequency.

Number of beats per second

$n = |f1 - f2|$ per second

Additional information (Not for examination): Mathematical treatment of beats

For mathematical treatment, let us consider two sound waves having same amplitude and slightly different frequencies f_1 and f_2, superimposed on each other.

Since the sound wave (pressure wave) is a longitudinal wave, let us consider $y_1 = A \sin(\omega_1 t)$ and $y_2 = A \sin(\omega_2 t)$ to be displacements of the two waves at a point x = 0 with same amplitude (region having high pressures) and different angular frequencies ω_1 and ω_2, respectively. Then when they are allowed to superimpose we get the net displacement

$$y = y_1 + y_2$$

$$y = A \sin(\omega_1 t) + A \sin(\omega_2 t)$$

But

$$\omega_1 = 2\pi f_1 \text{ and } \omega_2 = 2\pi f_2$$

Then

$$y = A \sin(2\pi f_1 t) + A \sin(2\pi f_2 t)$$

Using trigonometry formula

$$\sin C + \sin D = 2\cos\left(\frac{C-D}{2}\right)\sin\left(\frac{C+D}{2}\right)$$

$$y = 2A\cos\left(2\pi\left(\frac{f_1-f_2}{2}\right)t\right)\sin\left(2\pi\left(\frac{f_1+f_2}{2}\right)t\right)$$

Let, $y_p = 2A\cos\left(2\pi\left(\frac{f_1-f_2}{2}\right)t\right)$ (11.57)

and if f1 is slightly higher value than f2 then,

$\left(\frac{f_1-f_2}{2}\right) \ll \left(\frac{f_1+f_2}{2}\right)$ means yp in equation (11.57) varies very slowly when compared to $\left(\frac{f_1+f_2}{2}\right)$. Therefore $y = y_P \sin(2\pi f_{avg} t)$ (11.58)

This represents a simple harmonic wave of frequency which is an arithmetic average of frequencies of the individual waves, $f_{avg} = \left(\frac{f_1 + f_2}{2}\right)$ and amplitude yp varies with time t.

Case (A):

The resultant amplitude is maximum when y_p is maximum. Since $y_p\, a \cos\left(2\pi\left(\frac{f_1 - f_2}{2}\right)t\right)$, this means maximum amplitude occurs only when cosine takes ±1,

$$\cos\left(2\pi\left(\frac{f_1 - f_2}{2}\right)t\right) = \pm 1$$

$$\Rightarrow\ 2\pi\left(\frac{f_1 - f_2}{2}\right)t = n\pi,$$

or, $t = \frac{n}{(f_1 - f_2)}$ n = 0,1,2,3, ...

Hence, the time interval between two successive maxima is

$$t_2 - t_1 = t_3 - t_2 = \ldots = \frac{l}{(f_1 - f_2)} n = |f_1 - f_2| = \frac{1}{|t_1 - t_2|}$$

Therefore, the number of beats produced per second is equal to the reciprocal of the time interval between two consecutive maxima i.e., |f1 - f2|.

Case (B):

The resultant amplitude is minimum i.e., it is equal to zero when yp is minimum. Since $y_p\, a \cos\left(2\pi\left(\frac{f_1 - f_2}{2}\right)t\right)$, this means, minimum occurs only when cosine takes 0,

$$\cos\left(2\pi\left(\frac{f_1 - f_2}{2}\right)t\right) = 0$$

$$\Rightarrow\ 2\pi\left(\frac{f_1 - f_2}{2}\right)t = (2n+1)\frac{\pi}{2},$$

$$\Rightarrow\ (f_1 - f_2)t = \frac{1}{2}(2n+1)$$

or, $t = \frac{1}{2}\left(\frac{2n+1}{f_1 - f_2}\right)$, where f1 ≠ f2 n = 0,1,2,3,.....

Hence, the time interval between two successive minima is $t_2 - t_1 = t_3 = \ldots = \frac{1}{(f_1 - f_2)}; n = |f_1 f_2| = \frac{1}{|t_1 - t_2|}$

Therefore, the number of beats produced per second is equal to the reciprocal of the time interval between two consecutive minima i.e., $|f1 - f2|$.

Example

Consider two sound waves with wavelengths 5 m and 6 m. If these two waves propagate in a gas with velocity 330 ms^{-1}. Calculate the number of beats per second.

Solution

Given $\lambda_1 = 5_m$ and $\lambda_2 = 6m$

Velocity of sound waves in a gas is v = 330 ms-1

The relation between wavelength and velocity is $v = \lambda f \Rightarrow f = \frac{v}{\lambda}$

The frequency corresponding to wavelength λ_1 is $f_1 = \frac{v}{\lambda 1} = \frac{330}{5} = 66Hz$

The frequency corresponding to wavelength

λ_2 is $f_2 = \frac{v}{\lambda 2} = \frac{330}{6} = 55Hz$

The number of beats per second is

$|f1 - f2| = |66 - 55| = 11$ beats per sec

Example

Two vibrating tuning forks produce waves whose equation is given by $y_1 = 5 \sin(240\pi t)$ and $y_2 = 4 \sin(244\pi t)$. Compute the number of beats per second.

Solution

Given $y_1 = 5 \sin(240\pi t)$ and $y_2 = 4 \sin(244\pi t)$

Comparing with $y = A \sin(2\pi f_1 t)$, we get

$$2\pi f_1 = 240\pi \Rightarrow f_1 = 120Hz$$

$$2\pi f_2 = 244\pi \Rightarrow f_2 = 122Hz$$

The number of beats produced is $|f1 - f2| = |120 - 122| = |-2| = 2$ beats per sec

Standing Waves

Explanation of stationary waves

When the wave hits the rigid boundary it bounces back to the original medium and can interfere with the original waves. A pattern is formed, which are known as standing waves or stationary waves. Consider two harmonic progressive waves (formed by strings) that have the same amplitude and same velocity but move in opposite directions. Then the displacement of the first wave (incident wave) is

$$y_1 = A \sin(kx - \omega t) \quad (11.59)$$
(waves move toward right)

and the displacement of the second wave (reflected wave) is

$$y2 = A \sin(kx + \omega t) \quad (11.60)$$
(waves move toward left)

both will interfere with each other by the principle of superposition, the net displacement is

$$= y1 + y2 \quad (11.61)$$

Substituting equation (11.59) and equation (11.60) in equation (11.61), we get

$$y = A \sin(kx - \omega t) + A \sin(kx + \omega t) \quad (11.62)$$

Using trigonometric identity, we rewrite equation (11.62) as

$$y\,(x, t) = 2A \cos(\omega t) \sin(kx) \quad (11.63)$$

This represents a stationary wave or standing wave, which means that this wave does not move either forward or backward, whereas progressive or travelling waves will move forward or backward. Further, the displacement of the particle in equation (11.63) can be written in more compact form,

$$y(x,t) = A' \cos(\omega t)$$

where, $A' = 2A\sin(kx)$, implying that the particular element of the string executes simple harmonic motion with amplitude equals to A'. The maximum of this amplitude occurs at positions for which

$$\sin(kx) = 1 \Rightarrow kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = m\pi$$

where m takes half integer or half integral values. The position of maximum amplitude is known as antinode. Expressing wave number in terms of wavelength, we can represent the anti-nodal positions as

$$x_m = \left(\frac{2m+1}{2}\right)\frac{\lambda}{2}, \text{ where, m = 0,1,2... (11.64)}$$

For m = 0 we have maximum at $x_0 = \frac{\lambda}{2}$

For m = 1 we have maximum at $x_1 = \frac{3\lambda}{4}$

For m = 2 we have maximum at $x_2 = \frac{5\lambda}{4}$ and so on.

The distance between two successive antinodes can be computed by

$$x_m - x_{m-1} = \left(\frac{2m+1}{2}\right)\frac{\lambda}{2} - \left(\frac{(2m+1)+1}{2}\right)\frac{\lambda}{2} = \frac{\lambda}{2}$$

Similarly, the minimum of the amplitude A' also occurs at some points in the space, and these points can be determined by setting

$$\sin(kx) = 0 \Rightarrow k\,x = 0, \pi, 2\pi, 3\pi, \ldots = n\,\pi$$

where n takes integer or integral values. Note that the elements at these points do not vibrate (not move), and the points are called nodes. The n^{th}nodal positions is given by,

$$x_n = n\frac{\lambda}{2} \text{ where, n = 0,1,2,... (11.65)}$$

For n = 0 we have minimum at

$$x0 = 0$$

For n = 1 we have minimum at

$$x_1 = \frac{\lambda}{2}$$

For n = 2 we have maximum at

$$x2 = \lambda$$

and so on.

The distance between any two successive nodes can be calculated as

$$x_n - x_{n-1} = n\frac{\lambda}{2} - (n-1)\frac{\lambda}{2} = \frac{\lambda}{2}$$

Example

Compute the distance between anti-node and neighbouring node.

Solution

For nth mode, the distance between antinode and neighbouring node is

$$\Delta x_n = \left(\frac{2n+1}{2}\right)\frac{\lambda}{2} - n\frac{\lambda}{2} = \frac{\lambda}{4}$$

Characteristics of stationary waves

(1) Stationary waves are characterised by the confinement of a wave disturbance between two rigid boundaries. This means, the wave does not move forward or backward in a medium (does not advance), it remains steady at its place. Therefore, they are called "stationary waves or standing waves".

(2) Certain points in the region in which the wave exists have maximum amplitude, called as anti-nodes and at certain points the amplitude is minimum or zero, called as nodes.

(3) The distance between two consecutive nodes (or) anti-nodes is $\frac{\lambda}{2}$.

(4) The distance between a node and its neighbouring anti-node is $\frac{\lambda}{4}$.

(5) The transfer of energy along the standing wave is zero.

Comparison between progressive and stationary waves		
S.No	Progressive waves	Stationary waves
1.	Crests and troughs are formed in transverse progressive waves, and compression and rarefaction are formed in longitudinal progressive waves. These waves move forward or backward in a medium i.e., they will advance in a medium with a definite velocity.	Crests and troughs are formed in transverse stationary waves, and compression and rarefaction are formed in longitudinal stationary waves. These waves neither move forward nor backward in a medium i.e., they will not advance in a medium.
2.	All the particles in the medium vibrate such that the amplitude of the vibration for all particles is same.	Except at nodes, all other particles of the medium vibrate such that amplitude of vibration is different for different particles. The amplitude is minimum or zero at nodes and maximum at antinodes.
3.	These wave carry energy while propagating.	These waves do not transport energy.

Stationary waves in sonometer

Sono means sound related, and sonometer implies sound-related measurements. It is a device for demonstrating the relationship between the frequency of the sound produced in the transverse standing wave in a string, and the tension, length and mass per unit length of the string. Therefore, using this device, we can determine the following quantities:

(a) the frequency of the tuning fork or frequency of alternating current
(b) the tension in the string
(c) the unknown hanging mass

Construction:

The sonometer is made up of a hollow box which is one meter long with a uniform metallic thin string attached to it. One end of the string is connected to a hook and the other end is connected to a weight hanger through a pulley. Since only one string is used, it is also known as monochord. The weights are added to the free end of the wire to increase the tension of the wire. Two adjustable wooden knives are put over the board, and their positions are adjusted to change the vibrating length of the stretched wire.

Working :

A transverse stationary or standing wave is produced and hence, at the knife edges P and Q, nodes are formed. In between the knife edges, anti-nodes are formed.

If the length of the vibrating element is l then

$$l = \frac{\lambda}{2} \Rightarrow \lambda = 2l$$

Let f be the frequency of the vibrating element, T the tension of in the string and μ the mass per unit length of the string. Then using equation (11.13), we get

$$f = \frac{v}{\lambda} = \frac{1}{2l}\sqrt{\frac{T}{\mu}} \text{ in Hertz (11.66)}$$

Let ρ be the density of the material of the string and d be the diameter of the string. Then the mass per unit length μ,

$$\mu = \text{Area} \times \text{density} = \pi r2\rho = \frac{\pi\rho d^2}{4}$$

Frequency $f = \frac{v}{\lambda} = \frac{1}{2l}\sqrt{\frac{T}{\frac{\pi d^2 \rho}{4}}}$

$$f = \frac{1}{ld}\sqrt{\frac{T}{\pi\rho}}$$

Example

Let f be the fundamental frequency of the string. If the string is divided into three segments l_1, l_2 and l_3 such that the fundamental frequencies of each segments be f_1, f_2 and f_3, respectively. Show that

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}$$

Solution

For a fixed tension T and mass density μ, frequency is inversely proportional to the string length i.e.

$$f \propto \frac{1}{l} \Rightarrow f=\frac{v}{2l} \Rightarrow l=\frac{v}{2f}$$

For the first length segment

$$f_1=\frac{v}{2l_1} \Rightarrow l_1=\frac{v}{2f_1}$$

For the second length segment

$$f_2=\frac{v}{2l_2} \Rightarrow l_2=\frac{v}{2f_2}$$

Therefore, the total length

$$l=l_1+l_2+l_3$$

$$\frac{v}{2f}=\frac{v}{2f_1}+\frac{v}{2f_2}+\frac{v}{2f} \Rightarrow \frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}$$

Fundamental frequency and overtones

Let us now keep the rigid boundaries at x = 0 and x = L and produce a standing waves by wiggling the string (as in plucking strings in a guitar). Standing waves with a specific wavelength are produced. Since, the amplitude must vanish at the boundaries, therefore, the displacement at the boundary must satisfy the following conditions y(x = 0, t) = 0 and y(x = L, t) = 0. Since the nodes formed are at a distance $\frac{\lambda_n}{2}$ apart, we have $n\left(\frac{\lambda_n}{2}\right)=L$, where n is an integer, L is the length between the two boundaries and λn is the specific wavelength that satisfy the specified boundary conditions. Hence,

$$\boxed{\lambda_n=\left(\frac{2L}{n}\right)}$$

Therefore, not all wavelengths are allowed. The (allowed) wavelengths should fit with the specified boundary conditions, i.e., for n = 1, the first mode of vibration has specific wavelength

$$\lambda_2 = \left(\frac{2L}{2}\right) = L$$

For n = 3, the third mode of vibration has specific wavelength

$$\lambda_3 = \left(\frac{2L}{3}\right)$$

and so on.

The frequency of each mode of vibration (called natural frequency) can be calculated.
We have,

$$f_n = \frac{v}{\lambda_n} = n\left(\frac{v}{2L}\right)$$

The lowest natural frequency is called the fundamental frequency.

$$f_1 = \frac{v}{\lambda_1} = \left(\frac{v}{2L}\right)$$

The second natural frequency is called the first over tone.

$$f_2 = 2\left(\frac{v}{2L}\right) = \frac{1}{L}\sqrt{\frac{T}{\mu}}$$

The third natural frequency is called the second over tone.

$$f_3 = 3\left(\frac{v}{2L}\right) = 3\left(\frac{1}{2L}\sqrt{\frac{T}{\mu}}\right)$$

and so on.

Therefore, the nth natural frequency can be computed as integral (or integer) multiple of fundamental frequency, i.e.,

$f_n = nf_1$, where n is an integer. If natural frequencies are written as integral multiple of fundamental frequencies, then the frequencies are called harmonics. Thus, the first harmonic is $f_1 = f_1$ (the fundamental frequency is called first harmonic), the second harmonic is $f_2 = 2f_1$, the third harmonic is $f_3 = 3f_1$ etc.

Example

Consider a string in a guitar whose length is 80 cm and a mass of 0.32 g with tension 80 N is plucked. Compute the first four lowest frequencies produced when it is plucked.

Solution

The velocity of the wave

$$v = \sqrt{\frac{T}{\mu}}$$

The length of the string, L = 80 cm=0.8 m The mass of the string, m = 0.32 g =0.32 × 10^{-3}kg

Therefore, the linear mass density, $m = \frac{0.32 \times 10^{3}}{0.8} = 0.4 \times 10^{-3} kgm^{-1}$

The tension in the string, T = 80 N

$$v = \sqrt{\frac{80}{0.4 \times 10^{-3}}} = 447.2 \text{ m s}^{-1}$$

The wavelength corresponding to the fundamental frequency f_1 is λ_1 = 2L = 2 × 0.8 = 1.6 m

The fundamental frequency f_1 corresponding to the wavelength λ_1

$$f_1 = \frac{v}{\lambda_1} = \frac{447.2}{1.6} = 279.5 Hz$$

Similarly, the frequency corresponding to the second harmonics, third harmonics and fourth harmonics are

$f_2 = 2f_1$ = 559 Hz

$f_3 = 3f_1$ = 838.5 Hz

$f_4 = 4f_1$ = 1118 Hz

Laws of transverse vibrations in stretched strings

There are three laws of transverse vibrations of stretched strings which are given as follows:

(i) The law of length :

For a given wire with tension T (which is fixed) and mass per unit length μ (fixed) the frequency varies inversely with the vibrating length. Therefore,

$$f \propto \frac{1}{l} \Rightarrow f = \frac{C}{l}$$

⇒l×f = C, where C is a constant

(ii) The law of tension:

For a given vibrating length l (fixed) and mass per unit length μ (fixed) the frequency varies directly with the square root of the tension T,

$$f a \sqrt{T}$$

Þ $f a \sqrt{T}$ .where A is a constant

(iii) The law of mass:

For a given vibrating length l (fixed) and tension T (fixed) the frequency varies inversely with the square root of the mass per unit length μ,

$$f a \frac{1}{\sqrt{u}}$$

Þ $f = \frac{B}{\sqrt{m}}$, where B is a constant.

INTENSITY AND LOUDNESS

Consider a source and two observers (listeners). The source emits sound waves which carry energy. The sound energy emitted by the source is same regardless of whoever measures it, i.e., it is independent of any observer standing in that region. But the sound received by the two observers may be different; this is due to some factors like sensitivity of ears, etc. To quantify such thing, we define two different quantities known as intensity and loudness of sound.

Intensity of sound

When a sound wave is emitted by a source, the energy is carried to all possible surrounding points. The average sound energy emitted or transmitted per unit time or per second is called sound power. Therefore, the intensity of sound is defined as “the sound power transmitted per unit area taken normal to the propagation of the sound wave”.

For a particular source (fixed source), the sound intensity is inversely proportional to the square of the distance from the source.

$$I = \frac{power\ of\ the\ source}{4\pi r^2} \Rightarrow I \propto \frac{1}{r^2}$$

This is known as inverse square law of sound intensity.

Example

A baby cries on seeing a dog and the cry is detected at a distance of 3.0 m such that the intensity of sound at this distance is 10^{-2} W m^{-2}. Calculate the intensity of the baby's cry at a distance 6.0 m.

Solution

I_1 is the intensity of sound detected at a distance 3.0 m and it is given as 10^{-2} W m^{-2}. Let I_2 be the intensity of sound detected at a distance 6.0 m. Then, r_1 = 3.0 m, r_2 = 6.0 m

and since $I a \frac{1}{r^2}$

the power output does not depend on the observer and depends on the baby. Therefore,

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

$$I_2 = I_1 \frac{r_1^2}{r_2^2}$$

$$I_2 = 0.25 \times 10^{-2}\ W\ m^{-2}$$

Loudness of sound

Two sounds with same intensities need not have the same loudness. For example, the sound heard during the explosion of balloons in a silent closed room is very loud when compared to the same explosion happening in a noisy market. Though the intensity of the sound is the same, the loudness is not. If the intensity of sound is increased then loudness also increases. But additionally, not only does intensity matter, the internal and subjective experience of "how loud a sound is" i.e., the sensitivity of the listener also matters here. This is often called loudness. That is, loudness depends on both intensity of sound wave and sensitivity of the ear (It is purely observer dependent quantity which varies from person to person) whereas the intensity of sound does not depend on the observer. The loudness of sound is defined as "the degree of sensation of sound produced in the ear or the perception of sound by the listener".

Intensity and loudness of sound

Our ear can detect the sound with intensity level ranges from 10^{-2} Wm^{-2} to 20 W m^{-2}.

According to Weber-Fechner's law, "loudness (L) is proportional to the logarithm of the actual intensity (I) measured with an accurate non-human instrument". This means that

$$L \propto \ln I$$
$$L = k \ln I$$

where k is a constant, which depends on the unit of measurement. The difference between two loudnesses, L1 and L0 measures the relative loudness between two precisely measured intensities and is called as sound intensity level. Mathematically, sound intensity level is

$$\Delta L = L_1 - L_0 = k \ln I_1 - k \ln I_0 = k \ln \left[\frac{I_1}{I_0}\right]$$

If k = 1, then sound intensity level is measured in bel, in honour of Alexander Graham Bell. Therefore,

$$\Delta L = \ln \left[\frac{I_1}{I_0}\right] bel$$

However, this is practically a bigger unit, so we use a convenient smaller unit, called decibel. Thus, decibel $= \frac{1}{10}$ bel. Therefore, by multiplying and dividing by 10, we get

$$\Delta L = 10 \left(\ln \left[\frac{I_1}{I_0}\right]\right) \frac{1}{10} \text{ bel}$$

$$\Delta L = 10 \ln \left[\frac{I_1}{I_0}\right] \text{ decibel with } k = 10$$

For practical purposes, we use logarithm to base 10 instead of natural logarithm,

$$\Delta L = 10 \log_{10} \left[\frac{I_1}{I_0}\right] \text{ decibel}$$

Example

The sound level from a musical instrument playing is 50 dB. If three identical musical instruments are played together then compute the total intensity. The intensity of the sound from each instrument is 10^{-12} W m^{-2}

Solution

$$\Delta L = 10 \log_{10}\left[\frac{I_1}{I_0}\right] = 50\ dB$$

$$\log_{10}\left[\frac{I_1}{I_0}\right] = 5\ dB$$

$\frac{I_1}{I_0} = 10^5 \Rightarrow I_1 = 10^5 I_0 = 10^5 \times 10^{-12} Wm^{-2}$

$I_1 = 10^{-7} Wm^{-2}$

Since three musical instruments are played, therefore, $I_{total} = 3I_1 = 3 \times 10^{-7}$ Wm^{-2}.

VIBRATIONS OF AIR COLUMN

Musical instruments like flute, clarinet, nathaswaram, etc are known as wind instruments. They work on the principle of vibrations of air columns. The simplest form of a wind instrument is the organ pipe. It is made up of a wooden or metal pipe which produces the musical sound. For example, flute, clarinet and nathaswaram are organ pipe instruments. Organ pipe instruments are classified into two types:

(a) Closed organ pipes:

It is a pipe with one end closedand the other end open. If one end of a pipeis closed, the wave reflected at this closedend is 180° out of phase with the incomingwave. Thus there is no displacement of the particles at the closed end. Therefore, nodes are formed at the closed end and anti-nodes are formed at open end.

Let us consider the simplest mode of vibration of the air column called the fundamental mode. Anti-node is formed at the open end and node at closed end. From the Figure, let L be the length of the tube and the wavelength of the wave produced. For the fundamental mode of vibration, we have,

$$L = \frac{\lambda_1}{4} \ or \ \lambda_1 = 4L$$

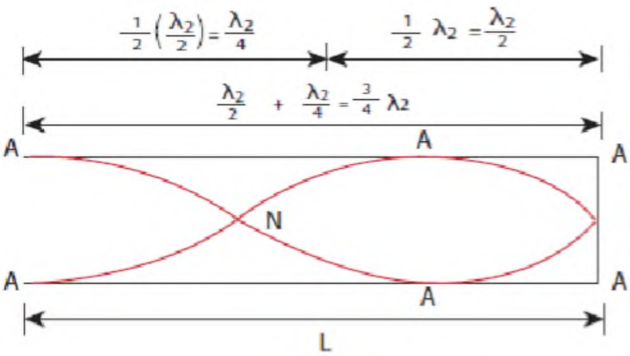
The frequency of the note emitted is

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$$

which is called the fundamental note.

The frequencies higher than fundamental frequency can be produced by blowing air strongly at open end. Such frequencies are called overtones.

The Figure shows the second mode of vibration having two nodes and two antinodes, for which we have, from example.



second mode of vibration
having two nodes and two anti-nodes

$$4L = 3l_2$$

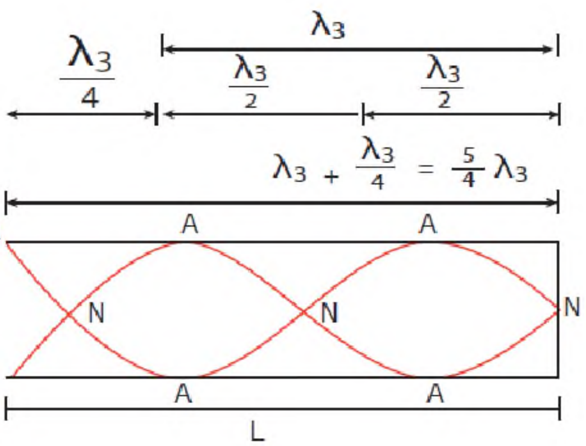
$$L = \frac{3l_2}{4} \; or \; l_2 = \frac{4L}{3}$$

The frequency for this,

$$f_2 = \frac{v}{l_2} = \frac{3v}{4L} = 3f_1$$

is called first over tone, since here, the frequency is three times the fundamental frequency it is called third harmonic.

The Figure shows third mode of vibration having three nodes and three anti-nodes.



Third mode of vibration
having three nodes and three anti-nodes

we have,

$$4L = 5l_3$$

$$L = \frac{5l_3}{4} \text{ or } l_3 = \frac{4L}{5}$$

The frequency

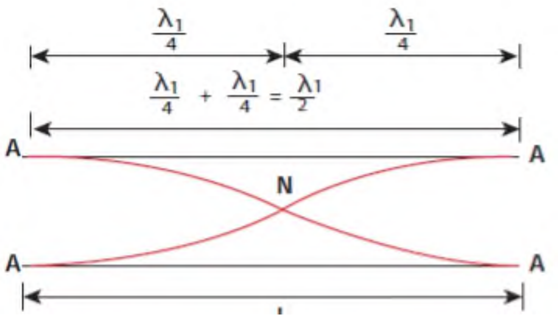
$$f^3 = \frac{v}{l_3} = \frac{5v}{4L} = 5f_1$$

is called second over tone, and since n = 5 here, this is called fifth harmonic. Hence, the closed organ pipe has only odd harmonics and frequency of the n^{th} harmonic is $f_n = (2n+1)f_1$. Therefore, the frequencies of harmonics are in the ratio

$$f_1 : f_2 : f_3 : f_4 : \ldots = 1 : 3 : 5 : 7 : \ldots$$

(b) Open organ pipes:

It is a pipe with both the ends open. At both open ends, anti-nodes are formed. Let us consider the simplest mode of vibration of the air column called fundamental mode. Since anti-nodes are formed at the open end, a node is formed at the mid-point of the pipe.



Antinodes are formed at
the open end and a node is formed at
the middle of the pipe.

From Figure, if L be the length of the tube, the wavelength of the wave produced is given by

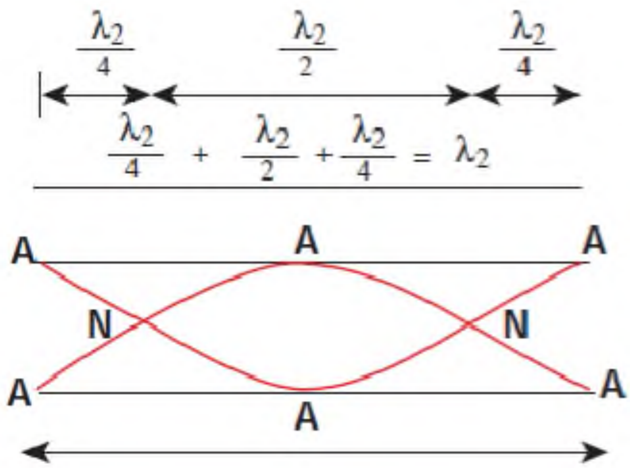
$$L = \frac{\lambda_1}{2} \text{ or } \lambda_1 = 2L$$

The frequency of the note emitted is

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

which is called the fundamental note.

The frequencies higher than fundamental frequency can be produced by blowing air strongly at one of the open ends. Such frequencies are called overtones.



Second mode of
vibration in open pipes having two
nodes and three anti-nodes

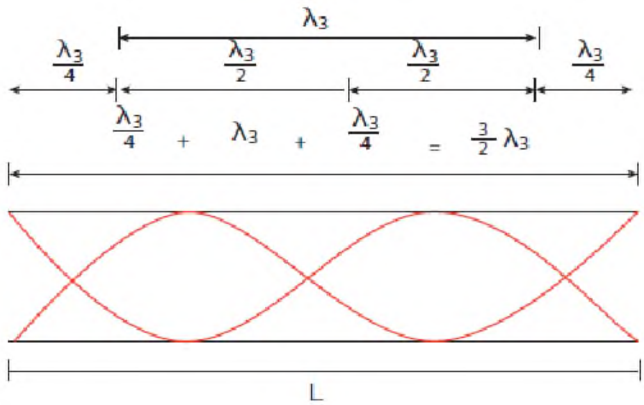
The Figure shows the second mode of vibration in open pipes. It has two nodes and three anti-nodes, and therefore,

$$L = \lambda_2 \text{ or } \lambda_2 = L$$

The frequency

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2 \times \frac{v}{2L} = 2 f_1$$

is called first over tone. Since n = 2 here, it is called the second harmonic.



Third mode of vibration
having three nodes and four anti-nodes

The Figure above shows the third mode of vibration having three nodes and four anti-nodes

$$L = \frac{3}{2}\lambda_3 \ or \ \lambda_3 = \frac{2L}{3}$$

The frequency

$$f_3 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3\,f_1$$

is called second over tone. Since n = 3 here, it is called the third harmonic.

Hence, the open organ pipe has all the harmonics and frequency of nth harmonic is $f_n = nf_1$. Therefore, the frequencies of harmonics are in the ratio

$$f_1 : f_2 : f_3 : f_4 : \ldots = 1 : 2 : 3 : 4 : \ldots$$

Example

If a flute sounds a note with 450Hz, what are the frequencies of the second, third, and fourth harmonics of this pitch?. If the clarinet sounds with a same note as 450Hz, then what are the frequencies of the lowest three harmonics produced ?.

Solution

For a flute which is an open pipe, we have
Second harmonics f2 = 2 f1 = 900 Hz
Third harmonics f3 = 3 f1 = 1350 Hz
Fourth harmonics f4 = 4 f1 = 1800 Hz

For a clarinet which is a closed pipe, we have
Second harmonics f2 = 3 f1 = 1350 Hz
Third harmonics f3 = 5 f1 = 2250 Hz
Fourth harmonics f4 = 7 f1 = 3150 Hz

Example

If the third harmonics of a closed organ pipe is equal to the fundamental frequency of an open organ pipe, compute the length of the open organ pipe if the length of the closed organ pipe is 30 cm.

Solution

Let l_2 be the length of the open organ pipe, with l_1 =30 cm the length of the closedorgan pipe.It is given that the third harmonic of closed organ pipe is equal to the fundamental frequency of open organ pipe.
The third harmonic of a closed organ pipe is

$$f_2 = \frac{v}{\lambda_2} = \frac{3v}{4l_1} = 3f_1$$

The fundamental frequency of open organ pipe is $f_1 = \frac{v}{\lambda_1} = \frac{v}{2l_2}$

Therefore,

$$\frac{v}{2l_2} = \frac{3v}{4l_1} \Rightarrow l_2 = \frac{2l_1}{3} = 20\,cm$$

Resonance air column apparatus

The resonance air column apparatus is one of the simplest techniques to measure the speed of sound in air at room temperature. It consists of a cylindrical glass tube of one meter length whose one end A is open and another end B is connected to the water reservoir R through a rubber tube as shown in Figure. This cylindrical glass tube is mounted on a vertical stand with a scale attached to it. The tube is partially filled with water and the water level can be adjusted by raising or lowering the water in the reservoir R. The surface of the water will act as a closed end and other as the open end. Therefore, it behaves like a closed organ pipe, forming nodes at the surface of water and antinodes at the closed end. When a vibrating tuning fork is brought near the open end of the tube, longitudinal waves are formed inside the air column. These waves move downward as shown in Figure, and reach the surfaces of water and get reflected and produce standing waves. The length of the air column is varied by changing the water level until a loud sound is produced in the air column. At this particular length the frequency of waves in the air column resonates with the frequency of the tuning fork (natural frequency of the tuning fork). At resonance, the frequency of sound waves produced is equal to the frequency of the tuning fork. This will occur only when the length of air column is proportional to $\left(\frac{1}{4}\right)^{th}$ of the wavelength of the sound waves produced.

Let the first resonance occur at length L_1, then

$$\frac{1}{4}\lambda = L_1$$

But since the antinodes are not exactly formed at the open end, we have to include a correction, called end correction e, by assuming that the antinode is formed at some small distance above the open end. Including this end correction, the first resonance is

$$\frac{1}{4}\lambda = L_1 + e$$

Now the length of the air column is increased to get the second resonance. Let L_2 be the length at which the second resonance occurs. Again taking end correction into account, we have

$$\frac{3}{4}\lambda = L_2 + e$$

In order to avoid end correction, let us take the difference of equation we get

$$\frac{3}{4}\lambda - \frac{1}{4}\lambda = (L_2 + e) - (L_1 + e)$$

$$\Rightarrow \frac{1}{2}\lambda = L_2 - L_1 = \Delta L$$

$$\Rightarrow \lambda = 2\Delta L$$

The speed of the sound in air at room temperature can be computed by using the formula

$$v = f\lambda = 2f\,\Delta L$$

Further, to compute the end correction, we use equations, we get

$$e = \frac{L_2 - 3L_1}{2}$$

Example

A frequency generator with fixed frequency of 343 Hz is allowed to vibrate above a 1.0 m high tube. A pump is switched on to fill the water slowly in the tube. In order to get resonance, what must be the minimum height of the water?. (speed of sound in air is 343 m s^{-1})

Solution

The wavelength, $\lambda = \frac{c}{f}$

$$\lambda = \frac{343ms^{1}}{343Hz} = 1.0m$$

Let the length of the resonant columns be L_1, L_2 and L_3. The first resonance occurs at length L_1

$$L_1 = \frac{l}{4} = \frac{1}{4} = 0.25m$$

The second resonance occurs at length L_2

$$L_2 = \frac{3l}{4} = \frac{3}{4} = 0.75m$$

The third resonance occurs at length

$$L_3 = \frac{5l}{4} = \frac{5}{4} = 1.25m$$

and so on.

Since total length of the tube is 1.0 m the third and other higher resonances do not occur. Therefore, the minimum height of water H_{min} for resonance is,

H_{min} = 1.0 m − 0.75 m = 0.25 m

Example

A student performed an experiment to determine the speed of sound in air using the resonance column method. The length of the air column that resonates in the fundamental mode with a tuning fork is 0.2 m. If the length is varied such that the same tuning fork resonates with the first overtone at 0.7 m. Calculate the end correction.

Solution

End correction

$$e = \frac{L_2 - 3L_1}{2} = \frac{0.7 - 3(0.2)}{2} = 0.05m$$

Example

Consider a tuning fork which is used to produce resonance in an air column. A resonance air column is a glass tube whose length can be adjusted by a variable piston. At room temperature, the two successive resonances observed are at 20 cm and 85 cm of the column length. If the frequency of the length is 256 Hz, compute the velocity of the sound in air at room temperature.

Solution

Given two successive length (resonance) to be L_1 = 20 cm and L_2 = 85 cm
The frequency is f = 256 Hz

$$v = f\lambda = 2f\,\Delta L = 2f\,(L_2 - L_1)$$

$$= 2 \times 256 \times (85 - 20) \times 10^{-2}\ m\ s^{-1}$$

$v = 332.8\ cm^{-1}$

DOPPLER EFFECT

Often we have noticed that the siren sound coming from a police vehicle or ambulance increases when it comes closer to us and decreases when it moves away from us. When we stand near any passing train the train whistle initially increases and then it will decrease. This is known as Doppler Effect, named after Christian Doppler (1803 – 1853). Suppose a source produces sound with some frequency, we call it the as source frequency fs. If the source and an observer are at a fixed distance then the observer observes the sound with frequency f_0. This is the same as the sound frequency produced by the source f_s, i.e., $f_0 = f_s$. Hence, there is no difference in frequency, implying no Doppler effect is observed.

Both source and observer are stationary. No Doppler effect is observed.

What happens if either source or an observer or both move?. Certainly, $f_o \neq f_s$. That is, when the source and the observer are in relative motion with respect to each other and to the medium in which sound propagates, the frequency of the sound wave observed is different from the frequency of the source. This phenomenon is called Doppler Effect. The frequency perceived by the observer is known as apparent frequency. We can consider the following situations for the study of Doppler effect in sound waves

(a) Source and Observer: We can consider either the source or observer in motion or both are in motion. Further we can treat the motion to be along the line joining the source and the observer, or inclined at an angle θ to this line.

(b) Medium: We can treat the medium to be stationary or the direction of motion of the medium is along or opposite to the direction of propagation of sound.

(c) Speed of Sound: We can also consider the case where speed of the source or an observer is greater or lesser than the speed of sound.

In the following section, we make the following assumptions: the medium is stationary, and motion is along the line joining the source and the observer, and the speeds of the source and the observer are both less than the speed of sound in that medium.

We consider three cases:
(i) Source in motion and Observer is at rest.
(a) Source moves towards observer
(b) Source moves away from the observer

(ii) Observer in motion and Source is at rest.
(a) Observer moves towards Source
(b) Observer receding away from the Source

(iii) Both are in motion
(a) Source and Observer approach each other
(b) Source and Observer recede from each other
(c) Source chases Observer
(d) Observer chases Source

Stationary observer and stationary source means the observer and source are both at rest with respect to medium respectively

Source in motion and the observer at rest

(a) Source moves towards the observer Suppose a source S moves to the right (as shown in Figure) with a velocity vs and let the frequency of the sound waves produced by the source be fs. We assume the velocity of sound in a medium is v. The compression (sound wave front) produced by the source S at three successive instants of time are shown in the Figure. When S is at position x1 the compression is at C_1. When S is at position x2, the compression is at C_2 and similarly for x_3 and C_3. Assume that if C_1 reaches the observer's position A then at that instant C_2 reaches the point B and C_3 reaches the point C as shown in the Figure 11.46. It is obvious to see that the distance between compressions C_2 and C_3 is shorter than distance between C_1 and C_2. This means the wavelength decreases when the source S moves towards the observer O (since sound travels longitudinally and wavelength is the distance between two consecutive compressions). But frequency is inversely related to wavelength and therefore, frequency increases.

Source S moves towards an
Observer O (right) with velocity

Let λ be the wavelength of the source S as measured by the observer when S is at position x_1 and λ' be wavelength of the source observed by the observer when S moves to position x_2. Then the change in wavelength is $\Delta\lambda = \lambda - \lambda' = v_s t$, where t is the time taken by the source to travel between x_1 and x_2. Therefore,

$$\lambda' = \lambda - v_s t$$

$$\text{But} \quad t = \frac{\lambda}{v}$$

On substituting equation (11.84) in equation (11.83), we get

$$\lambda' = \lambda\left(1 - \frac{v_s}{v}\right)$$

Since frequency is inversely proportional to wavelength, we have

$$f' = \frac{v_s}{\lambda'} \quad \text{and} \quad f = \frac{v_s}{\lambda}$$

$$\text{Hence,} \quad f' = \frac{f}{\left(1 - \frac{v_s}{v}\right)}$$

Since, $\frac{v_s}{v} \ll 1$, we use the binomial expansion and retaining only first order in $\frac{v_s}{v}$, we get

$$f' = f\left(1 + \frac{v_s}{v}\right) \text{v}$$

(b) Source moves away from the observer:

Since the velocity here of the source is opposite in direction when compared to case (a), therefore, changing the sign of the velocity of the source in the above case i.e, by substituting ($v_s \rightarrow -v_s$) in equation (11.83), we get

$$f' = \frac{f}{\left(1 + \frac{v_s}{v}\right)}$$

Using binomial expansion again, we get,

$$f' = f\left(1 - \frac{v_s}{v}\right)$$

Observer in motion and source at rest

(a) Observer moves towards Source

Observer moves towards Source

Let us assume that the observer O moves towards the source S with velocity v_o. The source S is at rest and the velocity of sound waves (with respect to the medium) produced by the source is v. From the Figure, we observe that both v_o and v are in opposite direction. Then, their relative velocity is $v_r = v + v_0$. The wavelength of the sound wave is $l = \frac{v}{f}$, which means the frequency observed by the observer O is $f_1 = \frac{v_r}{l}$. Then

$$f' = \frac{v_r}{\lambda} = \left(\frac{v + v_0}{v}\right) f = f\left(1 + \frac{v_0}{v}\right)$$

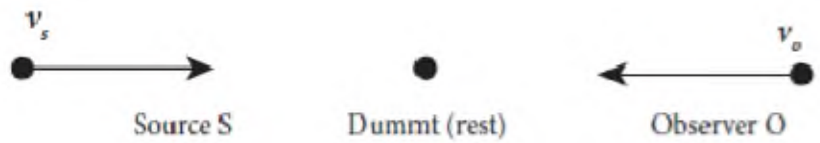
(b) Observer recedes away from the Source

If the observer O is moving away (receding away) from the source S, then velocity v_0 and v moves in the same direction. Therefore, their relative velocity is $v_r = v - v_0$. Hence, the frequency observed by the observer O is

$$f' = \frac{v_r}{\lambda} = \left(\frac{v - v_0}{v}\right) f = f\left(1 - \frac{v_0}{v}\right)$$

Both are in motion

(a) Source and observer approach each other



Source and Observer approach towards each other.

Let vs and v0 be the respective velocities of source and observer approaching each other. In order to calculate the apparent frequency observed by the observer, as a simple calculation, let us have a dummy (behaving as observer or source) in between the source and observer. Since the dummy is at rest, the dummy (observer) observes the apparent frequency due to approaching source as given in equation as

$$f_d = \frac{f}{\left(1 - \frac{v_s}{v}\right)}$$

At that instant of time, the true observer approaches the dummy from the other side. Since the source (true source) comes in a direction opposite to true observer, the dummy (source) is treated as stationary source for the true observer at that instant. Hence, apparent frequency when the true observer approaches the stationary source (dummy source), from equation is

$$f' = f_d\left(1+\frac{v_0}{v}\right) \Rightarrow f_d = \frac{f'}{\left(1+\frac{v_0}{v}\right)}$$

Since this is true for any arbitrary time, therefore, comparing equation (11.91) and equation (11.92), we get

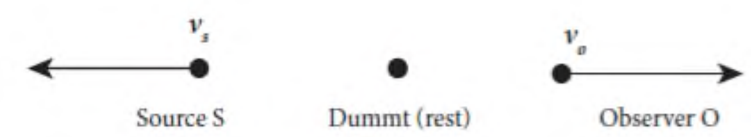
$$\frac{f}{\left(1-\frac{v_s}{v}\right)} = \frac{f'}{\left(1+\frac{v_0}{v}\right)}$$

$$\Rightarrow \frac{v f'}{(v+v_o)} = \frac{v f}{(v-v_s)}$$

Hence, the apparent frequency as seen by the observer is

$$f' = \left(\frac{v+v_0}{v-v_s}\right) f$$

(b) Source and observer recede from each other

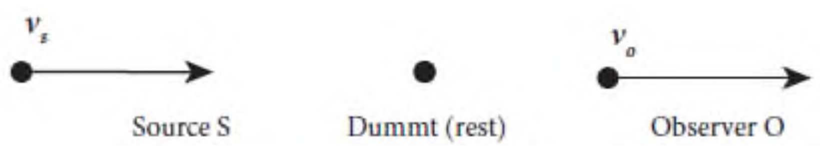


Source and Observer resides from each other

Here, we can derive the result as in the previous case. Instead of a detailed calculation, by inspection from Figure, we notice that the velocity of the source and the observer each point in opposite directions with respect to the case in (a) and hence, we substitute ($v_s \rightarrow -v_s$) and ($v_0 \rightarrow -v_0$) in equation, and therefore, the apparent frequency observed by the observer when the source and observer recede from each other is

$$f' = \left(\frac{v+v_0}{v-v_s}\right) f$$

(c) Source chases the observer

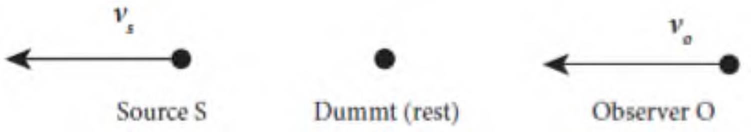


Source chases observer

Only the observer's velocity is oppositely directed when compared to case (a). Therefore, substituting ($v0 \rightarrow -v0$) in equation, we get

$$f' = \left(\frac{v - v_0}{v - v_s}\right) f$$

(d) Observer chases the source



Observer chases Source

Only the source velocity is oppositely directed when compared to case (a). Therefore, substituting $v_s \rightarrow -v_s$ in equation, we get

$$f' = \left(\frac{v + v_0}{v + v_s}\right) f$$

Discuss with your teacher
"Doppler effect in light"
"Doppler effect in sound is asymmetrical where as Doppler effect in light is symmetrical"

Applications of Doppler effect

Doppler effect has many applications. Specifically Doppler effect in light has many applications in astronomy. As an example, while observing the spectra from distant objects like stars or galaxies, it is possible to determine the velocities at which distant objects like stars or galaxies move towards or away from Earth. If the spectral lines of the star are found to shift towards red end of the spectrum (called as red shift) then the star is receding away from the Earth. Similarly, if the spectral lines of the star are found to shift towards the blue end of the spectrum (called as blue shift) then the star is approaching Earth. Let $\Delta\lambda$ be the Doppler shift. Then $\Delta\lambda = \frac{v}{c}\lambda$ where v is the velocity of the star. It may be noted that Doppler shift measures only the radial component (along the line of sight) of the relative velocity v.

Example

A sound of frequency 1500 Hz is emitted by a source which moves away from an observer and moves towards a cliff at a speed of 6 ms^{-1}.
(a) Calculate the frequency of the sound which is coming directly from the source.

(b) Compute the frequency of sound heard by the observer reflected off the cliff. Assume the speed of sound in air is 330 m s-1

Solution

(a) Source is moving away and observer is stationary, therefore, the frequency of sound heard directly from source is

$$f' = \frac{f}{\left(1+\frac{v_s}{v}\right)} = \frac{1500}{\left(1+\frac{6}{330}\right)} = 1473\,Hz$$

(b) Sound is reflected from the cliff and reaches observer, therefore,

$$f' = \frac{f}{\left(1-\frac{v_s}{v}\right)} = \frac{1500}{\left(1-\frac{6}{330}\right)} = 1528\,Hz$$

Example

An observer observes two moving trains, one reaching the station and other leaving the station with equal speeds of 8 m s^{-1}. If each train sounds its whistles with frequency 240 Hz, then calculate the number of beats heard by the observer.

Solution:

Observer is stationary

(i) Source (train) is moving towards an observer:

Apparent frequency due to train arriving station is

$$f_{in} = \frac{f}{\left(1-\frac{v_s}{v}\right)} = \frac{240}{\left(1-\frac{8}{330}\right)} = 246\,Hz$$

(ii) Source (train) is moving away form an observer:

Apparent frequency due to train leaving station is

$$f_{out} = \frac{f}{\left(1-\frac{v_s}{v}\right)} = \frac{240}{\left(1-\frac{8}{330}\right)} = 234\,Hz$$

So the number of beats = $|f_{in} - f_{out}|$ = (246-234) = 12